

# ON WEAKLY CONVEX STAR-SHAPED POLYHEDRA

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ABSTRACT. Weakly convex polyhedra which are star-shaped with respect to one of their vertices are infinitesimally rigid. This is a partial answer to the question whether every decomposable weakly convex polyhedron is infinitesimally rigid. The proof uses a recent result of Izestiev on the geometry of convex caps.

## 1. INTRODUCTION

**1.1. The rigidity of convex polyhedra.** The rigidity of Euclidean polyhedra has been of interest to geometers since Legendre [10] and Cauchy [4] proved that convex polyhedra are globally rigid. This result was an important source of inspiration in subsequent geometry, for instance for the theory of convex surfaces, and was a key tool in Alexandrov's theory of isometric embeddings of polyhedra [1, 12].

The notion of global rigidity leads directly to the related notion of infinitesimal rigidity; a polyhedron is *infinitesimally rigid* if any non-trivial first-order deformation induces a non-zero variation of the metric on one of its faces. Infinitesimal rigidity is important in applications since a structure which is rigid but not infinitesimally rigid is likely to be physically unreliable. Although Cauchy's argument can be used to prove that convex polyhedra are infinitesimally rigid, this result was proved much later by M. Dehn [8], by completely different methods.

**1.2. Non-convex polyhedra.** Cauchy's theorem left open the question of rigidity of non-convex polyhedra, until examples of flexible polyhedra were constructed by Connelly [5]. It would however be interesting to know a class of rigid polyhedra wider than the convex ones. We say that a polyhedron is *weakly convex* if its vertices are the vertices of a convex polyhedron, and that it is *decomposable* if it can be cut into convex polyhedra without adding any vertex.

*Question 1.1.* Let  $P$  be a weakly convex, decomposable polyhedron. Is  $P$  infinitesimally rigid ?

This question came up naturally in [16], where it was proved, using hyperbolic geometry tools, that the result is positive if the vertices of  $P$  are on an ellipsoid, or more generally if there exists an ellipsoid which contains no vertex of  $P$  but intersects all its edges. It was proved in [6] that the answer is also positive for two other classes of polyhedra: suspensions – which can be cut into simplices with only one interior edge – and polyhedra which have at most one non-convex edge, or two non-convex edges sharing a vertex.

**1.3. Main result.** Here we extend the result of [6] to a wider class of weakly convex decomposable polyhedra. From here on all the polyhedra we consider are triangulated; it is always possible to reduce to that situation by adding “flat” edges in non-triangular faces. For a polyhedron  $P$  which is star-shaped with respect to a vertex  $v_0$ , we do this subdivision by decomposing all non-triangular faces adjacent  $v_0$  by adding only diagonals containing  $v_0$ , so that this refinement of the triangulation of the boundary of  $P$  is compatible with a triangulation of the interior of  $P$  for which all simplices contain  $v_0$ .

**Theorem 1.2.** *Let  $P$  be a weakly convex polyhedron, which is star-shaped with respect to one of its vertices. Then  $P$  is infinitesimally rigid.*

Here by “star-shaped” with respect to  $v_0$  we mean that the interior of  $P$  has a decomposition as the union of finitely many non-degenerate simplices, all containing  $v_0$  as one of their vertices, of disjoint interior, and such that the intersection of each with  $P$  is a face of both.

**1.4. A refined statement.** There is a slightly refined version of Theorem 1.2, giving a better understanding on the reasons for which rigidity holds. Let  $P$  be star-shaped with respect to a vertex  $v_0$ , and let  $P = S_1 \cup \dots \cup S_N$  be a triangulation of  $P$  as a union of non-degenerate simplices all containing  $v_0$  (and having disjoint interior). Let  $e_1, \dots, e_m$  be the interior edges of this triangulation, i.e., the edges of the  $S_j$  which are not contained in faces of  $P$ . Let  $l_i$  be the length of  $e_i$ .

