Technical online appendix to “A Structured Argumentation Framework for Modeling Debates in the Formal Sciences”

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Abstract

This technical online appendix to the paper “A Structured Argumentation Framework for Modeling Debates in the Formal Sciences” consists of two parts: In the first part, we provide a detailed description of our formal ASPIC-END model of parts of the debate that mathematicians had about the Axiom of Choice in the early 20th century, and we briefly discuss the insight into the strengths and drawbacks of the modeling capacities of ASPIC-END that we have gained from producing this model. In the second part, we present, motivate and prove six rationality postulates that ASPIC-END satisfies.

1 Modelling argumentaton on Axiom of Choice

In this part of this technical online appendix, we present a model that formalizes parts of the debate that mathematicians had about the Axiom of Choice in the early 20th century [see Moore, 1982]. In 1904, the German mathematician Ernst Zermelo published a proof of the Well-Ordering Theorem, in which he explicitly referred to a set-theoretic principle that came to be known as the Axiom of Choice Zermelo [1904]. In the first years after its publication, this proof received a lot of critique, a significant part of which questioned the general validity of the Axiom of Choice (see Moore [1982]). In the long run, however, the proof got accepted, as the Axiom of Choice got accepted as a valid part of the de-facto standard foundational theory for mathematics, *Zermelo-Fraenkel set theory with the Axiom of Choice* (*ZFC*).

The two critiques of Zermelo's Axiom of Choice that we consider in this paper are those of Peano [1906] and Lebesgue [see Hadamard et al., 1905]. Furthermore, we consider the counterarguments to these critiques put forward by Zermelo [1908a] and by Hadamard [see Hadamard et al., 1905]. Given that the model still leaves out many contributions to that debate and additionally simplifies some of the contributions that it does take into account, we consider it to only be a preliminary model that we plan to extend in the future. However, we hope that this more extensive model gives some insight into the strengths and drawbacks of the modeling capacities of ASPIC-END, as well as inspiration for further research into this direction.

In Section 1.9, we discuss how the model could be extended in order to provide a more complete picture of the debate and to link it to debates on related topics within the foundations of mathematics and logic.

1.1 Some general remarks of the model

We are modeling some sophisticated argumentation that often involves a lot of implicit reasoning steps that are not made explicit. We attempt our formalization of the arguments to be as faithful as possible to the original intention of the authors of the arguments in question, but we cannot avoid making choices about the implicit reasoning steps that could potentially be made differently.

Generally, the purely mathematical and purely logical demonstrations and reasoning is formalized using intuitively strict rules, while the philosophical and metamathematical argumentation and reasoning is formalized using defeasible rules. Most of the attacks between arguments attack defeasible arguments, i.e. philosophical or metamathematical arguments. But given that some of the mathematical and logical principles that were applied in the mathematical and logical reasoning that we model, e.g. the Axiom of Choice and the non-constructivist parts of classical logic, are attacked by some philosophical or metamathematical arguments, there are also some arguments using only intuitively strict rules that get attacked. Of course, by the design of ASPIC-END, all such attacks have to be undercuts.

All arguments have to start from some assumptions, which are not explicitly backed up by further arguments. Such an assumption is formalized in our model as a premise, i.e. as a rule with no antecedent. Depending on whether this premise is of a purely mathematico-logical nature or has philosophical/metamathematical aspects, it gets modeled either as an intuitively strict rule without antecedent, also called an *axiom*, or as a defeasible rule without antecedent, also called a *defeasible premise*.

Instead of presenting the language and the set of rules of our model at once, we introduce them step by step as we show how to formalize various arguments put forward during the debate. We explicitly mention all the rules needed in out model. The language of our model is a standard first-order language over a vocabulary of predicate symbols, function symbols and constants. This vocabulary is not explicitly listed, but is evident from the list of rules that we put forward and from the explanations we provide about the formalization. We have chosen the names of all predicate symbols, function symbols and constants in such a way that they resemble either the actual words used in the debate[1] or the words of some reformulation of the cited arguments that we use when explaining the debate and our formalization of it.

The name of a rule, i.e. the formula that expresses the acceptability of a rule and whose negation can be used to undercut an argument using the rule, is denoted with $\text{accept}(\rho)$, where ρ is a constant symbol that refers to the rule as a syntactic object. For some rules we explicitly specify the constant ρ that refers

[1] All the arguments from the debate cited in this paper were originally presented in languages other than English. We generally work with their translations to English provided by Moore [1982] and Kennedy [1973]. In very few cases we have made minor modifications to the translation that Moore made of German texts in order to make the translation more faithful to the German original. By "actual words used in the debate" we here mean the English translation of these words.

to it by writing (ρ) (for some constant symbol ρ) in front of the rule. However, for other rules we do not specify such a constant symbol to refer to it, as it is mostly not needed.

The inference rules of intuitionistic logic (which are also included in classical logic) are not called into question by any mathematician involved in the debate that we model, so these rules never get undercut. Here are the schemes of intuitively strict rules[2] that are required to model intuitionistic logic in ASPIC-END:

$$
\begin{aligned}
(\neg\text{-Elim})\ & \varphi, \neg\varphi \rightsquigarrow \bot; \\
(MP)\ & \varphi, (\varphi \supset \psi) \rightsquigarrow \psi; \\
(\wedge\text{-Intro})\ & \varphi, \psi \rightsquigarrow (\varphi \wedge \psi); \\
(\wedge\text{-Elim}_{\mathrm{L}})\ & (\varphi \wedge \psi) \rightsquigarrow \varphi; \\
(\wedge\text{-Elim}_{\mathrm{R}})\ & (\varphi \wedge \psi) \rightsquigarrow \psi; \\
(\vee\text{-Intro}_{\mathrm{L}})\ & \varphi \rightsquigarrow (\varphi \vee \psi); \\
(\vee\text{-Intro}_{\mathrm{R}})\ & \psi \rightsquigarrow (\varphi \vee \psi); \\
(=\text{-Intro})\ & \rightsquigarrow t = t; \\
(=\text{-Elim})\ & \varphi, t_1 = t_2 \rightsquigarrow \varphi[t_1/t_2] \\
(\forall\text{-Elim})\ & \forall x\ \varphi \rightsquigarrow \varphi[t/x]; \\
(\exists\text{-Intro})\ & \varphi[t/x] \rightsquigarrow \exists x\ \varphi; \\
(\exists\text{-Elim})\ & \exists x\ \varphi, \forall x\ (\varphi \supset \psi) \rightsquigarrow \psi \text{ for any } \psi \text{ that does not} \\
& \qquad\qquad\qquad\qquad\qquad \text{contain } x \text{ as a free variable}
\end{aligned}
$$

For the final rule scheme in this list, i.e. ($\exists$-Elim), we only include in our model those instances for which ψ does not contain x as a free variable.

In order to simplify the exposition of our model, we sometimes omit implicit reasoning steps that involve only these inference rules of intuitionistic logic. We use the notation $(A_1, \ldots, A_n \vdash \psi)$ for an argument that uses multiple of these rules to get from the conclusions of arguments A_1, ..., A_n to the conclusion ψ. Since these rules can never be undercut, this omission does not lead to any attacks on such arguments being overlooked.

Furthermore, we will in one place refer to the rule scheme of double negation elimination, which, when added to the above rule schemes, gives a formalization of classical first-order logic in ASPIC-END:

$$(\neg\neg\text{-Elim}_{\varphi})\ \neg\neg\varphi \rightsquigarrow \varphi$$

1.2 Zermelo's explicitation of the Axiom of Choice

Since 1871, various mathematicians had produced proofs which relied on making infinitely many arbitrary choices, i.e. relied on what came to be known as the Axiom of Choice (see Moore [1982], Chapter 1). However, before 1904, these

[2]In this techincal report, we use the word *rule* in the way in which it is usually used in the structured argumentation literature. There is one important difference between this usage of *rule* and the way the word is usually used in the logical literature outside of structured argumentation theory: A *rule*, as the word is used in structured argumentation theory, is what would normally be called an instance of a rule. For this reason, it makes sense to speak of a *rule scheme*, which is what would normally be just called a rule.

mathematicians were not aware of the fact that these proofs require a novel mathematical principle (*ibid.*). The first mathematician who explicitly talked about the problem of making infinitely many arbitrary choices was Giuseppe Peano Peano [1890] (page 280), but Peano talked of it as something that cannot be done, i.e. he rejected the kind of inferences that the Axiom of Choice allows. This particular detail of his work did not influence any mathematicians other than some of his Italian colleagues (see Moore [1982], page 76). The first time that a mathematician made explicit reference to the problem of making infinitely arbitrary choices while considering this a valid form of inference was the paper of Zermelo that presented a proof of the Well-Ordering Theorem Zermelo [1904].

