

FIXED POINTS OF COMPOSITIONS OF EARTHQUAKES

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ABSTRACT. Let S be a closed surface of genus at least 2, and let λ and μ be two laminations that fill S . Let E_r^λ and E_r^μ be the right earthquakes on λ and μ respectively. We show that the composition $E_r^\lambda \circ E_r^\mu$ has a fixed point in the Teichmüller space of S . Another way to state this result is that it is possible to prescribe any two measured laminations that fill a surfaces as the upper and lower measured bending laminations of the convex core of a globally hyperbolic AdS manifold. The proof uses some estimates from the geometry of those AdS manifolds.

1. INTRODUCTION, MAIN RESULTS

1.1. Earthquakes. In this paper we consider a closed surface S of genus at least 2. We denote by \mathcal{T}_S , or sometimes simply by \mathcal{T} , the Teichmüller space of S , which we consider to be the space of hyperbolic metrics on S considered up to isotopy. We denote by \mathcal{ML}_S , or simply by \mathcal{ML} , the space of measured laminations on S .

Given a measured lamination $\lambda \in \mathcal{ML}_S$, we denote by E_l^λ the *left earthquake* along λ on S . E_l^λ is a continuous map from \mathcal{T}_S to \mathcal{T}_S , see e.g. [30, 22]. Given two measured laminations λ, μ in S , we say that λ and μ *fill* S if any closed curve c in S which is not homotopically trivial has non-zero intersection with either λ or μ .

The main result of this paper concerns fixed points of compositions of earthquakes on S .

Theorem 1.1. *Let $\lambda, \mu \in \mathcal{ML}_S$ be two measured laminations which fill S . Then $E_l^\lambda \circ E_l^\mu : \mathcal{T}_S \rightarrow \mathcal{T}_S$ has a fixed point in \mathcal{T}_S .*

There are some reasons to believe that this fixed point is unique. It is explained below why this statement is equivalent to a conjecture made by G. Mess [25, 2] concerning globally hyperbolic anti-de Sitter 3-manifolds.

Theorem 1.1 shows a contrast between earthquakes and Dehn twists. Let λ and μ be two closed curves that fill S , and let D^λ and D^μ be the Dehn twists along λ and μ , respectively. Thurston proved (see [31, Section 6]) that for n and m large enough, the composition $(D^\lambda)^n \circ (D^\mu)^m$ is a pseudo-Anosov diffeomorphism of S , so that it acts on \mathcal{T} without fixed point. This does not contradict Theorem 1.1 since Dehn twists do not act on \mathcal{T} as earthquakes. (The shearing distance of a Dehn twist depends on the hyperbolic metric.)

1.2. Quasifuchsian hyperbolic 3-manifolds. Let $M := S \times \mathbb{R}$, let \mathcal{QF}_S be the space of quasifuchsian hyperbolic metrics on M , that is, complete hyperbolic metrics containing a non-empty compact subset which is convex, considered up to isotopy. Note that, here and elsewhere, we say that a subset $K \subset M$ is *convex* if any geodesic segment with endpoints in K is contained in K . If K is convex then its lift to the universal cover of M is also convex and therefore connected, and, if K is non-empty, it is a deformation retract of M .

Quasifuchsian hyperbolic metrics have a boundary at infinity which is topologically the union of two copies of S , endowed with a complex structure. According to a classical theorem of Bers [6], this complex structure defines a parameterization of \mathcal{QF}_S by the product of two copies of \mathcal{T}_S , one for each boundary component.

Given $g \in \mathcal{QF}_S$, (M, g) contains a smallest non-empty closed convex subset $C(M)$, called its convex core. Except when $C(M)$ is a totally geodesic surface, its boundary is the disjoint union of two closed surfaces each homeomorphic to S . We call them the “upper” and “lower” boundary components of $C(M)$. Each has an induced metric which is hyperbolic, and is “bent” along a measured geodesic lamination, see [30, 16].

Let m_+, m_- be the induced hyperbolic metrics and λ_+, λ_- be the measured bending laminations on the upper and lower boundary components of $C(M)$. Then it is well known that λ_- and λ_+ fill S . Moreover λ_+ and λ_- have no closed curve with weight larger than π .

Thurston conjectured that any two measured laminations on S satisfying these two conditions can be uniquely realized as (λ_-, λ_+) . The existence part of this conjecture was proved (in a more general form) by Bonahon and Otal.

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Theorem 1.2 (Bonahon, Otal [9]). *Let λ_+, λ_- be two measured laminations which fill S , which have no closed curve with weight at least equal to π . There exists a quasifuchsian metric $g \in \mathcal{QF}_S$ such that the measured bending lamination on the upper (resp. lower) boundary component of $C(M)$ is λ_+ (resp. λ_-).*

Thurston also conjectured that any two hyperbolic metrics can be obtained uniquely as (m_+, m_-) , and the existence part of this statement is also known (it follows from [24, 16]). This type of statement however does not appear explicitly here.

1.3. Globally hyperbolic AdS manifolds. The geometric theory of 3-dimensional globally hyperbolic anti-de Sitter (AdS) manifolds is quite similar to the theory of quasifuchsian hyperbolic 3-manifolds, a remarkable fact discovered by Mess [25, 2].

Recall that an AdS 3-manifold M is *globally hyperbolic* if it contains a space-like surface S which intersects any inextendible time-like curve exactly once. It is *globally hyperbolic maximal compact* (GHMC) if this space-like surface is closed, and moreover M is maximal (for the inclusion) under the previous condition.

Remark 1.3. Global hyperbolicity is a notion that makes sense for every Lorentzian manifold. It has strong consequences. If S is a Cauchy surface in M , then topologically $M = S \times \mathbb{R}$.

Moreover it can be shown that there is a foliation of M into spacelike slices parallel to S . More precisely, for a suitable product structure on M , the metric is of the form

$$(1) \quad -dt^2 + g_t$$

where t is the real parameter and g_t is a path of Riemann metrics on S .