It is then possible to consider a wider class of (small) deformations of the metric on the interior of  $P$ : those for which the  $l_i$  vary, while the length of the edges of  $P$  remain constant. Such a variation of the  $l_i$  determines a unique deformation of the metric on the  $S_j$ , which can however still be glued isometrically along their common faces. Under such a variation, cone singularities might appear along the  $e_i$ : the angles around those edges might become different from  $2\pi$ . We call  $\theta_i$  the angle around  $e_i$ .

**Definition 1.3.** Let

$$\Lambda_P := \left( \frac{\partial \theta_i}{\partial l_j} \right)_{1 \leq i, j \leq m} .$$

Note that  $\Lambda_P$  *a priori* depends also on the decomposition  $P = S_1 \cup \dots \cup S_N$  (and of the labeling of the  $e_j$ ). It is well-known that  $\Lambda_P$  is symmetric (see e.g. [6]), this follows from the fact that  $\Lambda_P$  is minus the Hessian of the total scalar curvature of the metrics obtained by varying the  $l_i$ . The following statement is also well known.

*Remark 1.4.*  $P$  is infinitesimally rigid if and only if  $\Lambda_P$  is non-degenerate.

The proof is elementary: isometric first-order deformations of  $P$  correspond precisely to first-order variations of the  $l_i$  which do not change, at first order, the  $\theta_i$ . Although the proof of this point requires some care, we do not include one here and refer the reader to [9, 2] where a similar problem is treated in full details.

**Theorem 1.5.** *Under the hypothesis of Theorem 1.2,  $\Lambda_P$  is positive definite.*

**1.5. A word on the proof.** The proof is only indirectly related to the argument used in [16], and different from those used in [6]. It is based on a recent result of Izmistiev [9], who gives a new proof of Alexandrov’s theorem on the existence and uniqueness of a polyhedral convex cap with a given induced metric, based on the concavity of a geometric function. We slightly extend his argument, to encompass weakly convex “caps”, by proving that “removing” a simplex to a (weakly) convex cap actually makes this function “more” concave – a point which we found somewhat surprising. We then use a classical projective argument to obtain Theorem 1.2. Theorem 1.5 follows from the same arguments.

## 2. WEAKLY CONVEX HATS

We use a notion of “convex cap” which is a little different from the one used by Izmistiev [9]. We will use a different name to avoid ambiguities. In the whole paper we consider a distinguished oriented plane, which can for convenience be taken to be the horizontal plane  $\{z = 0\}$ . We call  $\mathbb{R}_+^3$  the half-space bounded by this plane  $\{z = 0\}$  on the the side of its oriented normal.

**Definition 2.1.** Let  $E \subset \mathbb{R}_+^3$ , its *shadow*  $Sh(E)$  is the set of points  $m \in \mathbb{R}_+^3$  for which there exists a point  $m' \in E$  such that  $m$  is contained in the segment joining  $m'$  to its orthogonal projection on the horizontal plane  $\{z = 0\}$ .

**Definition 2.2.** A **convex hat** is a polyhedral surface  $H$  in  $\mathbb{R}_+^3$  such that

- (1)  $H$  is homeomorphic to a disk and has finitely many vertices,
- (2) no point of  $H$  is in the shadow of another,
- (3)  $H$  is contained in the boundary of the convex hull of  $Sh(H)$ .

Note that we do *not* demand that  $Sh(H)$  is convex, and that the projection of  $H$  on  $\{z = 0\}$  is not necessarily convex.

**Definition 2.3.** A **weakly convex hat** is a polyhedral surface  $H \subset \mathbb{R}_+^3$ , satisfying conditions (1) and (2) of the previous definition, and such that every vertex of  $H$  is an extremal point of the convex hull of  $Sh(H)$ .

A convex hat can be obtained by the following procedure. Start from a convex polyhedron  $P$ , and apply a projective transformation sending one of the vertices,  $v$ , to infinity in the vertical direction (towards  $z \rightarrow -\infty$ ). Then remove all edges and faces adjacent to  $v$ . Any convex hat which has as its projection on  $\{z = 0\}$  a convex polygon can be obtained in this manner. In the same way, one can start from a weakly convex polyhedron which is star-shaped with respect to one of its vertices, say  $v_0$ , and apply a projective map sending  $v_0$  to infinity, to obtain a weakly convex hat. Any weakly convex hat can be obtained in this manner.