Zermelo's formulation of the Axiom of Choice was the following: "even for an infinite totality of [non-empty] sets there always exist mappings by which each set corresponds to one of its elements"

We formalize this in ASPIC-END as follows:

$$(AC) \rightsquigarrow \forall M\ (\forall m \in M\ \text{non-empty\_set}(m)) \supset \exists f\ (\text{domain}(f) = M \wedge \forall m \in M\ f(m) \in m)$$

Here is the argument that Zermelo gives for accepting the Axiom of Choice:

> "this logical principle cannot be reduced to a still simpler one, but is used everywhere in mathematical deduction without hesitation. So for example the general validity of the theorem that the number of subsets into which a set is partitioned is less than or equal to the number of its elements, cannot be demonstrated otherwise than by assigning to each subset one of its elements." [Zermelo, 1904, p. 516]

Zermelo does not explicitly explain what he means by reducing a principle to a simpler principle, so we just take this to be a primitive notion of his meta-mathematical reasoning, rendered in our model by the unary predicate simple. We assume that there is a defeasible premise that asserts that (AC) is simple in this sense. This defeasible premise is justified by the fact that Zermelo presumably put some considerable mathematical effort into attempting to reduce (AC) to a simple principle before claiming in print that this cannot be done.[3]

$$\Rightarrow \text{simple}(AC)$$

We formalize Zermelo's claim that his principle "is used everywhere in mathematical deduction without hesitation" as a conjunction of two claims, one asserting that (AC) is widely used in mathematical practice (formalized using the unary predicate widely_used), and one asserting that no mathematician has called this usage to doubt (formalized using the unary predicate calls_to_doubt and the function symbol usage). The first claim is backed up by the example

[3] The proof of Cohen [1963] that if ZF is consistent, then it does not prove the Axiom of Choice, can be viewed as a confirmation of Zermelo's claim that (AC) cannot be reduced to a simpler principle. However, to get to this conclusion, one would still require some judgment about the simplicity of (AC) in comparison to some other principles that have turned out to imply it, and unlike Cohen's proof, these judgments are not of a purely mathematico-logical nature, and should thus be formalized using defeasible rather than intuitively strict rules. So while progress has been made since Zermelo's claim in 1904 that would allow us to put forward stronger and more elaborate arguments in favor of this claim, these arguments would still be of a defeasible nature, just like Zermelo's original claim.

that he produces in the second sentence of the quotation (see the formalization below), while the second claim is made without argument, and therefore gets formalized as a defeasible premise:

$$(\rho_2^{Z^{04}}) \Rightarrow \neg\exists x \text{ calls\_to\_doubt}(x, \text{usage}(AC))$$

The theorem mentioned in the second sentence of the quotation is nowadays usually called the Partition Principle. The content of this theorem is not relevant for the argumentative force of the argument that Zermelo puts forward here. All that is relevant is that he puts forward an example of a theorem from the mathematical literature for which a proof has been published and accepted by the community, and that this proof makes makes implicit use of the principle AC that Zermelo is defending here. As the content of the theorem is not relevant, we simplify the exposition of the model by replacing it by a constant symbol PP ("Partition Principle"). Zermelo makes the claim that this theorem "cannot be demonstrated otherwise than by assigning to each subset one of its elements", which we formalize as a conjunction of two claims: that there is a proof that *demonstrates* PP, and that any proof that demonstrates PP *uses* (AC). We do not assign any specific formal meaning to the words *demonstrate* and *uses*, but consider them primitive concepts of Zermelo's metamathematical reasoning.[4] These two claims are not supported by a further argument, so they get modeled as defeasible premises:[5] [6]

$$\Rightarrow \exists p \text{ demonstrates}(p, PP);$$
$$\Rightarrow \forall p \; (\text{demonstrates}(p, PP) \supset \text{uses}(p, AC))$$

Zermelo uses this example of the Partition Principle to substantiate his claim that his principle AC is widely used in mathematical deduction. Of course, an argument that concludes that a principle is widely used based on evidence for one single usage of the principle is a comparatively weak argument. But it is still stronger than making the same claim based on no evidence at all, which is what would have been the case if Zermelo had not given this single example. In order to formally account for this inference from a single example of a usage of AC to the conclusion that it is widely used, we add the following scheme[7] of defeasible rules to our model:

$$\exists p, t \; (\text{demonstrates}(p, t) \wedge \text{uses}(p, \rho)) \Rightarrow \text{widely\_used}(\rho)$$

[4] Even though he did not explicitly use a word that translates to the English word *uses*, his usage of "otherwise than by" suggests that he had in mind some notion of a principle being involved in a demonstration, which we decide to render with the word *uses*.

[5] The defeasible premise that there is a proof that demonstrates PP could be replaced by an intuitively strict argument, if we chose to extend the model by the actual mathematical proof from the literature that Zermelo is referring to here. But this would require also extending the model with intuitively strict rules that formalize our semi-formal reasoning about syntactical entities like proofs and their conclusions.

[6] Interestingly, a certain reasonable formalization of the terms *demonstrates* and *uses* in terms of the (historically later) formal systems ZF and ZFC gives the second defeasible premise a reading which is still an open problem in set theory to this day: It is still unknown whether the full force of the Axiom of Choice is needed to prove the Partition Principle, or whether a weaker choice principle is sufficient (though it is known that the Partition Principle cannot be proved in ZF alone).

[7] It is a scheme, as ρ may be substituted by an arbitrary term of our language. The particular instance of the scheme that we will make use of is the one where ρ is AC.

It is clear that Zermelo puts forward his claims about the simplicity of, the wide use of and the lack of doubt about his principle AC in order to corroborate the acceptability of AC as a basic mathematical principle that does not require mathematical proof. In other words, he implicitly makes use of a metamathematical principle, according to which the simplicity of, wide use of and lack of doubt about a mathematical principle allow one to defeasibly infer the acceptability of said mathematical principle. This is formalized through the following scheme of defeasible rules:

$$\text{simple}(\rho), \text{widely\_used}(\rho), \neg\exists x\ \text{calls\_to\_doubt}(x, \text{usage}(\rho)) \Rightarrow \text{accept}(\rho)$$

Now that we have presented all the rules require to formalize this argument of Zermelo in favor of the acceptability of AC, we describe the arguments that Zermelo constructs from these rules:

$$\begin{aligned}
Z_1^{04} &= (\Rightarrow \text{simple}(AC)) \\
Z_2^{04} &= (\Rightarrow \neg\exists x\ \text{calls_to_doubt}(x, \text{usage}(AC))) \\
Z_3^{04} &= (\Rightarrow \exists p\ \text{demonstrates}(p, PP)) \\
Z_4^{04} &= (\Rightarrow \forall p\ (\text{demonstrates}(p, PP) \supset \text{uses}(p, AC))) \\
Z_5^{04} &= (Z_3^{04}, Z_4^{04} \vdash \exists p, t\ (\text{demonstrates}(p, t) \wedge \text{uses}(p, AC))) \\
Z_6^{04} &= (Z_5^{06} \Rightarrow \text{widely_used}(AC)) \\
Z_7^{04} &= (Z_1^{06}, Z_6^{06}, Z_2^{06} \Rightarrow \text{accept}(AC))
\end{aligned}$$

Note that the derivation indicated with the symbol $\vdash$ in argument Z_5^{04} includes an application of $\supset$-Introduction and an application of $\forall$-Introduction, namely in order to derive $\forall p\ (\text{demonstrates}(p, PP) \supset \exists p, t\ (\text{demonstrates}(p, t) \wedge \text{uses}(p, AC)))$, which is needed in order to make use of ($\exists$-Elim) in order to derive $\exists p, t\ (\text{demonstrates}(p, t) \wedge \text{uses}(p, AC))$ from $\exists p\ \text{demonstrates}(p, PP)$.

1.3 Peano's response to Zermelo's proof

In 1906, the Italian mathematician Giuseppe Peano published a note in the Rendiconti del Circolo matematico di Palermo in which he responded to Zermelo's proof [Peano, 1906] by criticizing his principle AC. First of all, he points out that he had previously considered and rejected this inference pattern:

> "This assumption, which occurs in several books, was already considered by me in the year 1890, in Math. Ann., 37, p. 210: 'one may not apply an infinite number of times an arbitrary law according to which to a class a is made to correspond an individual of that class ...' " [Peano, 1906, p. 208]

Note that this can be viewed as a counterargument against Zermelo's claim that the principle has been applied in mathematics "without hesitation", based on evidence that Peano himself has previously called this principle to doubt. We formalize this counterargument as follows:

$$(\rho_1^{P^{06}}) \Rightarrow \text{calls\_to\_doubt}(\text{Peano}, \text{usage}(AC))$$

Peano then explains how a single arbitrary choice from a non-empty class can be formalized in his *Formulario mathematico*, a semi-formal notational system for mathematical propositions and proofs that he had devised:

> "the form of argument 'if I arbitrarily choose an element x of class a, then proposition p (which does not contain x) follows' is reducible to the form
>
> $$\begin{array}{ll} \exists\, a & (1) \\ x \,\epsilon\, a. \supset .p & (2) \\ (1).(2). \supset .p & \end{array}$$
>
> 'If there exists an a, and if from $x \epsilon a$ follows proposition p, then proposition p may be affirmed.'
> This is the form of argument called 'elimination of x' in Formulario, V, p. 12, Prop. 3.1." [Peano, 1906, p. 208]

Note that –apart from irrelevant notational differences – his elimination of x is the same as our rule scheme ∃-Elim.

The point that Peano is making here is that for any informal argument that makes one arbitrary choice, there is a formalization of this argument in his Formulario system that makes use of elimination of x once. We formalize this by the following defeasible premise:[8]

$$\Rightarrow \forall a, b\ (\text{arb\_choices}(a, 1) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 1))$$

Here $\text{arb\_choices}(a, n)$ means that argument a is an informal argument that contains an inference step in which n arbitrary choices are made. $\text{uses}(b, \rho, n)$ means that b is a derivation that makes use of rule ρ n times.

Peano continues:

> "The assumption of two successive arbitrary elements has the form:
>
> $$\begin{array}{ll} \exists\, a & (1) \\ x \,\epsilon\, a. \supset .\exists\, b & (2) \\ x \,\epsilon\, a.y \,\epsilon\, b. \supset .p & (3) \\ (1).(2).(3). \supset .p\ \text{''} & \end{array}$$
>
> [Peano, 1906, p. 208]

In this case, four propositions are involved (three hypotheses and a conclusion). Note that in his semi-formal system the intermediate step resulting from just one application of elimination of x does not need to be written down, whereas in our fully formal system such an omission is not allowed, so that there are actually five propositions involved in two consecutive applications of ∃-Elim).

The point that Peano is making here is that for any informal argument that makes two arbitrary choices, there is a formalization of this argument in his

[8] Given that it is a statement about the connection between something informal and something formal, it is not purely mathematico-logical, but has a metamathematical character that justifies the choice of a defeasible premise instead of an (intuitively strict) axiom.