Conversely, assuming S to be compact, any metric on $S \times \mathbb{R}$ that is of the form (1) is globally hyperbolic.

We refer to [4, 17] for a complete treatment of this topic.

We call \mathcal{GH}_S the space of globally hyperbolic maximal compact AdS metrics on $S \times \mathbb{R}$, with the Cauchy surface homeomorphic to S , considered up to isotopy.

Mess [25, 2] discovered that to any AdS metric $g \in \mathcal{GH}_S$ are associated two points in \mathcal{T}_S , its left and right hyperbolic metric, which can be defined through the decomposition of the identity component of the isometry group of AdS_3 as the product of two copies of $PSL_2(\mathbb{R})$. Moreover, these left and right metrics define a parameterization of \mathcal{GH}_S by $\mathcal{T}_S \times \mathcal{T}_S$. This can be construed as an analog of the Bers theorem mentioned above. Given $g \in \mathcal{GH}_S$, (M, g) also contains a smallest closed non-empty convex subset which is compact, which we also call $C(M)$ here.

We say that (M, g) is *Fuchsian* if $C(M)$ is a totally geodesic surface. As in the quasifuchsian setting, if M is not Fuchsian, then the boundary of $C(M)$ is the union of two surfaces homeomorphic to S , and each has a hyperbolic induced metric and is bent along a measured geodesic lamination. We call m_+, m_- the two hyperbolic metrics, and λ_+, λ_- the two measured laminations. Extending Thurston's conjectures, Mess [25] asked whether any two hyperbolic metrics on S can be uniquely obtained, and whether any two measured laminations that fill S can be uniquely obtained. The second result presented here is the proof of the existence part of the statement concerning the measured bending laminations.

Theorem 1.4. *Let $\lambda_+, \lambda_- \in \mathcal{ML}_S$ be two measured laminations that fill S . There exists a globally hyperbolic maximal AdS manifold M such that λ_+ and λ_- are the measured pleating laminations on the upper and lower boundary components of the convex core.*

1.4. From earthquakes to AdS manifolds. Theorem 1.4 is equivalent to Theorem 1.1 thanks to some key properties of GHMC AdS manifolds, which we recall briefly here and in more details in Section 2.

Theorem 1.5 (Mess [25]).

- Given a GHMC AdS manifold, its left and right metrics ρ_l, ρ_r are related to the induced metrics and measured bending laminations on the boundary of the convex core as follows:

$$\rho_l = E_l^{\lambda_+}(m_+) , \quad \rho_r = E_r^{\lambda_+}(m_+) ,$$

$$\rho_l = E_r^{\lambda_-}(m_-) , \quad \rho_r = E_l^{\lambda_-}(m_-) ,$$

so that

$$(2) \quad \rho_l = E_l^{2\lambda_+}(\rho_r) = E_r^{2\lambda_-}(\rho_r) .$$

- Given $\rho_l, \rho_r \in \mathcal{T}_S$ and $\lambda_+, \lambda_- \in \mathcal{ML}$ such that Equation (2) holds, there is a unique GHMC AdS manifold with left and right metrics ρ_l and ρ_r , and the measured bending laminations on the boundary components of its convex core are λ_+, λ_- .

The proof of Theorem 1.4 clearly follows from this and from Theorem 1.1 (and conversely): given two measured laminations λ_+, λ_- that fill S , the map $E_r^{2\lambda_+} \circ E_r^{2\lambda_-}$ has a fixed point, which we call ρ_r . Setting $\rho_l := E_r^{2\lambda_-}(\rho_r) = E_l^{2\lambda_+}(\rho_r)$, we see with Theorem 1.5 that ρ_l, ρ_r are the left and right metrics of a GHMC AdS manifold for which λ_+ and λ_- are the upper and lower measured bending laminations of the boundary of the convex core. The same argument can be used to prove Theorem 1.1 from Theorem 1.4.

1.5. Outline of the proof. The proof of Theorem 1.4 has two main parts.

The first is a description of elements of \mathcal{GH}_S near the “Fuchsian locus”, that, is the subset of AdS metrics on $S \times \mathbb{R}$ containing a totally geodesic Cauchy surface. In the quasifuchsian case, Bonahon [8] proved that any two measured laminations which fill S and are “small enough” in a suitable sense can be uniquely obtained as the measured bending laminations on the boundary of the convex core of a quasifuchsian manifold which is “almost-Fuchsian”. Series [29] then proved that those almost-Fuchsian metrics are the only ones realizing λ_+, λ_- as the bending lamination on the boundary of the convex core.

We present here an analog of those arguments for the GHMC AdS setting.

Theorem 1.6. *Let $\lambda, \mu \in \mathcal{ML}_S$ be two measured laminations that fill S . There exists $\epsilon > 0$ such that, for all $t \in (0, \epsilon)$, there exists a unique GHMC AdS manifold such that the measured bending laminations on the upper and lower boundary components of the convex core are $t\lambda$ and $t\mu$.*

The second tool of the proof of Theorem 1.4 is a compactness statement. First a definition.

Definition 1.7. Let $\mathcal{FML}_S \subset \mathcal{ML}_S \times \mathcal{ML}_S$ be the space of pairs of measured laminations that fill S . Let $\Phi : \mathcal{T}_S \times \mathcal{T}_S \rightarrow \mathcal{FML}_S$ be the map which associates to (ρ_l, ρ_r) the measured bending laminations (λ_+, λ_-) on the boundary of the convex core of the unique GHMC AdS manifold with left and right metrics ρ_l and ρ_r .

The compactness statement that is needed is the following statement, which is equivalent to Proposition 5.1.

Proposition 1.8. *Φ is proper.*

Since Φ is continuous, it is possible to define its degree, which by Theorem 1.6 is equal to 1. Therefore Φ is surjective, which is another way to state Theorem 1.4.