The proof of Theorem 1.2 will follow from the following lemma, using classical arguments relating projective transformations to infinitesimal rigidity.

**Lemma 2.4.** *Let  $H$  be a weakly convex hat. Any first-order isometric deformation of  $H$  which fixes the heights of the boundary vertices is trivial.*

The proof relies on an extension of the notion of (weakly) convex cap, to allow for cone singularities along vertical edges, a device which is common in the field known as ‘‘Regge calculus’’, or more specifically e.g. in [2], [9], or in the last part of [6]. This type of construction has been used successfully also in the contexts of hyperbolic polyhedra or circle patterns on surfaces, see e.g. [13, 14, 11, 3, 16, 17].

**Definition 2.5.** A *prism* is a non-degenerate convex polyhedron  $P$  in  $\mathbb{R}_+^3$  which is the shadow of a triangle in  $\mathbb{R}_+^3$ .

The faces which are neither the bottom or the upper face of  $P$  are its *vertical* faces. It is not difficult to check that a prism is uniquely determined, among prisms with the same induced metric on the upper face, by the heights of its vertical edges.

**Definition 2.6.** A *generalized hat* is a metric space obtained from a finite set of prisms  $P_1, \dots, P_N$  by isometrically identifying some of their vertical faces, so that

- each vertical face is glued to at most one other, so that singularities occur only at line segments corresponding to some vertical edges of the  $P_i$ ,
- the prisms containing a given vertical edge are pairwise glued along vertical faces in a cyclic way (with either all vertical faces containing the given vertical edges pairwise glued, for an interior edge, or with two faces not glued and corresponding to vertical boundary faces of the generalized hat, for a boundary edge),
- under the gluing of two vertical faces, the segments corresponding to the bottom (resp. upper) face of the  $P_i$  are identified.

Given a convex or weakly convex hat  $H$ , it can be used to construct a generalized hat  $G$  by gluing the shadows of the faces of  $H$ . Moreover it’s easy to characterize the generalized hats obtained in this manner. It is necessary that the angles around all interior ‘‘vertical’’ edges are equal to  $2\pi$ ; under this condition, generalized hats admit an isometric immersion into  $\mathbb{R}_+^3$ , with their bottom faces sent to  $\{z = 0\}$ , and a generalized hat  $G$  is obtained from a (weakly) convex hat  $H$  if and only if this image in  $\mathbb{R}^3$  is embedded and (weakly) convex. This simple construction allows us to consider convex or weakly convex hats as special cases of generalized hats.

We define a generalized hat to be *convex* if it is convex at each edge  $e$  which is shared by the upper faces of two of the prisms  $P_i$  and  $P_j$ , i.e., if the angles at  $e$  of  $P_i$  and  $P_j$  add up to at most  $\pi$ . It is *strictly convex* if those angles add up to strictly less than  $\pi$ .

Given a generalized hat  $G$ , one can consider the space  $\mathcal{M}_G$  of all generalized hats for which the upper boundary has the same combinatorics and the same induced metric. It is not difficult to check that  $\mathcal{M}_G$  is parametrized by the heights of the vertical edges  $h_1, \dots, h_n$ . We call  $\mathcal{M}_{G,0}$  the subspace of  $\mathcal{M}_G$  of generalized hat having the same boundary heights as  $G$ , so that  $\mathcal{M}_{G,0}$  is parametrized by the heights of the interior vertical edges,  $h_1, \dots, h_m$ .

## 3. THE RIGIDITY OF CONVEX HATS

We mainly recall in this section results of Izestiev [9], adapting the arguments to the proof of Lemma 2.4 for the special case of convex hats. In the next section it is shown how the argument can be extended to weakly convex hats.