Formulario system that makes use of elimination of x twice. We formalize this by the following defeasible premise:

$$\Rightarrow \forall a, b\ (\text{arb\_choices}(a, 2) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 2))$$

At this point, Peano is ready to make his main argument against Zermelo's principle AC:

> "The assumption of two arbitrary elements x and y leads to an argument with three hypotheses (1), (2), (3), and a thesis (4). In general the assumption of n successive arbitrary elements leads to an argument which consists of $n+2$ propositions. Therefore we may not suppose $n = \infty$, that is, we cannot construct an argument with an infinite number of propositions." [Peano, 1906, p. 209]

To formalize this argument, we first need some rules that allow him to conclude the claims of the first two sentences of this citation from the above mentioned defeasible premises. Note that he makes a generalization about n arbitrary choices from evidence about what happens in the case of one or two arbitrary choice. This kind of generalization is of course not a mathematical demonstration, but a common form of argumentation in the informal exposition of mathematical ideas. So we formalize it as a defeasible rule:

$$\begin{aligned}
&\forall a, b\ (\text{arb_choices}(a, 1) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 1)),\\
&\quad \forall a, b\ (\text{arb_choices}(a, 2) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 2))\\
&\quad \Rightarrow \forall n, a, b\ (\text{arb_choices}(a, n) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, n));
\end{aligned}$$

The property of his system that n applications of elimination of x lead to an argument that involves $n+2$ propositions, on the other hand, can be considered a mathematical statement about his system, so we formalize it using an intuitively strict rule (which, given the lack of proof of this statement, is here just treated as an axiom, i.e. an intuitively strict rule without antecedent):

$$\rightsquigarrow \forall n\ (\text{uses}(a, x\text{-elim}, n) \supset \text{involves\_propositions}(a, n+2))$$

For concluding the claim in the final sentence, we need two axioms that formalize his mathematical assumptions that $\infty + 2 = \infty$ and that there is no Formulario argument that involves infinitely many propositions:

$$\begin{aligned}
&\rightsquigarrow \infty + 2 = \infty;\\
&\rightsquigarrow \neg\exists a\ (\text{Formulario}(a) \wedge \text{involves_propositions}(a, \infty))
\end{aligned}$$

These rules allow us to conclude that no informal argument that makes infinitely many choices can be formalized in the Formulario. Peano considers this an attack on the usability of Zermelo's principle AC. Here Peano is making the implicit assumption that an informal mathematical argument is acceptable if and only if it can be formalized in the Formulario. To get an attack on AC, we additionally need the premise that the acceptability of the Axiom of Choice implies the acceptability of some informal argument that makes infinitely many arbitrary choices:

$$(\rho_4^{P^{06}}) \Rightarrow \forall a\ (\text{accept}(a) \equiv \exists b(\text{formalizes}(b,a) \wedge \text{Formulario}(b)));$$
$$\rightsquigarrow (\text{accept}(AC) \supset \exists a\ (\text{accept}(a) \wedge \text{arb\_choices}(a,\infty)))$$

Now that we presented all the rules we require to model Peano's argument against Zermelo's principle AC, let us describe the arguments that Peano constructs from these rules:

$P_1^{06} = (\Rightarrow \text{calls\_to\_doubt}(\text{Peano}, \text{usage}(AC)))$
$P_2^{06} = (P_1^{06} \rightsquigarrow \exists x\ \text{calls\_to\_doubt}(x, \text{usage}(AC)))$
$P_3^{06} = (\Rightarrow \forall a, b\ (\text{arb\_choices}(a,1) \wedge \text{formalizes}(b,a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 1)))$
$P_4^{06} = (\Rightarrow \forall a, b\ (\text{arb\_choices}(a,2) \wedge \text{formalizes}(b,a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 2)))$
$P_5^{06} = (P_3^{06}, P_4^{06} \Rightarrow \forall n, a, b\ (\text{arb\_choices}(a,n) \wedge \text{formalizes}(b,a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, n)))$
$P_6^{06} = (\rightsquigarrow \forall n\ (\text{uses}(a, x\text{-elim}, n) \supset \text{involves\_propositions}(a, n+2)))$
$P_7^{06} = (\rightsquigarrow \infty + 2 = \infty)$
$P_8^{06} = (\rightsquigarrow \neg\exists a\ (\text{Formulario}(a) \wedge \text{involves\_propositions}(a,\infty)))$
$P_9^{06} = (\Rightarrow \forall a(\text{accept}(a) \equiv \exists b\ (\text{formalizes}(b,a) \wedge \text{Formulario}(b))))$
$P_{10}^{06} = (P_5^{06}, P_6^{06}, P_7^{06}, P_8^{06}, P_9^{06} \vdash \neg\exists a\ (\text{accept}(a) \wedge \text{arb\_choices}(a,\infty)))$
$P_{11}^{06} = (\rightsquigarrow (\text{accept}(AC) \supset \exists a\ (\text{accept}(a) \wedge \text{arb\_choices}(a,\infty))))$
$P_{12}^{06} = (\mathsf{Assume}_\neg(\text{accept}(AC)))$
$P_{13}^{06} = (P_{10}^{06}, P_{11}^{06}, P_{12}^{06} \vdash \bot)$
$P_{14}^{06} = (\mathsf{ProofbyContrad}(P_{13}^{06}, \neg\text{accept}(AC)))$

1.4 Zermelo's response to Peano

In 1908, Zermelo wrote an article responding to multiple critiques of his proof of the Well-Ordering Theorem. The article contains the following response to Peano's arguments:

> "First of all, how does Peano arrive at his own fundamental principles and how does he justify admitting them into the *Formulaire*, since he cannot prove them either? Obviously, through analyzing the rules of inference that have historically been recognized as valid and by referring both to the intuitive evidence for the rules and to their necessity for science – considerations which may be argued just as well for the disputed Principles. This Axiom, without being formulated in a scholastic manner, has been applied successfully, and very frequently, in the most diverse mathematical fields, particularly set theory, by R. Dedekind, G. Cantor, F. Bernstein, A. Schoenflies, and J. König among others. Such extensive usage of a principle can only be explained through its self-evidence, which, naturally, must not be confused with its provability. While this self-evidence may be subjective to a certain degree, it is in any case an essential source of

> mathematical principles, though not a basis for mathematical proofs. Thus Peano's statement, that self-evidence has nothing to do with mathematics, does not do justice to obvious facts. However, what can be objectively decided, the question of necessity for science, I would like now to submit to judgment by presenting a series of elementary and fundamental theorems and problems, which, in my opinion, could not be settled without the Axiom of Choice."

Zermelo continues by listing seven theorems of set theory, which he believed not to be provable without the Axiom of Choice. Some of these theorems were already widely considered as proven among set theorists of his time, e.g. the theorem that a countable union of countable sets is contable. Others had been implicitly assumed by many set theorists without explicit proof, e.g. that every Dedekind finite set is finite. He also repeated in the list the Partition Principle mentioned in his 1904 article.

In the second sentence of this quotation, Zermelo mentions three criteria for admitting fundamental principles: being historically recognized as valid, being intuitively evident, and being necessary for science. In this passage he seems interested in providing strong evidence for admitting a fundamental principle by satisfying all three of these criteria. We formalize this by the following defeasible rule scheme:

$$\text{hist\_rec\_as\_valid}(\rho) \wedge \text{int\_evident}(\rho) \wedge \text{nec\_for\_science}(\rho) \Rightarrow \text{accept}(\rho)$$

Zermelo cites the frequent usage of the Axiom of Choice as evidence both for its being historically recognized as valid, and for it intuitive evidence. This is formlized by the following rules:

$$\begin{aligned}
&\Rightarrow \text{used}(\text{Dedekind}, AC);\\
&\Rightarrow \text{used}(\text{Cantor}, AC);\\
&\Rightarrow \text{used}(\text{Bernstein}, AC);\\
&\Rightarrow \text{used}(\text{Schoenflies}, AC);\\
&\Rightarrow \text{used}(\text{König}, AC);\\
&\text{used}(\text{Dedekind}, \rho) \wedge \text{used}(\text{Cantor}, \rho) \wedge \text{used}(\text{Bernstein}, \rho) \wedge\\
&\quad \text{used}(\text{Schoenflies}, \rho) \wedge \text{used}(\text{König}, \rho) \Rightarrow \text{widely_used}(\rho);\\
&\text{widely_used}(\rho) \Rightarrow \text{hist_rec_as_valid}(\rho);\\
&\text{widely_used}(\rho) \Rightarrow \text{int_evident}(\rho);
\end{aligned}$$

Finally, the seven theorems that Zermelo puts forward as examples for where the Axiom of Choice is needed serve as evidence for the Axiom of Choice being necessary for science. As in the case of the Particion Principle that Zermelo already mentioned in his 1904 article, the precise content of these theorems is not of great importance for the argumentative power of his argument. So we will replace the theorems other than PP, which we have already given a name in Section 1.2, by the placeholder names $Th2$, ..., $Th7$ corresponding to the numbering used by Zermelo in his paper:

$$\begin{aligned}
&\Rightarrow \exists p \ \text{demonstrates}(p, Th2);\\
&\Rightarrow \forall p \ (\text{demonstrates}(p, Th2) \supset \text{uses}(p, AC));
\end{aligned}$$

$\Rightarrow \exists p$ demonstrates$(p, Th3)$;

$\Rightarrow \forall p$ (demonstrates$(p, Th3) \supset$ uses(p, AC));

$\Rightarrow \exists p$ demonstrates$(p, Th4)$;

$\Rightarrow \forall p$ (demonstrates$(p, Th4) \supset$ uses(p, AC));

$\Rightarrow \exists p$ demonstrates$(p, Th5)$;

$\Rightarrow \forall p$ (demonstrates$(p, Th5) \supset$ uses(p, AC));

$\Rightarrow \exists p$ demonstrates$(p, Th6)$;