The proof of Proposition 1.8 is based on Proposition 4.1, a simple estimate on the intersection between measured laminations under some geometric assumptions.

Remark 1.9. The degree argument used here is possibly not limited to the AdS setting. It might also be used for hyperbolic quasifuchsian 3-manifolds to prove Theorem 1.2. The proof should then use the compactness result of [9] to prove that the map sending a quasifuchsian metric to the measured laminations on the boundary of the convex core (as seen as a map to the space of “admissible” pairs of laminations) is proper, so that its degree can be considered. Then the analysis made by Bonahon [8] concerning the behavior of this map near the Fuchsian locus, and the results of Series [29] showing that small laminations can be obtained only there, should indicate that the degree is one, so that the map is surjective. An interesting facet of this possible argument would be that it does not use the rigidity of hyperbolic cone-manifolds proved by Hodgson and Kerckhoff [21].

1.6. Cone singularities. The arguments used here can be extended, with limited efforts, to hyperbolic surfaces with cone singularities. Recall that Thurston’s Earthquake Theorem can be extended to this case [11]. We need some notations before stating the result.

We call $\mathcal{T}_{S,\theta}$ the Teichmüller space of conformal structures on S with N marked points x_1, \dots, x_N of cone angles given by $\theta = (\theta_1, \dots, \theta_N) \in (0, \pi)^N$. Any conformal class in $\mathcal{T}_{S,\theta}$ contains a unique hyperbolic metric with cone singularities of angle θ_i at each x_i (see [32]). Let $\mathcal{ML}_{S,N}$ be the space of measured laminations on the complement of $\{x_1, \dots, x_N\}$ in S . Given a hyperbolic metric g on S with cone singularities of angle θ_i at each x_i , any $\lambda \in \mathcal{ML}_{S,N}$ can be realized uniquely as a measured geodesic lamination for g (as long as the θ_i are less than π), see e.g. [15]. The notion of earthquake as defined by Thurston extends to this setting with cone singularities, see [11]. In this context, we (still) say that two measured laminations $\lambda, \mu \in \mathcal{ML}_{S,N}$ fill S if, given any closed curve c in $S \setminus \{x_1, \dots, x_N\}$ not homotopic to zero or to one of the x_i , the intersection of c with either λ or μ is non-zero.

Theorem 1.10. *Let $\lambda, \mu \in \mathcal{ML}_{S,N}$ be two measured laminations which fill S . Then $E_r^\lambda \circ E_r^\mu$ has at least one fixed point on $\mathcal{T}_{S,\theta}$.*

The proof follows the same line as the proof of Theorem 1.1. Theorem 1.5 still holds in the context of hyperbolic surfaces with cone singularities (of fixed angle less than π), in this case the GHMC AdS manifolds that are involved have “particles”, that is, cone singularities along time-like curves, see [11]. Those manifolds

have a convex core, its boundary is a pleated surface (outside the cone singularities) with an induced metric which is hyperbolic with cone singularities, and the pleating defines a measured lamination. Theorem 1.10 can therefore be stated in an equivalent way involving measured laminations on the boundary of the convex core.

Theorem 1.11. *Let $\lambda_+, \lambda_- \in \mathcal{ML}_{S,N}$ be two measured laminations which fill S (considered as a surface with N punctures). There exists a GHMC AdS manifold with particles, with angles $\theta_1, \dots, \theta_N$, such that the measured bending laminations on the upper and lower boundary components of the convex core are λ_+ and λ_- .*

The proof of Theorem 1.11 basically follows the same path as the proof of Theorem 1.4, with “particles” in the GHMC AdS manifolds which is considered.

1.7. Flat space-times. The Minkowski globally hyperbolic spacetimes have interesting properties which are reminiscent of, but different from, those of globally hyperbolic AdS manifolds. We recall those properties briefly here, more details are given in Appendix B.

Consider again a closed surface S of genus at least 2. Maximal globally hyperbolic spacetimes containing a space-like surface homeomorphic to S are quotient of a convex domain in the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$ by $\rho(\pi_1(S))$, where ρ is a morphism from $\pi_1(S)$ into the isometry group of $\mathbb{R}^{2,1}$. The linear part of ρ determines a point in \mathcal{T}_S ([25]).

In those spacetimes Mess pointed out a particular Cauchy surface that is obtained by grafting the hyperbolic surface corresponding to the linear holonomy along a measured geodesic lamination. In this way a measured geodesic lamination is associated to every MGHC flat spacetime ([25, 5]).

Maximal flat globally hyperbolic spacetimes come in pairs, with the elements of each pair sharing the same holonomy: each pair contains one future complete and one past complete spacetime.

Theorem 1.12. *For each $\lambda_-, \lambda_+ \in \mathcal{ML}_S$ which fill S , there is a unique pair of MGHC flat spacetimes, D_-, D_+ with the same holonomy, respectively past and future complete, such that λ_- and λ_+ are the laminations associated respectively with D_- and D_+ .*

The proof is in Appendix A.

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2. PRELIMINARIES

2.1. Earthquakes. According to Thurston’s Earthquake Theorem [22, 25], given two elements $u, v \in \mathcal{T}_S$, there is a unique $(\lambda, \mu) \in \mathcal{ML}_S \times \mathcal{ML}_S$ such that

$$E_l^\lambda(u) = E_r^\mu(u) = v.$$

So we can consider the map

$$(3) \quad \Phi' : \mathcal{T}_S \times \mathcal{T}_S \rightarrow \mathcal{ML}_S^2$$

associating to (u, v) the pair of measured lamination (λ, μ) . It will be clear below that this map Φ' differs from the map Φ introduced in the previous section only by a factor 2.

Recall that $(E_r^\lambda)^{-1} = E_l^\lambda$. So if u is a fixed point of $E_l^\mu \circ E_l^\lambda$ then

$$E_l^\lambda(u) = E_r^\mu(u)$$

which, in turn, is equivalent to

$$\Phi'(u, v) = (\lambda, \mu)$$

where we have put $v = E_l^\lambda(u)$.