The proof is based on a matrix very similar to the matrix  $\Lambda_P$  appearing in Definition 1.3. We consider a convex hat  $H$ , and the corresponding generalized hat  $G$ . Since a prism, with given induced metric on its upper face, is uniquely determined by the heights of its vertices, elements of  $\mathcal{M}_{G,0}$  are uniquely determined by the heights of the interior vertices. Conversely, each choice of those heights, close to the heights of the interior vertices in  $H$ , determines an element of  $\mathcal{M}_{G,0}$ . We call  $e_1, \dots, e_m$  the vertical edges ending at interior points of  $H$ ,  $(h_i)_{1 \leq i \leq m}$  their heights, and  $(\theta_i)_{1 \leq i \leq m}$  the angle around them. So the  $\theta_i$  are equal to  $2\pi$  for  $G$ , but not necessarily at other points of  $\mathcal{M}_{G,0}$ .

**Lemma 3.1** (Izestiev [9]). *Let*

$$\Lambda_G := \left( \frac{\partial \theta_i}{\partial h_j} \right)_{1 \leq i, j \leq m} .$$

*Then  $\Lambda_G$  is symmetric and positive definite.*

We only give a brief outline of the proof here. The symmetry of  $\Lambda_G$  follows from the fact that it is minus the Hessian of a natural ‘‘total scalar curvature’’ function appearing in this context, called  $S$  in [9]. The coefficients of  $\Lambda_G$  are computed explicitly in [9] (Proposition 4, note that the coefficients given there are minus the ones considered here), they are equal to:

- $a_{ij} = 0$  when  $i \neq j$  and  $e_i$  and  $e_j$  are not the endpoints of an interior edge of  $H$ .
- $a_{ij} = -(\cotan(\alpha_{ij}) + \cotan(\alpha_{ji}))/l_{ij} \sin^2(\rho_{ij})$  when  $i \neq j$  but  $e_i$  and  $e_j$  are the two endpoints of an interior edge of  $H$ . Here  $\alpha_{ij}$  and  $\alpha_{ji}$  are the angles between the shadow of the edge of  $G$  joining the endpoints of  $e_i$  and  $e_j$  with the two upper faces of  $G$  adjacent to that edge,  $l_{ij}$  is the length of that edge, and  $\rho_{ij}$  is its angle with the vertical.
- $a_{ii} = -\sum_{j \neq i} a_{ij}$ .

It follows from this explicit description that  $\Lambda_G$  has dominant diagonal, and therefore that it is positive definite.

Remark 1.4 still applies in this context, so that it follows from Lemma 3.1 that convex hats are infinitesimally rigid.

## 4. WEAKLY CONVEX HATS ARE RIGID

The proof of Lemma 2.4 follows from Lemma 3.1 by a simple argument, remarking that (1) it is possible to go from a weakly convex hat to a convex hat by adding a finite set of simplices (which are in a specific position with respect to the vertical direction) (2) when removing such a simplex, the matrix  $\Lambda$  defined above becomes ‘‘more’’ positive.

**Lemma 4.1.** *Let  $H$  be a weakly convex hat, and let  $H_c$  be the convex hat which is the union of the upper faces of the convex hull of  $Sh(H)$  which project orthogonally to  $\{z = 0\}$  as a polygon in the projection of  $H$ . There exists a finite sequence  $H_0, \dots, H_p$  of weakly convex hats in  $\mathbb{R}_+^3$  such that*

- $H_0 = H$  and  $H_p = H_c$ ,
- for all  $i \in \{1, \dots, p\}$ ,  $H_i$  has the same vertices as  $H_{i-1}$ , and  $Sh(H_i)$  is obtained from  $Sh(H_{i-1})$  by gluing a simplex  $S_i$ ,
- the projection of  $S_i$  on  $\{z = 0\}$  is a quadrilateral.

*Proof.* Set  $H_0 := H$ , and choose a concave edge  $e_0$  of  $H_0$  (which is thus not a boundary edge of  $H_0$ ) with vertices  $v_0$  and  $v_1$ . Let  $f$  and  $f'$  be the faces of  $H$  adjacent to  $e_0$ , and let  $v_3$  and  $v_4$  be the vertices of  $f$  and  $f'$  opposite to  $e_0$ . Let  $S_1$  be the simplex with vertices  $v_0, v_1, v_2, v_3$ , then  $S_1$  projects to  $\{z = 0\}$  as a quadrilateral.