$\Rightarrow \forall p$ (demonstrates$(p, Th6) \supset$ uses(p, AC));

$\Rightarrow \exists p$ demonstrates$(p, Th7)$;

$\Rightarrow \forall p$ (demonstrates$(p, Th7) \supset$ uses(p, AC));

$\exists p$ (demonstrates$(p, PP) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th2) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th3) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th4) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th5) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th6) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th7) \wedge$ uses(p, ρ)) $\Rightarrow$ nec_for_science(ρ)

Now the rules provided in this subsection can be combined into a new argument in favour of the acceptability of the Axiom of Choice:

$$Z_1^{08} = (\Rightarrow \text{used}(\text{Dedekind}, AC))$$
$$Z_2^{08} = (\Rightarrow \text{used}(\text{Cantor}, AC))$$
$$Z_3^{08} = (\Rightarrow \text{used}(\text{Bernstein}, AC))$$
$$Z_4^{08} = (\Rightarrow \text{used}(\text{Schoenflies}, AC))$$
$$Z_5^{08} = (\Rightarrow \text{used}(\text{König}, AC))$$
$$Z_6^{08} = (Z_1^{08}, Z_2^{08}, Z_3^{08}, Z_4^{08}, Z_5^{08} \Rightarrow \text{widely\_used}(AC))$$
$$Z_7^{08} = (Z_6^{08} \Rightarrow \text{hist\_rec\_as\_valid}(AC))$$
$$Z_8^{08} = (Z_6^{08} \Rightarrow \text{int\_evident}(AC))$$
$$Z_9^{08} = (Z_3^{04}, Z_4^{04} \vdash \exists p\ (\text{demonstrates}(p, PP) \wedge \text{uses}(p, AC)))$$
$$Z_{10}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th2))$$
$$Z_{11}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th2) \supset \text{uses}(p, AC)))$$
$$Z_{12}^{08} = (Z_{10}^{08}, Z_{11}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th2) \wedge \text{uses}(p, AC)))$$
$$Z_{13}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th3))$$
$$Z_{14}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th3) \supset \text{uses}(p, AC)))$$
$$Z_{15}^{08} = (Z_{13}^{08}, Z_{14}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th3) \wedge \text{uses}(p, AC)))$$
$$Z_{16}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th4))$$
$$Z_{17}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th4) \supset \text{uses}(p, AC)))$$
$$Z_{18}^{08} = (Z_{16}^{08}, Z_{17}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th4) \wedge \text{uses}(p, AC)))$$
$$Z_{19}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th5))$$
$$Z_{20}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th5) \supset \text{uses}(p, AC)))$$
$$Z_{21}^{08} = (Z_{19}^{08}, Z_{20}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th5) \wedge \text{uses}(p, AC)))$$
$$Z_{22}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th6))$$

$$Z^{08}_{23} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th6) \supset \text{uses}(p, AC)))$$
$$Z^{08}_{24} = (Z^{08}_{22}, Z^{08}_{23} \vdash \exists p\ (\text{demonstrates}(p, Th6) \wedge \text{uses}(p, AC)))$$
$$Z^{08}_{25} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th7))$$
$$Z^{08}_{26} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th7) \supset \text{uses}(p, AC)))$$
$$Z^{08}_{27} = (Z^{08}_{25}, Z^{08}_{26} \vdash \exists p\ (\text{demonstrates}(p, Th7) \wedge \text{uses}(p, AC)))$$
$$Z^{08}_{28} = (Z^{08}_{9}, Z^{08}_{12}, Z^{08}_{15}, Z^{08}_{18}, Z^{08}_{21}, Z^{08}_{24}, Z^{08}_{27} \Rightarrow \text{nec\_for\_science}(AC))$$
$$Z^{08}_{29} = (Z^{08}_{7}, Z^{08}_{8}, Z^{08}_{28} \Rightarrow \text{accept}(AC))$$

Even though the conclusion of Peano's argument P^{06}_{14} is the negation of the conclusion of Zermelo's new argument Z^{08}_{29}, this does not constitute a direct attack from Z^{08}_{29} to P^{06}_{14}. The reason for this is that the conclusion of P^{06}_{14} is attained by a proof by contradiction, and such a proof cannot be rebutted on the top level. However, it is possible to construct from Z^{08}_{29} an argument attack on P^{06}_{14} by making use of some of Peano's subarguments that Zermelo does not intend to attack:

$$Z^{08}_{30} = (\mathsf{Assume}_{\neg}(\forall a\ (\text{accept}(a) \equiv \exists b(\text{formalizes}(b, a) \wedge \text{Formulario}(b)))));$$
$$Z^{08}_{31} = (Z^{08}_{30}, P^{06}_{5}, P^{06}_{6}, P^{06}_{7}, P^{06}_{8}, P^{06}_{11} \vdash \neg\text{accept}(AC));$$
$$Z^{08}_{32} = (Z^{08}_{29}, Z^{08}_{31} \rightsquigarrow \bot);$$
$$Z^{08}_{33} = (\mathsf{ProofbyContrad}(Z^{08}_{32}, \neg\forall a\ (\text{accept}(a) \equiv \exists b(\text{formalizes}(b, a) \wedge \text{Formulario}(b)))));$$

Now Z^{08}_{33} directly rebuts P^{06}_{9}, which is a subargument of P^{06}_{14}, so Z^{08}_{33} indirectly attacks P^{06}_{14} as well. This is how our model formalizes Zermelo's attack on Peano's argument. Note that without taking into account preferences, there would also be an attack back from Peano's argument P^{06}_{14} onto Zermelo's argument Z^{08}_{29}. We explain in Section 1.6 how this is avoided through the use of preferences.

1.5 Lebesgue's and Hadamard's letters

In this subsection, we extend the model with a somewhat simplified formalization of arguments from two more mathematicians – Henri Lebesgue and Jacques Hadamard – who had participated in this debate before Peano's response. The purpose of this addition to the model is mainly to illustrate the possibility of attacks on classical inference rules in the context of such foundational debates. In order to not complicate the exposition of the model much more, we simplify the formalization of these arguments a bit, while acknowledging that this simplification makes the formalization less faithful to the wording used by Lebesgue and Hadamard than it could be.

In 1905, French mathematician Émile Borel, who himself had critiqued Zermelo's proof of the Well-Ordering Theorem, asked his colleague Henri Lebesgue to comment on the proof. Lebesgue responded in a letter to Borel that shortly afterwards got published together with four other letters on the topic in the Bulletin de la Société mathématique de France [Hadamard et al., 1905]. Lebesgue rejected Zermelo's statement that he had proved the Well-Ordering Theorem, and a central statement in his justification for this rejection is the following:

"I believe that we can only build solidly by granting that it is impossible to demonstrate the existence of an object without defining it."

Lebesgue attributes this principle to the German mathematician Leopold Kronecker, who is now often considered a forerunner of later constructivist and intuitionistic approaches to mathematics. Lebesgue does not make any precise statement about which forms of inference involving existential statements are acceptable and which ones are not. However, it is clear that he intends this to be an attack on the existence claim that the Axiom of Choice makes, namely that there exists a certain choice function. At the same time, it is fair to assume that this statement puts him at odds with any non-constructive proof of an existential statement. So in order to capture the argumentative force of this claim, we assume that it implies (through two intuitively strict rules) both a rejection of the Axiom of Choice (AC) and a rejection of double negation elimination applied to an existential statement ($\neg\neg$-Elim$_{\exists x\ \psi}$). But in order to keep the formalization simple, we do not formalize the internal structure of this claim, but instead formalize it as a propositional variable.

Lebesgue does not put forward any argument to support this belief other than attributing the idea to Kronecker. For this reason, we have decided to model it as a defeasible premise:

$$(\rho_1^{L^{05}}) \Rightarrow \text{existence\_proof\_requires\_definition};$$
$$\text{existence\_proof\_requires\_definition} \rightsquigarrow \neg\text{accept}(AC);$$
$$\text{existence\_proof\_requires\_definition} \rightsquigarrow \neg\text{accept}(\neg\neg\text{-Elim}_{\exists x\ \psi})$$

Borel sent a copy of Lebesgue's response to Jacques Hadamard, who reacted to it in another letter, which was published together with Lebesgue's letter in the Bulletin de la Société mathématique de France. In this letter, Hadamard defends Zermelo's proof against Lebesgue's critique. In this letter, he calls the following argument the "essence of the debate":

"From the invention of the infinitesimal calculus to the present, it seems to me, the essential progress in mathematics has resulted from successively annexing notions which, for the Greeks or the Renaissance geometers or the predecessors of Riemann, were "outside mathematics" because it was impossible to describe them."

In order to keep the exposition of the model simple – just as for Lebesgue's argument – we will not analyse the internal structure of this claim, but just formalize it as a propositional variable and defeasible premise, to which we assign the argumentative force that it was intended to have by including rule $\rho_2^{H^{05}}$ that allows it to be used to attack Lebesgue's argument:

$$\Rightarrow \text{progress\_by\_accepting\_existence\_of\_undescribables};$$
$$(\rho_2^{H^{05}})\ \text{progress\_by\_accepting\_existence\_of\_undescribables} \Rightarrow \neg\text{existence\_proof\_requires\_definition}$$

We can describe the arguments that Lebesgue and Hadamard construct:

$$L_1^{05} = (\Rightarrow \text{existence\_proof\_requires\_definition})$$

$L_2^{05} = (L_1^{05} \rightsquigarrow \neg\text{accept}(AC))$
$L_3^{05} = (L_1^{05} \rightsquigarrow \neg\text{accept}(\neg\neg\text{-Elim}_{\exists x\ \psi}))$
$H_1^{05} = (\Rightarrow \text{progress\_by\_accepting\_existence\_of\_undescribables})$
$H_2^{05} = (H_1^{05} \Rightarrow \neg\text{existence\_proof\_requires\_definition})$

1.6 Preferences in our model

Without imposing preferences on the set of rules, all attacks in our model other than the undercuts on the applications of the Axiom of Choice and double-negation elimination would become *practically* bidirectional. By this we mean that even though there can be a unidirectional attack from some argument A to some argument B, in such a case there will always be an attack back onto A from some argument B' that is closely related to B and accepted in the same circumstances as B.