Conversely, if $\Phi'(u, v) = (\lambda, \mu)$ then

$$v = E_l^\lambda(u) = E_r^\mu(u)$$

so that $u = E_l^\mu \circ E_l^\lambda(u)$. Therefore, $E_l^\mu \circ E_l^\lambda$ admits a fixed point if and only if (λ, μ) is contained in the image of Φ' . Moreover fixed points of $E_l^\mu \circ E_l^\lambda$ are obtained by projecting $\Phi'^{-1}(\lambda, \mu)$ on the first factor. As a consequence, our fixed point problem can be reduced to studying the image of Φ' .

2.2. Some AdS geometry. We briefly recall here some basic geometric properties of the anti-de Sitter space. More details can be found e.g. in [25, 2, 12].

Definition. The n -dimensional anti-de Sitter space can be defined as a quadric in the $(n + 1)$ -dimensional flat space $\mathbb{R}^{n-1,2}$ of signature $(n - 1, 2)$:

$$AdS_n = \{x \in \mathbb{R}^{n-1,2} \mid \langle x, x \rangle = -1\},$$

with the induced metric. It is a geodesically complete Lorentzian manifold of constant curvature -1 , with isometry group $O(n - 1, 2)$. It can be considered as the Lorentzian analog of the n -dimensional hyperbolic space.

One difference however is that AdS_n is not simply connected, its fundamental group is isomorphic to \mathbb{Z} , and AdS_n is homeomorphic to the product of a $(n - 1)$ -dimensional ball and a circle. It is often useful to consider its universal cover, denoted by \tilde{AdS}_n here.

Space-like, time-like and light-like vectors. We have mentioned that the metric induced by $\mathbb{R}^{n-1,2}$ is Lorentzian. So a non-zero vector v tangent to AdS_n can be of three types:

- *Space-like* if $\langle v, v \rangle > 0$.
- *Time-like* if $\langle v, v \rangle < 0$.
- *Light-like* or *isotropic* if $\langle v, v \rangle = 0$.

This terminology originates from relativity theory, see e.g. [19].

Geodesics and hyperplanes. By an elementary symmetry argument, the intersection of AdS_n with a 2-planes $P \subset \mathbb{R}^{n-1,2}$ containing the origin is a geodesic g . When the restriction to P of the metric of $\mathbb{R}^{n-1,2}$ is negative definite, the restriction to g of the induced metric of AdS_n is negative definite. Those geodesics are called time-like, and their non-zero tangent vectors are time-like. Conversely, time-like geodesics in AdS_n are exactly the intersections of AdS_n with the 2-dimensional planes containing 0 with negative definite induced metrics. It follows that those time-like geodesics are closed, of length equal to 2π . Each of those time-like geodesics is a retract by deformation of AdS_n .

Let P be a 2-dimensional plane containing 0 in $\mathbb{R}^{n-1,2}$ on which the induced metric is of signature $(1, 1)$. The intersection of P with AdS_n is a complete geodesic, on which the induced metric is positive definite – those geodesics are called space-like.

The intersections of AdS_n with the hyperplanes in $\mathbb{R}^{n-1,2}$ containing 0 and of signature $(n - 1, 1)$ are totally geodesic, and the induced metric is Riemannian. They are isometric to the hyperbolic $(n - 1)$ -dimensional space. Those hyperplanes are called *space-like*, and their non-zero tangent vectors are all space-like.

The intersections of AdS_n with the hyperplanes containing 0 and of signature $(n - 2, 2)$ are also totally geodesic, but of Lorentzian signature. They are isometric to AdS_{n-1} , and are called *time-like* hyperplanes. Note however that their non-zero tangent vectors are not all time-like, but they can be either space-like, time-like or light-like (isotropic). Actually a totally geodesic hyperplane is time-like if and only if it contains at least one time-like tangent vector.

Causal structure. We consider in the sequel a time orientation in AdS_n . In other terms, we choose one of the two connected components of the space of time-like vectors in AdS_n , and consider those vectors to be future-oriented, while those in the other connected component are past-oriented.

We can then define the future of a subset $\Omega \subset AdS_n$, as the subset of points in AdS_n which can be obtained as the endpoint of a future-oriented time-like curve starting from Ω . We will denote the future of Ω by $I^+(\Omega)$. Analogously we define the past of Ω as the set of endpoints of past-oriented time-like curves originating from Ω , and denote it by $I^-(\Omega)$.

This notion is not too helpful in AdS_n , because there are closed time-like curves. For instance it is not difficult to check that the future of a totally geodesic space-like hyperplane is the whole space. However it is more interesting in the universal cover \tilde{AdS}_n . For instance, the future of a space-like totally geodesic hyperplane P_0 is a connected component of $\tilde{AdS}_n \setminus P_0$.

Projective model, boundary at infinity. There is a natural action of $\mathbb{Z}/2\mathbb{Z}$ on AdS_n , given by $x \mapsto -x$ in the quadric model above. The quotient space $AdS_n/(\mathbb{Z}/2\mathbb{Z})$ has a projective model, which is often useful to obtain a heuristic idea of its geometry. It is obtained in the same manner as the Klein model of hyperbolic space, by projecting from the quadric in $\mathbb{R}^{n-1,2}$ to the tangent hyperplane P_0 at one of its points x_0 , in the direction of 0. This projection map, restricted to the points $x \in AdS_n$ such that $\langle x, x_0 \rangle < 0$, is projective — it sends geodesics to line segments — and its image is the interior of a quadric of signature $(n - 2, 1)$ in \mathbb{R}^n . It is sometimes more convenient to consider this model in the projective space \mathbb{RP}^n , rather than in \mathbb{R}^n . One gets in this manner a projective model of the quotient of AdS_n by the “antipodal” action of $\mathbb{Z}/2\mathbb{Z}$.