We can add to  $Sh(H)$  the simplex  $S_1$ , this yields a polyhedron in  $\mathbb{R}_+^3$  which is the shadow of a weakly convex hat  $H_1$  (which by construction has the same vertices as  $H_0$ ).

If  $H_1$  is convex, the lemma is proved. Otherwise,  $H_1$  has at least one concave edge and one can choose one of those edges, say  $e_1$ , and repeat the construction, adding a simplex  $S_2$ .

After a finite number of steps the weakly convex hat  $H_p$  obtained in this way will be convex, because the number of simplices that can be added is bounded from above, for instance by the number of Euclidean simplices having as vertices some vertices of  $H$ .  $\square$

The next step is to describe in what manner the matrix  $\Lambda$  associated to a weakly convex hat changes when a simplex is removed. We consider a simplex  $S$  with vertices  $v_1, v_2, v_3, v_4$  which projects on the plane  $\{z = 0\}$  as a quadrilateral. Then the boundary of  $S$  is the union of two surfaces, each made by gluing two triangles, and each of which has injective projection on  $\{z = 0\}$ : the “lower” surface  $S_-$ , and the “upper” surface  $S_+$ , with  $S_- \subset Sh(S_+)$ . We suppose for instance that  $S_-$  is the union of the triangles  $(v_1, v_3, v_4)$  and  $(v_2, v_3, v_4)$ , while  $S_+$  is the union of  $(v_1, v_2, v_3)$  and  $(v_1, v_2, v_4)$ . Let  $h_i$  be the height of  $v_i$  over  $\{z = 0\}$ . Any first order variation of the  $h_i$ ,  $1 \leq i \leq 4$ , determines a first-order displacement of the  $v_i$  which preserves the lengths of the five segments in  $S_-$  (including the diagonal), which is unique up to horizontal translation and rotation with vertical axis. Similarly a first-order variation of the  $h_i$  determines a displacement of the  $v_i$  which preserves the lengths of the five segments in  $S_+$ .

**Definition 4.2.** Let

$$M_S = \left( \frac{\partial(\theta_i^- - \theta_i^+)}{\partial h_j} \right)_{1 \leq i, j \leq 4},$$

where, for heights of the  $v_i$  close to the  $h_i$ ,  $\theta_i^+$  is the angle of the projection of  $S_+$  on  $\{z = 0\}$  at the projection of  $v_i$ , and  $\theta_i^-$  is the angle of the projection of  $S_-$  at the projection of  $v_i$ .

**Lemma 4.3.** *Let  $H$  and  $H'$  be two weakly convex hats, with  $Sh(H')$  obtained by removing from  $Sh(H)$  a simplex  $S$ . Then  $\Lambda_{H'}$  is obtained by adding  $M_S$  to  $\Lambda_H$  (with the lines/columns of  $M_S$  added to the lines/columns of  $\Lambda_H$  corresponding to the same vertices).*

*Proof.* This follows from the definitions, since  $\Lambda_{H'}$  is equal to  $\Lambda_H$  except that the variation of the curvature at the vertical edges ending on the vertices of  $S$  are given by the lower surface  $S_-$  rather than by the upper surface  $S_+$ .  $\square$

The interesting point is that  $M_S$  is always positive semi-definite, so that adding it to a positive definite matrix yields another positive definite matrix.

**Lemma 4.4.** *For any simplex  $S$  projecting on  $\{z = 0\}$  as a quadrilateral,  $M_S$  is positive semi-definite of rank 1.*

*Proof.* The space of Killing fields in  $\mathbb{R}^3$  has dimension 6. It contains a 3-dimensional subspace fixing  $\{z = 0\}$ , and therefore acting on  $S$  without changing any of the heights. There remains a 3-dimensional vector space of Killing fields which do change the heights  $h_i$ . Each acts by deforming globally  $S$ , so that, at each vertex, the angles of the projection of the upper and the lower surface change in the same way, and therefore those first-order variations of the  $h_i$  are in the kernel of  $M_S$ . So the rank of  $M_S$  is at most 1.