In order to make the model more interesting and more realistic, it is therefore a good idea to include in it some preference order on the rules, which gives rise to a preference order on the arguments. One drawback of our methodology is that it gives no methodological guidance on how to select a preference order on the rules. So for now, we just have to follow our common sense of the relative strength of different rules and different argument. We will just specify some instances of rules being preferred to other rules, leaving most pairs of rules incomparable on the preference order, as comparison is only needed for some pairs of rules.

The defeasible rule ($\rho_2^{Z^{04}}$) of Zermelo's 1904 argument, which claims that no one has called the Axiom of Choice into doubt, is clearly weaker than Peano's defeasible rule ($\rho_1^{P^{06}}$) that claims that Peano has called the Axiom of Choice into doubt, as Peano can know better than Zermelo what he has called into doubt, and can even provide a reference to a publication, where he has called this inference pattern into doubt in print. So we assume $\rho_2^{Z^{04}} < \rho_1^{P^{06}}$.

Furthermore, the rules that Zermelo requires for his 1908 argument are comparatively strong: For example, he makes claims about certain people having used the Axiom of Choice implicitly, which can be verified by reading the proofs produced by the mathematicians in question. The rule that allows him to conclude frequent usage of the Axiom of Choice from five cited instances of such usage is clearly stronger than the similar rule from 1904, by which he made this conclusion based on one instance of such usage. Also the central premise used to conclude the acceptability of the Axiom of Choice based on three criteria seems to be a philosophically strong point of his argument. In contrast, Peano's rule $\rho_4^{P^{06}}$, which claims that all acceptable informal mathematical arguments can be formalized in the Formulario is weaker than those rules from Zermelo's 1908 argumentation. So we assume $\rho_4^{P^{06}}$ to be weaker than all the rules introduced in Section 1.4. In Lebegue's text analyzed in this report, there is no explicit philosophical support for Lebesgue's rule $\rho_1^{L^{05}}$, which claims that proving the existence of an object requires defining it, which could be considered to be a reason for preferring Zermelo's rules over Lebesgue's rule $\rho_1^{L^{05}}$. On the other hand, with the benefit of hindsight, we know that there exist some compelling philosophical arguments in favor of constructivist approaches to mathematics and thus supportive of Lebesgue's rule $\rho_1^{L^{05}}$. For this reason, we consider $\rho_1^{L^{05}}$ and the rules introduced in Section 1.4 to be incomparable according to the preference relation between rules.

1.7 A relevant implicit argument

While the preference relation between rules explained in the previous section does not lead to Lebesgue's argument L_2^{05} against accepting the Axiom of Choice to be preferred over Zermelo's 1908 argument Z_{29}^{08} in favor of the Axiom of Choice, Zermelo's argument by itself does not work as a counterargument against Lebesgue's argument. The reason for this is a technical detail of ASPIC-END that it has inherited from ASPIC+ and that helps to ensure the rationality postulate of *closure under accepted intuitively strict rules* that is explained and proved as Theorem 2 in Section 2 below, namely the fact that arguments ending in an intuitively strict rule cannot be rebutted (i.e. ASPIC+ has *restricted rebuttal* rather than *unrestricted rebuttal*; see Caminada et al. [2014] for an explanation of the distinction). However, by making use of the proof by contradiction construct, we can actually construct the following attack on Lebesgue's argument based on argument Z_{29}^{08} for the Axiom of Choice:

$$I_1 = (\mathsf{Assume}_{\neg}(\text{existence\_proof\_requires\_definition}))$$
$$I_2 = (I_1 \rightsquigarrow \neg\text{accept}(AC))$$
$$I_3 = (Z_{29}^{08}, I_2 \rightsquigarrow \bot)$$
$$I_4 = (\mathsf{ProofbyContrad}(I_3, \neg\text{existence\_proof\_requires\_definition}))$$

Now I_4 directly rebuts L_1^{05} and thus indirectly rebuts L_2^{05}.

This is an example of an implicit argument that is not explicitly stated in the debate that we model, but that can be derived from other rules in the model. One of the strengths of our methodological approach is precisely that it allows to identify such implicit arguments that no one has put forward, but that could be put forward and that could have a relevance influence on the outcome of the debate.

Note that a similar implicit argument can be built based on Z_7^{04} (Zermelo's first argument for the Axiom of Choice), but given that we will in Section 1.6 introduce a preference relation that gives preference to the rules needed for constructing L_3^{05} over the rules needed for constructing Z_7^{04}, this alternative implicit argument is not going to successfully rebut L_1^{05}, so we can safely ignore it.

While the model described here has not led to philosophically relevant implicit arguments, we believe that the methodology we are proposing has the potential to bring to light such arguments once more sophisticated formal models of debates in the formal sciences are constructed. We expect the use of automated theorem provers to be helpful in order to discover philosophically relevant implicit arguments in more sophisticated models, just like they already have been used by Benzmüller and Woltzenlogel Paleo [2016] and Benzmüller et al. [2017] to discover philosophically relevant mistakes and insights in axiomatic theories of metaphysics.

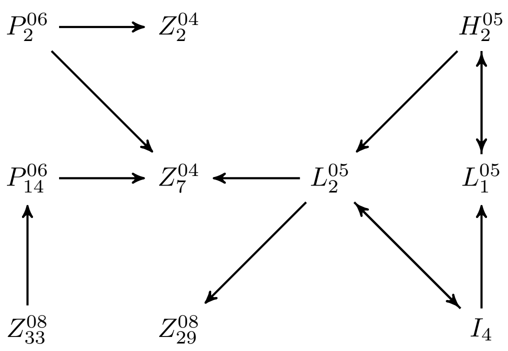

Figure 1: The relevant arguments and attacks from the example

1.8 Depicting the relevant arguments

The specified set of rules of our model allow for infinitely many arguments to be constructed, so that the EAF corresponding to the model will also be infinite. However, only a small finite subset of this infinite EAF contains attacks that are relevant for the overall status of the acceptability of the Axiom of Choice, which was the focus of attention of the debate that we have formally modeled. For example, an implicit argument similar to I_4 could be built based on Z_7^{04} instead of on Z_{29}^{08}, but as Z_7^{04} will not be accepted in any argumentative core extension of the overall EAF, this implicit argument will also never be accepted. In Figure 1, we depict the small subset of relevant arguments and the defeats between them.

Restricted to this set of relevant arguments, there are two argumentative core (AC) extensions: $S_1 = \{P_2^{06}, Z_{33}^{08}, Z_{29}^{08}, H_2^{05}, I_4\}$, and $S_2 = \{P_2^{06}, Z_{33}^{08}, L_1^{05}, L_2^{05}\}$. This means that arguments P_2^{06} and Z_{33}^{08} are accepted in every AC-extension of our model, while P_{14}^{06}, Z_2^{04} and Z_7^{04} are rejected in every AC-extension, and the status of the arguments Z_{29}^{08}, I_4, L_1^{05} and L_2^{05} depends on the choice of AC-extension. Note that this set of relevant arguments contains two arguments with conclusion accept(AC), namely Z_7^{04} and Z_{29}^{08}. While the first one gets rejected in both extensions, the second one gets accepted in one and rejected in the other extension, so that overall, the status of the claim accept(AC) depends on the choice of the AC-extension.

These properties of our formal model intuitively correspond to the situation that on the one hand there are compelling arguments both in favor and against the Axiom of Choice, and purely formal methods will not decide which of the two stands is "correct" (if there even is a single "correct" answer here), while on the other hand certain arguments in favor or against the axiom of choice are so weak that they do not hold up against the scrutiny provided by certain counterarguments against them.

Of course, the fact that the status of the Axiom of Choice in our formal model of the debate is not determined but depends on the choice of the AC-extension is to a certain extent an artifact of the choice of arguments that we formalized and of the preference order that we imposed. We could have gotten a different result, for example if we had chosen to formalize only strong arguments in favor of the Axiom of Choice and weak arguments against it, or if we had just made significantly different judgments about the preference order on the rules involved in our model. So at the current level of development, such a model cannot be seriously defended as a method for deciding which side in a debate is right. What it can do, however, is to help us discover relevant implicit arguments like argument I_4 in our model (and hopefully with a more developed model also philosophically more relevant implicit arguments), to help us get a more precise understanding of what assumptions are made and what is at stake in a given debate, and to point towards weaknesses of the current methodology of structured argumentation theory, like the lack of a methodological guidance for choosing a preference order on the rules.

1.9 Conclusion and proposed extensions to the model

The parts of the debate presented and formalized in this subsection were, of course, only a small part of the debate that mathematicians had about the Axiom of Choice in the early 20th century, and additionally some of the considered arguments have been formalized in a simplified way. So it is obvious that the model could be expanded to a more extensive formal model of that debate. One obvious extension that has already been alluded to above is to include the proof of the Well-Ordering Theorem, so that attacks on the Axiom of Choice would also be attacks on the Well-Ordering Theorem, as actually intended by the mathematicians involved in the debate.

Some of the points that were raised during the debate touch on other issues from the foundations of mathematics that were discussed at the time. For example, the German mathematician Felix Bernstein criticized Zermelo's proof of the Well-Ordering Theorem not for the usage of the Axiom of Choice, but for its similarity to Burali-Forti's Paradox [see Moore, 1982, p. 110]. Bernstein had somewhat peculiar ideas about how Burali-Forti's Paradox should be resolved, ideas which later turned out not to be tenable, but which at the time led him to think that the resolution of Burali-Forti's Paradox also blocks the possibility of a construction that Zermelo used in his proof of the Well-Ordering Theorem. The face that this idea of Bernstein, unlike rejection of the Axiom of Choice, turned out to not be a viable position, should be explainable by a formal model that incorporates his argumentation.