The *boundary at infinity* of $AdS_n/(\mathbb{Z}/2\mathbb{Z})$ can be defined in this projective model, as the quadric bounding the projective model of $AdS_n/(\mathbb{Z}/2\mathbb{Z})$ in $\mathbb{R}P^n$. Equivalently, we can define the boundary of AdS_n , seen as a quadric in $\mathbb{R}^{n-1,2}$, as the quotient of the space of non-zero isotropic vectors in $\mathbb{R}^{n-1,2}$ by $\mathbb{R}_{>0}$.

This boundary projects to $\mathbb{R}P^n$, and we obtain in this way the boundary of the projective model in $\mathbb{R}P^n$ of the quotient $AdS_n/(\mathbb{Z}/2\mathbb{Z})$.

Normal vectors to hyperplanes. The metric on AdS_n is non-degenerate. It follows that, given any totally geodesic hyperplane $P \subset AdS_n$ and any point $x \in P$, the orthogonal to $T_x P$ in $T_x AdS_n$ is a well-defined line in $T_x AdS_n$. If P is oriented and non-isotropic, then we can define its unit oriented normal vector at x , which is a vector n such that $\langle n, n \rangle = 1$ if P is time-like, and such that $\langle n, n \rangle = -1$ if P is space-like.

It is useful to remark that the unit normal vector field defined in this way is actually a parallel vector field along P , as can be checked by using the quadric model of AdS_n as defined above.

Angles between space-like hyperplanes. Let P_1, P_2 be two space-like totally geodesic hyperplanes in AdS_n , which intersect along a codimension 2 plane. Then P_1 and P_2 are the intersections with AdS_n of two hyperplanes H_1 and H_2 in $\mathbb{R}^{n-1,2}$ containing 0, each of signature $(n-1, 1)$.

We can define the angle θ between P_1 and P_2 as the angle between H_1 and H_2 in $\mathbb{R}^{n-1,2}$. Let N_1, N_2 be the two unit orthogonal vectors to H_1 and H_2 which are in the same connected component of $\mathbb{R}^{n-1,2} \setminus H_1$, then θ is the non-negative number defined by the equation

$$(4) \quad \cosh(\theta) = |\langle N_1, N_2 \rangle|.$$

If $x \in P_1 \cap P_2$ and n_1, n_2 are the unit future-oriented normals to P_1 and P_2 at x , then θ can also be defined by the fact that $\cosh(\theta) = |\langle n_1, n_2 \rangle|$.

Orthogonality between space-like and time-like hyperplanes. There is a well-defined notion of angle between a space-like and time-like hyperplane in AdS_n (see e.g. [27]), but we will not really need this notion here. What we do need, however, is the notion of orthogonality between a time-like hyperplane and a space-like hyperplane, and also between a time-like hyperplane and a space-like geodesic line or between a space-like hyperplane and a time-like line.

Those notions can be defined as follows. Given a time-like hyperplane T and a space-like hyperplane S , we say that they are orthogonal at a point $x \in P \cap S$ if, at x , the unit vector normal to T is orthogonal to the unit vector normal to S . Since the normal vector fields are parallel along T and along S , it then follows that T and S are also orthogonal at any other intersection point, and we will say simply that they are orthogonal.

Given now a space-like geodesic line D intersecting T , we say that D is orthogonal to T if, at the intersection point, D is parallel to the unit vector normal to T . In the same way, a time-like line D is orthogonal to a space-like hyperplane S if, at the intersection point, D is parallel to the unit vector normal to S .

AdS vs de Sitter. It can be useful to recall that the de Sitter n -dimensional space dS_n is defined as a quadric in the Minkowski $(n+1)$ -dimensional space:

$$dS_n = \{x \in \mathbb{R}^{n,1} \mid \langle x, x \rangle = 1\}.$$

It is also a Lorentzian constant curvature space, but of curvature 1 rather than -1 . The de Sitter space is dual to the hyperbolic space, rather than analogous to it, see e.g. [26].

For $n = 2$, however, AdS_2 and dS_2 are very similar, since AdS_2 is isometric to dS_2 with a reversed sign (in other terms, there is a map from AdS_2 to dS_2 which only changes the sign of the scalar product).

AdS₃. The 3-dimensional AdS space, AdS_3 , has some very specific properties, see [25]. One reason for this is that $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ is isometric to $PSL_2(\mathbb{R})$ with its Killing metric. As a consequence, $PSL_2(\mathbb{R})$ acts isometrically on $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ by left and right multiplication. This identifies $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ with the identity component of the isometry group of $AdS_3/(\mathbb{Z}/2\mathbb{Z})$.

2.3. AdS geometry and hyperbolic surfaces. G. Mess, in his celebrated work [25], showed that the map Φ' is meaningful in AdS context. We recall here how Mess connected the map Φ' to AdS geometry. This connection will play a fundamental role in the rest of the paper.

The relation between AdS and hyperbolic surfaces arises from the fact that AdS manifolds are locally modeled on the model $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ whose isometry group is $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. Moreover, the boundary at infinity of $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ is naturally identified to $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. The action of $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ on $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ extends to the boundary. In fact, the action on the boundary coincides with the product action.

Given two Fuchsian representations $\rho_l, \rho_r : \pi_1(S) \rightarrow PSL_2(\mathbb{R})$, there exists a unique homeomorphism ϕ of $\partial\mathbb{H}^2$ conjugating their actions on $\mathbb{R}P^1 = \partial\mathbb{H}^2$ ([1]).

Notice that the pair $\rho = (\rho_l, \rho_r)$ can be regarded as a representation in $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. The graph Γ_ρ of ϕ turns out to be an invariant subset of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$. In fact, it is the minimal non-empty invariant closed subset, like the limit set of a quasifuchsian representation.

Mess showed that there exists an open convex subset $\Omega = \Omega(\rho)$ in $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ that is invariant under ρ such that the action of ρ on Ω is free and properly discontinuous and the quotient $M(\rho) = \Omega/\rho$ is a globally hyperbolic AdS-manifold diffeomorphic to $S \times \mathbb{R}$. In fact, such Ω can be chosen maximally, in the sense that any other domain satisfying the same properties is contained in Ω .