The same argument can be used, conversely, to show that the rank of  $M_S$  can not be zero. Otherwise the kernel of  $M_S$  would have dimension 4, which would mean that there exists a non-trivial first-order deformation of  $S$  leaving invariant the lengths of all edges in both the upper and lower surfaces, and such that the angles of the projections of the upper and lower surface vary in the same way. One could then consider the first-order deformations of the upper and of the lower surface, and add a trivial deformation so that they match at all four vertices, because the first-order variations of both the heights of the vertices and the projections of the two surfaces on  $\{z = 0\}$  match. This would mean that there is a non-trivial isometric deformation of  $S$ , and this is well known to be impossible – all simplices are infinitesimally rigid.

So the signature of  $M_S$  is constant over the space of simplices which projects on  $\{z = 0\}$  as quadrilaterals. This means that  $M_S$  is either positive semi-definite or negative semi-definite for all such simplices. To decide which happens, it is sufficient to check for one simplex, for instance a maximally symmetric one. Consider the

first-order deformations pictured in Figure 1, with the heights of  $v_1$  and  $v_2$  raised and the heights of  $v_3$  and  $v_4$  lowered.

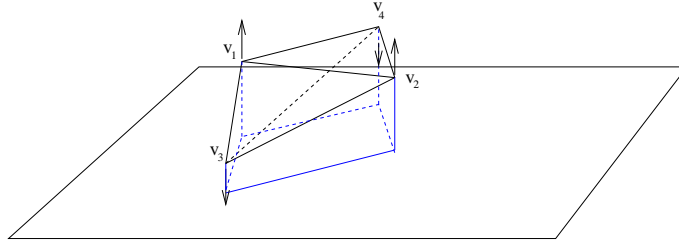


FIGURE 1. A positive deformation of a simplex

It is easy to check that in this case:

- $\theta_1^+$  and  $\theta_2^+$  decrease: the angles of the projection of the upper surface at the projections of  $v_1$  and  $v_2$  decrease,
- $\theta_3^+$  and  $\theta_4^+$  increase,
- $\theta_1^-$  and  $\theta_2^-$  increase,
- $\theta_3^-$  and  $\theta_4^-$  decrease,

It follows that the first-order variation of  $\theta_i^- - \theta_i^+$  is positive at  $v_1$  and  $v_2$  and negative at  $v_3$  and  $v_4$ , so that  $M_S$  has at least one positive eigenvalue. So  $M_S$  is positive semi-definite.  $\square$

**Lemma 4.5.** *Let  $H$  be any weakly convex hat, then  $\Lambda_H$  is positive definite.*

*Proof.* Let  $H_c$  be the convex hat obtained as the upper boundary of  $Sh(H)$ . Lemma 4.1 shows that  $Sh(H)$  is obtained from  $Sh(H_c)$  by removing a finite sequence of simplices. But  $\Lambda_{H_c}$  is positive definite by Lemma 3.1, and Lemma 4.3 shows that, each time a simplex is removed, the matrix  $\Lambda_H$  changes by the addition of a  $4 \times 4$  matrix, which is positive semi-definite by Lemma 4.4. It follows that  $\Lambda_H$  is also positive definite.  $\square$

*Proof of Lemma 2.4.* We have already seen that the fact that  $\Lambda_H$  is non-degenerate implies that  $H$  is infinitesimally rigid: any isometric first-order deformation of  $H$  which fixes the boundary heights is trivial.  $\square$

## 5. PROJECTIVE MAPS

The goal of this section is to prove Theorem 1.2, concerning polyhedra which are star-shaped with respect to one of their vertices, using Lemma 2.4, which deals with weakly convex hats. The basic idea here is old, going back at least to Darboux [7] and Sauer [15]: infinitesimal rigidity is a property which is invariant under projective maps. The particular case of this property which is used here can be stated more precisely as follows.