Zermelo wrote his 1908 response to his critiques [Zermelo, 1908a] in conjunction with another paper [Zermelo, 1908b], in which he proposed an axiomatization of set theory including his new Axiom of Choice as well as other set-theoretic principles. This axiomatization, which after later modifications by Fraenkel gave rise to ZFC, also had to avoid the two set-theoretic paradoxes that were hotly discussed at the time, namely Russel's Paradox and Burali-Forti's Paradox. An interesting extension of the model from this paper would be one that covers these as well as other competing resolutions to these paradoxes. This will also bring into the picture the notion of explanation of a paradox defined in this paper.

A model of the debate about these paradoxes could also be naturally combined with a model of the debate about semantic paradoxes like the Liar Paradox, which we have already looked at superficially in the model in Section 4 of Dauphin and Cramer [2017]. Semantic paradoxes are a topic that many philosophical logicians continue to work on and that has given rise to a number of relatively novel non-classical logics like paraconsistent logic [see Priest, 2006a], paracomplete logic [see Field, 2008] and substructural logics [see Beall and Murzi, 2013]. This area of research is characterized by a combination of formal rigor, philosophical depth and debate about the acceptability of various logical principles, which is likely to make it a fruitful field for testing the applicability of structured argumentation theory to debates in the formal sciences. As works like that of Field [2006] show, the topic of semantic paradoxes is also connected to the philosophical interpretation of Gödel's Second Incompleteness Theorem, which has also been studied intensively within the philosophy of mathematics.

An overarching formal model of these foundational debates across multiple formal sciences is certainly still a distant goal. But given the potential insights that it could provide into foundational research in the long run, this distant goal could become a driving force for research on structured argumentation models of debates in the formal sciences.

2 Closure and rationality postulates

In this section, we present four rationality postulates that ASPIC-END satisfies and that are analogous to the four postulates that Modgil and Prakken [2013] have established for ASPIC+, as well as two new postulates motivated by the application of structured argumentation to debates in the formal sciences.

The first postulate concerns the closure of the extensions under the sub-argument relation. The idea is that one cannot accept an argument while rejecting part of it.

Theorem 1. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ and S be an AC-extension of Δ. Then, for all $A \in S$, $\mathsf{Sub}(A) \subseteq S$.

The proof of Theorem 1 rests on the following lemma, which can be proven in a straightforward way as in the case of ASPIC+ (see Lemma 35 of Modgil and Prakken [2013]):

Lemma 1. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ, $S \subseteq \mathcal{A}$ and $A, B \in \mathcal{A}$. We have that:

1. If S defends A and $S \subseteq S'$, then S' defends A.
2. If A defeats B' and $B' \in \mathsf{Sub}(B)$, then A defeats B.
3. If S defends A and $A' \in \mathsf{Sub}(A)$, then S defends A'.

We now show another intuitive result which will be needed in the proof of the postulates. This result is that given a satisfactory set of arguments, including additional arguments which do not interfere with the admissibility of the set, does not prevent the set from being satisfactory.

Lemma 2. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ and $S, S' \subseteq \mathcal{A}$ with S satisfactory. If S' is admissible and $S \subseteq S'$, then S' is also satisfactory.

Proof:

Assume $S' \supseteq S$ is admissible. Now, suppose for a contradiction that S' is not satisfactory. Then, since S' is admissible, there exists $S'' \supset S'$ such that $S'' >_p S'$ and S'' is admissible.

We will now show that $S'' >_p S$. Since $S'' \supset S'$ and $S' \supseteq S$, we have $S'' \supset S$. For each explanandum e for which S offers an explanation $X[e]$, $X[e] \in S''$, so S'' also offers an explanation for e. Hence, S'' offers an explanation for at least as many explananda as S. However, since $S'' >_p S'$, there exists an explanandum e' for which S'' offers an explanation but for which S' does not offer an explanation. $S \subseteq S'$, hence S does not offer an explanation for e' either. Therefore, S'' offers an explanation for strictly more explananda as S and thus $S'' >_p S$.

So we have $S'' \supset S$, $S'' >_p S$ and S'' is admissible. However, since S is satisfactory, this is a contradiction. Hence, S' is satisfactory. □

Proof of Theorem 1:

Let $A \in S$ and $A' \in \mathcal{A}$. Assume $A' \in \mathsf{Sub}(A)$. Suppose for a contradiction that $S \cup \{A'\}$ is not conflict-free. Since S is an AC-extension of Δ, S is conflict-free. Hence, either A' defeats some argument $B \in S$, or some argument $B \in S$ defeats A'.

- Suppose first that A' defeats some argument $B \in S$. Then, since S is an AC-extension of Δ, there exists some argument $B' \in S$ which defeats A'. Thus, by Lemma 1.2, B' also defeats A. But S is conflict-free. We have a contradiction.

- Suppose now that some argument $B \in S$ defeats A'. Then, by Lemma 1.2, B also defeats A. But S is conflict-free. We have a contradiction.

Since both cases lead to a contradiction, we can conclude that $S \cup \{A'\}$ is conflict-free.

Now, S defends A and so, by Lemma 1.3, S defends A'. Since S is an AC-extension of Δ, S also defends S. Thus, S defends $S \cup \{A'\}$. Hence, by Lemma 1.1, $S \cup \{A'\}$ defends $S \cup \{A'\}$. Since $S \cup \{A'\}$ is also conflict-free, $S \cup \{A'\}$ is admissible.

Also, by Lemma 2, since S is satisfactory and $S \cup \{A'\}$ is admissible, $S \cup \{A'\}$ is also satisfactory.

Now suppose for a contradiction that $A' \notin S$. Then, $S \cup \{A'\}$ is a proper superset of S which is also satisfactory. Hence, S is not an AC-extension of Δ. So we have a contradiction, and thus $A' \in S$. □

Notice that this postulate does not hold for EC-extensions, as they are by definition minimal in their inclusion of arguments, and thus will often leave out low-level sub-arguments.

The second postulate concerns the closure of the conclusions under intuitively strict rules. In the case of ASPIC+, the corresponding postulate concerned the closure of the conclusions under all strict rules (see Theorem 13 in Modgil and Prakken [2013]). But since ASPIC-END allows for the rejection of intuitively strict rules, it is undesirable to consider the closure under all of them. Instead, we consider the closure under a set of intuitively strict rules which are deemed acceptable. The following two definitions define the set of *accepted* intuitively strict rules and the *closure* under a given set of intuitively strict rules:

Definition 1. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ and S be an extension of Δ. The *set of intuitively strict rules accepted by* S is $\mathcal{R}_{isa}(S) = \{r \in \mathcal{R}_{is} | \forall A \in \mathcal{A}$ s.t. $\mathsf{As}(A) = \emptyset$ and $\mathsf{Conc}(A) = \neg n(r)$ or $\neg\mathsf{Conc}(A) = n(r), \exists B \in S$ s.t. B defeats $A\}$.

Definition 2. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $P \subseteq \mathcal{L}$ and $R' \subseteq \mathcal{R}_{is}$. We define the *closure of* P *under the set of rules* R', denoted $Cl_{R'}(P)$, as the smallest set such that $P \subseteq Cl_{R'}(P)$, and when $(\varphi_1, ..., \varphi_n \rightsquigarrow \psi) \in R'$ and $\varphi_1, ..., \varphi_n \in Cl_{R'}(P)$, then $\psi \in Cl_{R'}(P)$.

Now the postulate on the closure under accepted intuitively strict rules can be formulated as follows:

Theorem 2. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ and S be an AC-extension of Δ. Then, $\mathsf{Conc}(S) = Cl_{\mathcal{R}_{isa}(S)}(\mathsf{Concs}(S))$.

Proof:
Let S be an AC-extension of Δ. We want to show that $\mathsf{Concs}(S) = Cl_{\mathcal{R}_{isa}(S)}(\mathsf{Concs}(S))$. First, notice that $\mathsf{Concs}(S) \subseteq Cl_{\mathcal{R}_{isa}(S)}(\mathsf{Concs}(S))$. Hence, we only need to show that if $(\varphi_1, ..., \varphi_n \rightsquigarrow \psi) \in \mathcal{R}_{isa}(S)$ and $\varphi_1, ..., \varphi_n \in \mathsf{Concs}(S)$, then $\psi \in \mathsf{Concs}(S)$.

Suppose that $(\varphi_1, ..., \varphi_n \rightsquigarrow \psi) \in \mathcal{R}_{isa}(S)$ and $\varphi_1, ..., \varphi_n \in \mathsf{Concs}(S)$. Then, by the definition of Concs, there exists $A_1, ..., A_n \in S$ such that $\mathsf{Conc}(A_i) = \varphi_i$ and $\mathsf{As}(A_i) = \emptyset$ for $1 \leq i \leq n$. Hence, we can construct the argument $A = A_1, ..., A_n \rightsquigarrow \psi$, and thus $A \in \mathcal{A}$ with $\mathsf{As}(A) = \emptyset$.

Assume $B \in \mathcal{A}$ defeats A. Then, B either undercuts, assumption-attacks or successfully rebuts A. Let us first consider the case of undercut. Then, $\mathsf{As}(B) = \emptyset$ and either $\mathsf{Conc}(B) = \neg n(\varphi_1, ..., \varphi_n \rightsquigarrow \psi)$ or $\neg\mathsf{Conc}(B) = n(\varphi_1, ..., \varphi_n \rightsquigarrow \psi)$. However, since $(\varphi_1, ..., \varphi_n \rightsquigarrow \psi) \in \mathcal{R}_{isa}(S)$, there exists $C \in S$ such that C defeats B. Also, since A cannot be rebutted nor assumption-attacked, S defends A.