Spacetimes $M(\rho)$ constructed in this way completely classify globally hyperbolic AdS structures on $S \times \mathbb{R}$.

Proposition 2.1. [25] *Let M be a globally hyperbolic AdS spacetime diffeomorphic to $S \times \mathbb{R}$. Then there is a pair of Fuchsian representations $\rho = (\rho_l, \rho_r)$ of $\pi_1(S)$ such that M isometrically embeds into $M(\rho)$.*

Given a pair of Fuchsian representations $\rho = (\rho_l, \rho_r)$, the set Γ_ρ is contained in the closure of $\Omega(\rho)$ in $AdS_3/(\mathbb{Z}/2\mathbb{Z})$, see [25]. Thus the convex hull $C(\tilde{M}(\rho))$ of Γ_ρ in $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ is contained in Ω and it projects to a convex subset $C(M(\rho))$ of $M(\rho)$. This subset turns out to be the *convex core* of $M(\rho)$, in the sense that it is a non-empty convex closed subset of $M(\rho)$ that is contained in every non-empty closed convex subset of $M(\rho)$. Here we say that $K \subset M(\rho)$ is *convex* if any any geodesic segment with endpoints in K is contained in K . With this definition any non-empty convex subset is a deformation retract of $M(\rho)$.

If ρ_l and ρ_r are conjugate, then $C(M(\rho))$ is a totally geodesic space-like surface homeomorphic to S , and the restriction of the metric on $C(M(\rho))$ makes it isometric to \mathbb{H}^2/ρ_l . If ρ_l and ρ_r represent two different points in \mathcal{T} then $C(M(\rho))$ is homeomorphic to $S \times [-1, 1]$ and it is a strong deformation retract of $M(\rho)$. The boundary components of $C(M(\rho))$ are achronal $C^{0,1}$ -surfaces (achronal means that time-like paths meet these sets in at most one point). We denote by $\partial_- C(M(\rho))$ (resp. $\partial_+ C(M(\rho))$) the past (resp. future) component of $\partial C(M(\rho))$.

The minimality condition implies that $\partial_- C(M(\rho))$ and $\partial_+ C(M(\rho))$ are totally geodesic space-like surfaces bent along geodesic laminations L_- and L_+ respectively. Since space-like planes in AdS_3 and in $AdS_3/(\mathbb{Z}/2\mathbb{Z})$ are isometric to \mathbb{H}^2 , the boundary components of $C(M(\rho))$ are equipped with a hyperbolic structure. Moreover, it is possible to equip L_- and L_+ with transverse measures μ_- and μ_+ that measure the amount of bending [25].

Mess [25] noticed that the right earthquake along $2\lambda_+ = (L_+, 2\mu_+)$ transforms \mathbb{H}^2/ρ_l into \mathbb{H}^2/ρ_r and analogously the left earthquake along $2\lambda_- = (L_-, 2\mu_-)$ transforms \mathbb{H}^2/ρ_l into \mathbb{H}^2/ρ_r . With our notation, we have

$$(5) \quad \Phi'(\rho_l, \rho_r) = (2\lambda_-, 2\lambda_+)$$

so that Φ' appears as twice the map Φ in Section 1.

Thus, the problem of determining the image of Φ' is equivalent to the problem of determining which pairs of laminations can be realized as bending laminations of some AdS globally hyperbolic structure on $S \times \mathbb{R}$. A first constraint for (λ_-, λ_+) to lie in the image of Φ' is that they have to fill the surface.

Lemma 2.2. *If $u \neq v$ then $\Phi'(u, v)$ is a pair of measured laminations that fill the surface S .*

Proof. By Formula (5), the statement is equivalent to the fact that the bending laminations of every GHMC AdS spacetime homeomorphic to $S \times \mathbb{R}$ fill S .

This fact can be proved along the same line as the analogous result for quasifuchsian manifold. Suppose that a curve c meets neither λ_+ nor λ_- . Then c can be realized as an unbroken geodesic both in $\partial_+ C(M(\rho))$ and in $\partial_- C(M(\rho))$. Thus two geodesic representatives of c should exist in M . But this contradicts the fact that for any free homotopy class of loop in M there is a unique geodesic representative (see [25, 5]). \square

Remark 2.3. The description of globally hyperbolic AdS manifolds, including the geometry of the boundary of the convex core, is strongly reminiscent of quasifuchsian hyperbolic manifolds. In the quasifuchsian case the convex core also exists; its boundary has an induced metric which is hyperbolic and it is pleated along a measured lamination. However the measured bending laminations on the boundary satisfy an additional condition: the weight of any closed curve is less than π .

The reason why such a restriction is not needed in AdS case lies on the fact that in AdS geometry, and more generally in Lorentzian geometry, the angle between two space-like totally geodesic planes is a well defined number in $[0, +\infty)$ (see Equation (4)).

3. EXISTENCE OF FIXED POINTS FOR SMALL LAMINATIONS

3.1. General setup. Let Δ denote the diagonal of $\mathcal{T} \times \mathcal{T}$. In this section we will prove that there is a neighborhood \hat{U} of Δ in $\mathcal{T} \times \mathcal{T}$ such that the restriction of Φ to $U = \hat{U} \setminus \Delta$ is a homeomorphism onto an open

set $V \subset \mathcal{FML}_S$. Moreover, we will show that \hat{U} can be chosen such that for any $(\lambda, \mu) \in \mathcal{FML}_S$ we have that $(t\lambda, t\mu) \in V$ for t sufficiently small.