**Lemma 5.1.** *Let  $v_0 \in \mathbb{R}^3 \subset \mathbb{R}P^3$ , and let  $\phi : \mathbb{R}P^3 \rightarrow \mathbb{R}P^3$  be a projective transformation sending  $v_0$  to the point at infinity corresponding to the vertical direction in  $\mathbb{R}^3$ . There exists a map  $\Phi : T\mathbb{R}^3 \rightarrow T\mathbb{R}^3$  sending  $(x, v) \in T\mathbb{R}^3$  to  $(\phi(x), \psi_x(v)) \in T\mathbb{R}^3$  such that:*

- *the image by  $\Phi$  of any Killing vector field in  $\mathbb{R}^3$  is a Killing vector field,*
- *Killing fields which are infinitesimal rotations of axis containing  $v_0$  are sent to the translations along horizontal directions and the infinitesimal rotations of vertical axis.*

The proof of this Lemma is left to the reader, since it is quite classical. The map  $\psi_x$  can be explicitly described as follows: it sends vectors parallel to the direction of  $v_0$  to vertical vectors of the same norm, while acting on vectors orthogonal to the direction of  $v_0$  as the differential of the projective map  $\phi$ .

*Proof of Theorem 1.2.* Let  $P$  be a weakly convex polyhedron which is star-shaped with respect to a vertex  $v_0$ . Let  $U$  be an isometric first-order deformation of  $P$ , i.e., the restriction of  $V$  to each face of  $P$  is a Killing field. Adding a global Killing field if necessary, we can assume that the restriction of  $U$  to all faces of  $P$  containing  $v_0$  is a Killing field fixing  $v_0$ , i.e., an infinitesimal rotation with axis containing  $v_0$ .

Let  $Q = \phi(P)$ , then  $Q$  is an infinite polyhedron with infinite vertical faces corresponding to the faces of  $P$  containing  $v_0$ . Applying a vertical translation if necessary, we can suppose that the intersection of  $Q$  with  $\mathbb{R}_+^3$  is of the form  $Sh(H)$ , where  $H$  is a weakly convex hat with one face in its upper boundary corresponding to each face of  $P$  not containing  $v_0$  (and conversely).

Now let  $V = \Phi(U)$ , then, by Lemma 5.1, the restriction of  $V$  to each face of  $Q$  is a Killing field, so that  $V$  is a first-order isometric deformation of  $Q$ . Moreover, since the restriction of  $U$  to each face of  $P$  containing  $v_0$  fixes  $v_0$ , the restriction of  $V$  to the vertical faces of  $Q$  are horizontal translations or rotations around a vertical axis. So  $V$  does not change the heights of the boundary vertices of  $H$ . It follows from Lemma 2.4 that  $V$  is a trivial deformation – the restriction to  $Q$  of a global Killing vector field – and therefore, again from Lemma 5.1, that  $U$  is a trivial deformation of  $P$ . So  $P$  is infinitesimally rigid.  $\square$

Note that this argument – along with the results recalled in section 3, but without the need of section 4 – gives a direct proof of the infinitesimal rigidity of convex polyhedra.

*Proof of Theorem 1.5.* Let again  $Q = \phi(P)$ , so that, applying a vertical translation again if necessary,  $Q \cap \mathbb{R}_+^3 = Sh(H)$ , where  $H$  is a weakly convex hat. Let  $(\phi_t)_{t \in [0,1]}$  be a one-parameter family of projective transformation, chosen such that

- $\phi_0$  is the identity, while  $\phi_1 = \phi$ ,
- $\phi_t(P)$  is a compact polyhedron in  $\mathbb{R}^3$  for all  $t \in [0, 1)$ .

Let  $P_t = \phi_t(P)$ ,  $0 \leq t < 1$ . We know by Theorem 1.2 that  $P_t$  is infinitesimally rigid for all  $t \in [0, 1)$ . This means by Remark 1.4 that  $\Lambda_{P_t}$  has maximal rank, so that the signature of  $\Lambda_{P_t}$  is constant for  $t \in [0, 1)$ .

But a quick look at the definitions shows that  $\lim_{t \rightarrow 1} \Lambda_{P_t} = \Lambda_H$ , which is positive definite by Lemma 2.4. It follows that  $\Lambda_P = \Lambda_{P_0}$  is also positive definite.  $\square$

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