Now suppose there is an argument $D \in S$ such that D defeats A. Then, since S defends A, there is an argument $C \in S$ which defeats D. However, S is conflict-free, so we have a contradiction. Hence, there is no argument in S which defeats A.

Let us now assume A defeats some argument $D \in S$. Since S is admissible, there is an argument in S which defeats A. However, we have just concluded that no such argument exists, hence we have a contradiction. Therefore, A does not defeat any of the arguments in S.

Thus, $S \cup \{A\}$ is conflict-free. Also, since S defends A, $S \cup \{A\}$ is admissible. By Lemma 2 and since S is satisfactory, $S \cup \{A\}$ is also satisfactory.

Assume $A \notin S$. Then, since $(S \cup \{A\}) \supset S$ is satisfactory, S is not an AC-extension of Δ. This is a contradiction of one of our initial assumptions. Hence, $A \in S$. Therefore, $\psi \in \mathsf{Concs}(S)$ and thus $\mathsf{Concs}(S) = Cl_{\mathcal{R}_{isa}(S)}(\mathsf{Concs}(S))$. $\square$

The last two postulates presented by Modgil and Prakken [2013] are direct and indirect consistency, which state that when the set of strict rules is consistent, the set of conclusions and the closure of this set under strict rules are consistent.

While consistency postulates are not relevant for the application of ASPIC-END to argumentation about paradoxes, we also want ASPIC-END to be applicable to more standard domains in which the consistency postulates are relevant. For this reason, we also establish consistency postulates for ASPIC-END.

In order to show the consistency of the conclusions, we will have to show that no two arguments with contradictory conclusions may co-exist in the same extension. While these two arguments may have intuitively strict TopRules, and thus not attack each other, we will show that one of their sub-arguments is attacked and undefended. For the purpose of gradual inspection of the sub-arguments, we first define direct sub-arguments.

Definition 3. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ the EAF defined by Σ and $A, A' \in \mathcal{A}$. We say that A' is a *direct sub-argument* of A iff $A' \in \mathsf{Sub}(A)$ and there is no $A'' \in \mathsf{Sub}(A)$ s.t. $\mathsf{Sub}(A') \subset \mathsf{Sub}(A'')$.

Then, in order to identify those potential points of attack, we define maximal fallible sub-arguments, which represent the top-most sub-arguments with defeasible top rules.

Definition 4. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ the EAF defined by Σ and $A \in \mathcal{A}$. We define the multiset $M(A)$ of the *maximal fallible sub-arguments* of A as:

$$M(A) := \begin{cases} \{A\} & \text{if } \mathsf{TopRule}(A) \in \mathcal{R}_d \\ \emptyset & \text{if } \mathsf{DefRules}(A) = \emptyset \\ \biguplus_{i=1}^{k} M(A_i) & \text{Otherwise, where } \{A_1, ..., A_k\} \text{ is} \\ & \text{the set of direct sub-arguments of A.} \end{cases}$$

For a set of arguments S, we write $\mathsf{Subs}(S)$ as a shorthand for $\bigcup_{A \in S} \mathsf{Sub}(A)$.

Definition 5. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ the EAF defined by Σ and $S \subseteq \mathcal{A}$. We say that $A \in \mathcal{A}$ is an *intuitively strict continuation* of S iff:

- $\mathsf{Subs}(S) \subseteq \mathsf{Sub}(A)$;
- $\{r |$ for some $X \in \mathsf{Sub}(A) \setminus \mathsf{Subs}(S), r = \mathsf{TopRule}(X)\} \subseteq R_{is}$

We then show some intuitive results from our preference lifting. These results are closely related to the properties of a reasonable argument ordering as defined in Modgil and Prakken [2013].

Lemma 3. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory and $\prec$ the preference relation over $\mathcal{A}$ lifted from $<$. We have that:

1. for all $A, B \in \mathcal{A}$, if $\mathsf{DefRules}(A) = \emptyset$, then $A \not\prec B$;
2. for all $A, B \in \mathcal{A}$, if $\mathsf{DefRules}(A) = \emptyset$ and $\mathsf{DefRules}(B) \neq \emptyset$, then $B \prec A$;
3. for any finite multiset $\{C_1, ..., C_n\}$ of arguments, it is not the case that for all $i \in \{1, ..., n\}$, $C^{+\setminus i} \prec C_i$ (where $C^{+\setminus i}$ is an intuitively strict continuation of $\{C_1, ..., C_n\} \setminus \{C_i\}$).

Proof:

1. Suppose for a contradiction that $A \prec B$. Then, by definition, there exists $r_a \in \mathsf{DefRules}(A)$ such that for all $r_b \in \mathsf{DefRules}(B)$, $r_a < r_b$. However, since $\mathsf{DefRules}(A) = \emptyset$, no such r_a exists. Hence, $A \not\prec B$. □

2. Take any $r_b \in \mathsf{DefRules}(B)$. Since $\mathsf{DefRules}(A) = \emptyset$, it holds that for all $r_a \in \mathsf{DefRules}(A)$, $r_b < r_a$. Hence, $B \prec A$. □

3. Suppose for a contradiction that for all $i \in \{1, ..., n\}$, there exists an intuitively strict continuation $C^{+\backslash i}$ such that $C^{+\backslash i} \prec C_i$. Take an arbitrary C_j with $1 \leq j \leq n$. We have that there exists $C^{+\backslash j}$ such that $C^{+\backslash j} \prec C_j$. Hence, there exists $r \in \mathsf{DefRules}(C^{+\backslash j})$ such that for all $r_j \in C_j$, $r < r_j$. Select a least preferred such r (for all $r_l \in \mathsf{DefRules}(C^{+\backslash j})$, $r_l \not< r$). Take any argument $C_k \in \{C_1, ..., C_n\}$ such that $r \in \mathsf{DefRules}(C_k)$. Since $r < r_j$ for all $r_j \in C_j$ and $r_l \not< r$ for all $r_l \in \mathsf{DefRules}(C^{+\backslash j}) = \bigcup_{i=1, i \neq j}^{i=n} \mathsf{DefRules}(C_i)$, we have that $r_m \not< r$ for all $r_m \in \bigcup_{i=1}^{i=n} \mathsf{DefRules}(C_i)$, and hence $r_m \not< r$ for all $r_m \in \bigcup_{i=1, i \neq k}^{i=n} \mathsf{DefRules}(C_i)$. For all intuitively strict continuations $C^{+\backslash k}$ of $\{C_1, ..., C_{k-1}, C_{k+1}, ..., C_n\}$, we have $\mathsf{DefRules}(C^{+\backslash k}) = \bigcup_{i=1, i \neq k}^{i=n} \mathsf{DefRules}(C_i)$. Hence, we have $C^{+\backslash k} \not\prec C_k$. This is a contradiction, and hence it is not the case that for all $i \in \{1, ..., n\}$, $C^{+\backslash i} \prec C_i$. □

We have three requirements for applying the consistency postulates. The first is that there cannot be non-defeasible arguments which contradict each other. The second requirement ensures that a formula and its negation are considered as contradictory and the third guarantees that no assumptions for proof by contradiction are prevented. The last two requirements are motivated by the consideration that in the applications of ASPIC-END not related to paradoxes, one would likely accept classical or intuitionistic logic, for both of which these requirements hold.

Definition 6. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory. We say that Σ is *consistency-inducing* iff:

1. there are no $A, B \in \mathcal{A}$ such that $\mathsf{DefRules}(A) = \mathsf{DefRules}(B) = \emptyset = \mathsf{As}(A) = \mathsf{As}(B)$ and $\mathsf{Conc}(A) = \neg\mathsf{Conc}(B)$,
2. for each $\varphi \in \mathcal{L}$ there is a rule r_φ of the form $\varphi, \neg\varphi \rightsquigarrow \bot \in \mathcal{R}_{is}$ such that $n(r_\varphi)$ is undefined,
3. there is no rule $r \in \mathcal{R}$ such that $\mathit{Assumable}_\neg(\varphi)$ appears in r.

The following theorem establishes direct consistency for ASPIC-END:

Theorem 3. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be a consistency-inducing argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ and S be an AC or EC-extension of Δ. Then, there does not exist $\varphi \in \mathsf{Concs}(S)$ such that $\neg\varphi \in \mathsf{Concs}(S)$.

Proof:

Suppose for a contradiction that there exists $\varphi \in \mathsf{Conc}(S)$ such that $\neg\varphi \in \mathsf{Conc}(S)$. Then, there exist two arguments $A, B \in S$ such that $\mathsf{Conc}(A) = \varphi$, $\mathsf{Conc}(B) = \neg\varphi$ and $\mathsf{As}(A) = \emptyset = \mathsf{As}(B)$.

Consider the multiset $S = M(A) \uplus M(B)$. By Lemma 3.3, there exists $C \in S$ such that for all intuitively strict continuations S' of $S \setminus \{C\}$, we have $S' \not< C$. Without loss of generality, suppose $C \in M(A)$. Let $C' = Assume_{\neg}(\mathsf{Conc}(C))$ and construct A' from A by replacing C with C'. We now have that $\mathsf{Conc}(A') = \varphi$ and $\mathsf{As}(A') = \{\mathsf{Conc}(C)\}$. Since Σ is consistency-inducing, $\varphi, \neg\varphi \rightsquigarrow \bot \in \mathcal{R}_{is}$. Thus, we can construct $A'' = A', B \rightsquigarrow \bot$ with $\mathsf{As}(A'') = \{\mathsf{Conc}(C)\}$. Hence, we can also construct $D = \mathsf{ProofByContrad}(\neg\mathsf{Conc}(C), A'')$. Since $\mathsf{Conc}(D) = \neg\mathsf{Conc}(C)$, D attacks C. Also, D is an intuitively strict continuation of $S \setminus C$, thus we have $D \not< C$ and therefore D defeats C.

By Theorem 1, since $C \in \mathsf{Sub}(A)$, $C \in S$.