3.2. Infinitesimal earthquakes and the length function. For a given measured geodesic lamination λ on S , the following semi-group law holds:

$$E_l^{t\lambda} \circ E_l^{s\lambda} = E_l^{(t+s)\lambda}.$$

We consider the *infinitesimal left earthquake* $e_l^\lambda(u) = \frac{d}{dt} E_l^{t\lambda}(u)$, which is a vector field on \mathcal{T} . It follows from the semi-group relation that $(E_l^{t\lambda})_{t \geq 0}$ corresponds to the flow of e_l^λ for positive time.

Analogously one can define the *infinitesimal right earthquake* e_r^λ and remark that $(E_r^{t\lambda})_{t \geq 0}$ is the flow for positive time of this field. Moreover, since $E_r^\lambda = (E_l^\lambda)^{-1}$ we have that $e_r^\lambda = -e_l^\lambda$ for every λ .

A basic standard property is that these fields continuously depend on the lamination λ . For the convenience of the reader, we will sketch the proof of this fact.

Proposition 3.1. *Let $\Gamma(T\mathcal{T})$ denote the space of vector fields on \mathcal{T} equipped with the C^∞ -topology. The map*

$$\mathcal{ML}_S \ni \lambda \mapsto e_l^\lambda \in \Gamma(T\mathcal{T})$$

is continuous.

Proof. Let l_γ denote the length function of the lamination λ , which is a smooth function on \mathcal{T} . A celebrated result of Wolpert states that the field e_l^λ is the symplectic gradient of l_λ with respect to the Weil-Petersson form (see [33]). Thus it is sufficient to prove that the map

$$\mathcal{ML}_S \ni \lambda \mapsto l_\lambda \in C^\infty(\mathcal{T})$$

is continuous.

Consider \mathcal{T} as the Fuchsian locus of the space of quasifuchsian metrics \mathcal{QF} on $S \times \mathbb{R}$, that is in a natural way a complex manifold. Bonahon [7] proved that the function l_λ can be extended to a holomorphic function — still denoted by l_λ — on \mathcal{QF} .

Let $\mathcal{O}(\mathcal{QF})$ be the space of holomorphic functions on \mathcal{QF} . The map

$$l : \mathcal{ML}_S \ni \lambda \mapsto l_\lambda \in \mathcal{O}(\mathcal{QF})$$

is locally bounded and continuous with respect to the pointwise topology on $\mathcal{O}(\mathcal{QF})$. Montel's theorem shows that it is continuous if $\mathcal{O}(\mathcal{QF})$ is equipped with the C^∞ -topology. \square

The results of this section will be based on a transversality argument that has been developed in [23, 28, 8].

Proposition 3.2. [8] *Let $\lambda, \mu \in \mathcal{ML}_S$ be two measured laminations. The intersection between e_l^λ and e_r^μ , considered as submanifolds in the total space of the bundle $T\mathcal{T}$, is transverse. Moreover if λ and μ fill up the surface then these sections meet in exactly one point $k_0(\lambda, \mu)$. Otherwise they are disjoint.*

Remark 3.3. The point $k_0(\lambda, \mu)$ continuously depends on λ and μ and it is the unique minimum point for the proper function $l_\lambda + l_\mu$. The fact that e_l^λ and e_r^μ transversely intersect in $k_0(\lambda, \mu)$ is a consequence of the fact that the Hessian of $l_\lambda + l_\mu$ at $k_0(\lambda, \mu)$ is positive definite (see [23]).

3.3. Bending laminations near the Fuchsian locus. Through this section, for $(\lambda, \mu) \in \mathcal{FML}_S$, we denote by $k_0(\lambda, \mu)$ the unique point of \mathcal{T} where e_l^λ is equal to e_r^μ .

Using the transversality result of Proposition 3.2, we prove that for every pair of laminations λ, μ that fill the surface the composition $E_l^{t\mu} \circ E_l^{t\lambda}$ admits a fixed point for small t .

Proposition 3.4. *Let λ_0 and μ_0 be laminations that fill up the surface and $u_0 = k_0(\lambda_0, \mu_0)$. There exist $\epsilon > 0$, a neighborhood V of $(\lambda_0, \mu_0) \in \mathcal{FML}_S$, a neighborhood U of u_0 , and a continuous map*

$$k : V \times [0, \epsilon) \rightarrow U$$

such that:

- For $t > 0$, $k(\lambda, \mu, t)$ is the unique fixed point of $E_l^{t\mu} \circ E_l^{t\lambda}$ lying in U .
- For all $(\lambda, \mu) \in V$, $k(\lambda, \mu, 0) = k_0(\lambda, \mu)$.

The first idea to prove this proposition could be to consider the function $E_l^\lambda \circ E_l^\mu : \mathcal{T}_S \rightarrow \mathcal{T}_S$, considered as depending on the parameter $(\lambda, \mu) \in \mathcal{ML}_S \times \mathcal{ML}_S$, prove that it has a fixed point as $(\lambda, \mu) \rightarrow 0$, and apply the Implicit Function Theorem. This is broadly speaking the argument we use, but some care is needed. One reason is that the map is critical as $(\lambda, \mu) \rightarrow 0$, so that a blow-up procedure is necessary. Another, related reason is that $(0, 0) \notin \mathcal{FML}_S$.

A more exact description of the argument of the proof is as follows. First, for any $(\lambda, \mu) \in \mathcal{FML}_S$ and $t \geq 0$ we point out a smooth function $\phi_{\lambda, \mu, t} : \mathcal{T} \rightarrow \mathbb{R}^{6g-6}$ such that

- $\phi_{\lambda,\mu,0}^{-1}(0)$ contains only $k_0(\lambda, \mu)$.
- $\phi_{\lambda,\mu,t}^{-1}(0)$ is the set of fixed points of $E_l^{t\mu} \circ E_l^{t\lambda}$ when $t > 0$.

In particular, u_0 is a solution of the equation $\phi_{\lambda_0,\mu_0,0}(u) = 0$. Using the transversality given by Proposition 3.2 we show that the differential $(d\phi_{\lambda_0,\mu_0,0})_{u_0}$ is not singular. The proof is then concluded by a simple application of the Implicit Function Theorem. We state for the reader's convenience the Implicit Function Theorem in the form we will use below.