Similarly, for all $A_i \in M(A)$, $A_i \in S$ by Theorem 1. A' is an intuitively strict continuation of $M(A) \setminus C$ which uses the same rules as A. Hence, S defends A', and thus $A' \in S$.

Suppose an argument F defeats D. Then, F cannot defeat D on A'' by rebut since $\mathsf{TopRule}(A'') \in \mathcal{R}_{is}$. Also, F cannot defeat D on A'' by undercutting, since Σ is consistency-inducing and thus $n(\mathsf{TopRule}(A''))$ is undefined. F cannot defeat D nor A'' on C' by $\neg$-assumption-attack, again because Σ is consistency-inducing. Since $D = \mathsf{ProofByContrad}(\neg\mathsf{Conc}(C), A'')$, F cannot defeat D on D either.

So F defeats D on D', where $D' \neq A''$, $D' \neq D$ and $D \neq C'$. Hence, $D' \in \mathsf{Sub}(A')$ or $D' \in \mathsf{Sub}(B)$. By Theorem 1 and since $A', B \in S$, we have $D' \in S$. Hence, S defends D from F and so $D \in S$.

But D defeats C, so S is not conflict-free, which is a contradiction. Therefore, no such $\varphi \in \mathsf{Concs}(S)$ exists and thus $\mathsf{Concs}(S)$ is consistent. □

Indirect consistency of AC-extensions follows from closure under accepted intuitively strict rules together with direct consistency:

Theorem 4. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be a consistency-inducing argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ and S be an AC-extension of Δ. Then, there does not exist $\varphi \in Cl_{\mathcal{R}_{isa}(S)}(\mathsf{Concs}(S))$ such that $\neg\varphi \in Cl_{\mathcal{R}_{isa}(S)}(\mathsf{Concs}(S))$.

We want ASPIC-END to be applicable to debates in the formal sciences, in which the correctness of logical rules can be up for debate. For example, among the proposals made by philosophers of how to handle the semantic paradoxes, there is paraconsistent dialetheism [Priest, 2006b], which accepts some inconsistencies as true and uses a paraconsistent logic to avoid that everything can be derived. And in order to be able to show the internal structure of the paradox, we need to have an inconsistency arise from intuitively strict rules under no assumptions. For these reasons, the consistency postulates do not make sense for this kind of application of ASPIC-END.

However, there is a property similar to consistency that should still hold even when the intuitively strict rules lead to paradoxes and when the output extensions contain one that accepts paraconsistent dialetheism, namely that an extension should never be trivial, i.e. conclude everything.

For the non-triviality of the extensions, we require every intuitively strict rule, except for the ones of conjunction elimination from $\bot$, to have a name so that it can be attacked. We say that the argumentation theory is well-defined if it satisfies this requirement, and assume well-definedness in the non-triviality postulate stated in Theorem 5.

Definition 7. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be an argumentation theory. We say that Σ is *well-defined* if and only if for each rule $r' \in \mathcal{R}_{is} \setminus \mathcal{R}_{ce}$, $n(r') \in \mathcal{L}$.

Theorem 5. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be a well-defined argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ, and S be an AC or EC-extension of Δ. Then, $\bot \notin \mathsf{Concs}(S)$.

Proof: Suppose for a contradiction that $\bot \in \mathsf{Concs}(S)$. Then there exists a minimal (under sub-argument relation) argument $A \in S$ such that $\mathsf{Conc}(A) = \bot$ and $\mathsf{As}(A) = \emptyset$.

We have two cases:

1. $\mathsf{TopRule}(A)$ is undefined. Then, A must be of the form $\mathsf{ReasonByCases}(\bot, A_1, A_2, A_3)$. We have two sub-cases:
   (a) $\mathsf{DefRules}(A) \neq \emptyset$. Then, there must be a minimal (w.r.t. $\prec$) argument A' such that $\mathsf{TopRule}(A') \in \mathcal{R}_d$. Let $B = A \rightsquigarrow \neg\mathsf{Conc}(A')$. Then, since A' is minimal w.r.t. $\prec$, $B \not\prec A'$ and so B successfully rebuts A', so B defeats A.
   (b) $\mathsf{DefRules}(A) = \emptyset$. If A_3 is of the form $\mathsf{ReasonByCases}(\bot, A'_1, A'_2, A'_3)$, set $A_3 := A'_3$ and repeat this process until you obtain an argument A_3 which is not a reasoning by cases. Now A_3 is such that $\mathsf{Conc}(A_3) = \phi \vee \neg\phi$ for some $\phi \in \mathcal{L}$ and $\mathsf{DefRules}(A_3) = \emptyset$ since $\mathsf{DefRules}(A) = \emptyset$, so A_3 must be of the form $P_1, P_2, ..., P_n \rightsquigarrow \phi \vee \phi'$. Since Σ is well-defined, $\mathsf{TopRule}(A_3)$ is defined, and so let $B = A \rightsquigarrow \neg n(\mathsf{TopRule}(A_3))$. So B undercuts A_3 and thus defeats A.
2. $\mathsf{TopRule}(A)$ is defined. Let $r = \mathsf{TopRule}(A)$. If $r \in \mathcal{R}_{is}$, then $n(r) \in \mathcal{L}$ and so let $B = A \rightsquigarrow \neg n(r)$. Otherwise, let $B = A \rightsquigarrow \neg\bot$. By the definition of $\prec$ and the construction of B, $B \not\prec A$. Then B undercuts or successfully rebuts A on A, so B defeats A.

Since S is an AC- or EC-extension of Δ, it defends itself, so there exists $C \in S$ such that C defeats B. Suppose for a contradiction that C defeats B on $B' \neq B$. Since $\mathsf{Sub}(B) = \mathsf{Sub}(A) \cup \{B\}$, $B' \in \mathsf{Sub}(A)$. Then, by Lemma 1.2, C defeats A on B'. But S is conflict-free, so we have a contradiction. Hence, C defeats B on B. Since $B = A \rightsquigarrow \neg n(r)$, B cannot be rebutted nor assumption-attacked. Hence, C undercuts B on B. But since $\mathsf{TopRule}(B) \in \mathcal{R}_{ce}$, $n(\mathsf{TopRule}(B))$ is undefined, i.e. no argument undercuts B on B, a contradiction.

Hence, $\bot \notin \mathsf{Concs}(S)$. □

Indirect non-triviality of AC-extensions then follows from closure under accepted intuitively strict rules and direct non-triviality:

Theorem 6. Let $\Sigma = (\mathcal{L}, \mathcal{R}, n, <)$ be a well-defined argumentation theory, $\Delta = \langle \mathcal{A}, \mathcal{X}, \rightarrow, \dashrightarrow \rangle$ be the EAF defined by Σ and S be an AC-extension of Δ. Then, $\bot \notin Cl_{\mathcal{R}_{isa}(S)}(\mathsf{Concs}(S))$.

References

Jc Beall and Julien Murzi. Two Flavors of Curry's Paradox. *Journal of Philosophy*, 110(3):143–165, 2013.

Christoph Benzmüller and Bruno Woltzenlogel Paleo. The Inconsistency in Gödel's Ontological Argument: A Success Story for AI in Metaphysics. In Subbarao Kambhampati, editor, *IJCAI 2016*, volume 1-3, pages 936–942. AAAI Press, 2016. ISBN 978-1-57735-770-4. URL `http://www.ijcai.org/Proceedings/16/Papers/137.pdf`. (Acceptance rate $\leq 25\%$).

Christoph Benzmüller, Leon Weber, and Bruno Woltzenlogel Paleo. Computer-Assisted Analysis of the Anderson-Hájek Controversy. *Logica Universalis*, 11(1):139–151, 2017. doi: 10.1007/s11787-017-0160-9. URL `http://christoph-benzmueller.de/papers/J32.pdf`.

Martin Caminada, Sanjay Modgil, and Nir Oren. Preferences and Unrestricted Rebut. In *Computational Models of Argument - Proceedings of COMMA 2014*, pages 209–220, 2014.

Paul J Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences*, 50(6):1143–1148, 1963.

Jérémie Dauphin and Marcos Cramer. ASPIC-END: Structured Argumentation with Explanations and Natural Deduction. In *Proceedings of the 2017 International Workshop on Theory and Applications of Formal Argument*, 2017.

Hartry Field. Truth and the Unprovability of Consistency. *Mind*, pages 567–605, 2006.

Hartry Field. *Saving Truth from Paradox*. Oxford University Press, 2008.

Jacques Hadamard, R Baire, H Lebesgue, and E Borel. Cinq lettres sur la théorie des ensembles. *Bulletin de la Société mathématique de France*, 33: 261–273, 1905.

Hubert C. Kennedy. *Selected works of Giuseppe Peano*. University of Toronto Press, 1973.

Sanjay Modgil and Henry Prakken. A general account of argumentation with preferences. *Artificial Intelligence*, 195:361–397, 2013.

G.H. Moore. *Zermelo's axiom of choice: its origins, development, and influence*. Studies in the history of mathematics and physical sciences. Springer-Verlag, 1982. ISBN 9780387906706.

Giuseppe Peano. Démonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*, 37(2):182–228, 1890.

Guiseppe Peano. Additione. *Revista de mathematica*, 8:143–157, 1906.

Graham Priest. *In Contradiction: A Study of the Transconsistent*. Oxford University Press, 2006a.

Graham Priest. *In Contradiction: A Study of the Transconsistent*. Oxford University Press, 2006b.

Ernst Zermelo. Beweis, daß jede Menge wohlgeordnet werden kann. *Mathematische Annalen*, 59(4):514–516, 1904.

Ernst Zermelo. Neuer Beweis für die Möglichkeit einer Wohlordnung. *Mathematische Annalen*, 65(1):107–128, 1908a.

Ernst Zermelo. Untersuchungen über die Grundlagen der Mengenlehre. I. *Mathematische Annalen*, 65(2):261–281, 1908b.