Lemma 3.5. *Let X be a topological space and M be a differentiable manifold of dimension n . Consider a family of smooth maps $\{\phi_x : M \rightarrow \mathbb{R}^n\}_{x \in X}$ indexed by the elements of X so that the induced map*

$$X \ni x \mapsto \phi_x \in C^\infty(M, \mathbb{R}^n)$$

is continuous.

Suppose that for some $x_0 \in X$ and $u_0 \in \mathbb{R}^n$ we have $\phi_{x_0}(u_0) = 0$ and $d(\phi_{x_0})_{u_0}$ is not singular. Then, there is a neighborhood V of x_0 and a neighborhood U of u_0 such that for any $x \in V$ the equation

$$\phi_x(u) = 0$$

admits a unique solution $u(x)$ lying in U . Moreover the map

$$V \ni x \mapsto u(x) \in U$$

is continuous.

Proof of Proposition 3.4. Finding a fixed point of $E_l^{t\lambda} \circ E_l^{t\mu}$ is equivalent to finding a point u such that

$$(6) \quad E_l^{t\lambda}(u) = E_r^{t\mu}(u) .$$

Let us fix a global diffeomorphism

$$\mathbf{x} : \mathcal{T} \rightarrow \mathbb{R}^{6g-6} .$$

(We could for instance use the Fenchel-Nielsen coordinate for some pant decomposition of S , but the choice is irrelevant here.)

Consider the \mathbb{R}^{6g-6} -valued functions on \mathcal{T} defined by $\mathcal{E}_l^{t\lambda} = \mathbf{x} \circ E_l^{t\lambda}$ and $\mathcal{E}_r^{t\mu} = \mathbf{x} \circ E_r^{t\mu}$.

Notice that $\mathcal{E}_l^{t\lambda} - \mathcal{E}_r^{t\mu}$ forms a family of smooth maps from \mathcal{T} to \mathbb{R}^{6g-6} indexed by $\mathcal{FML}_S \times [0, \infty)$. Unfortunately, for $t = 0$, $\mathcal{E}_l^{t\lambda} = \mathcal{E}_r^{t\mu}$, so the difference is 0 everywhere and we cannot apply Lemma 3.5.

For this reason we blow up the difference. More precisely, we consider the family of maps

$$\phi_{\lambda,\mu,t}(u) = \frac{\mathcal{E}_l^{t\lambda}(u) - \mathcal{E}_r^{t\mu}(u)}{t} .$$

Clearly, if we fix (λ, μ) and $u \in \mathcal{T}$ then $\lim_{t \rightarrow 0} \phi_{\lambda,\mu,t}(u) = d\mathbf{x}_u(e_l^\lambda(u) - e_r^\mu(u))$, where we are implicitly using the canonical identification of $T_{\mathbf{x}(u)}\mathbb{R}^{6g-6}$ with \mathbb{R}^{6g-6} .

Moreover, putting

$$(7) \quad \phi_{\lambda,\mu,0}(u) = d\mathbf{x}_u(e_l^\lambda(u) - e_r^\mu(u)) ,$$

Proposition 3.1 implies that the induced application

$$\phi : \mathcal{FML}_S \times [0, +\infty) \rightarrow C^\infty(\mathcal{T}, \mathbb{R}^{6g-6})$$

is continuous.

By the choice of u_0 we have $\phi_{\lambda,\mu,0}(u_0) = 0$. On the other hand, since $d\mathbf{x}$ provides a trivialization of the tangent bundle of \mathcal{T} , Proposition 3.2 implies that the graphs of the functions

$$X_0(u) = d\mathbf{x}_u(e_l^{\lambda_0}) , \quad Y_0(u) = d\mathbf{x}_u(e_r^{\mu_0})$$

transversely meet over u_0 . Since $\phi_{\lambda_0,\mu_0,0} = X_0 - Y_0$ we conclude that $d(\phi_{\lambda_0,\mu_0,0})_{u_0}$ is non-singular.

From Lemma 3.5 there is a neighborhood V of (λ_0, μ_0) in \mathcal{FML}_S , $\epsilon > 0$, a neighborhood U of u_0 , and a continuous function

$$k : V \times [0, \epsilon) \rightarrow U$$

such that $k(\lambda, \mu, t)$ is the unique point in U such that

$$\phi_{\lambda,\mu,t}(k(\lambda, \mu, t)) = 0 .$$

Therefore, for $t > 0$, $k(\lambda, \mu, t)$ is the unique fixed point for $E_l^{t\mu} \circ E_l^{t\lambda}$ lying in U . On the other hand, Equation (7) implies that $k(\lambda, \mu, 0) = k_0(\lambda, \mu)$. \square

Let \mathcal{FML}_1 be a hypersurface in \mathcal{FML}_S that intersects every ray of \mathcal{FML}_S exactly once. By Proposition 3.4, there exists an open covering $\{V_i\}_{i \in I}$ of \mathcal{FML}_1 and a family of maps

$$(8) \quad k^{(i)} : V_i \times [0, \epsilon_i) \rightarrow U_i \subset \mathcal{T}$$

such that, for all $i \in I$,

- (1) $k^{(i)}(\lambda, \mu, 0) = k_0(\lambda, \mu)$.
- (2) If $t > 0$, then $k^{(i)}(\lambda, \mu, t)$ is the unique fixed point of $E_i^{t\mu} \circ E_i^{t\lambda}$ lying in U_i .

The following lemma shows how these maps can be glued to a global map.

Lemma 3.6. *There are an open neighborhood V^* of $\mathcal{FML}_1 \times \{0\}$ in $\mathcal{FML}_1 \times [0, +\infty)$ and a continuous map $k^* : V^* \rightarrow \mathcal{T}$ such that:*

- (1) *If $(\lambda, \mu, t_0) \in V^*$ then $(\lambda, \mu, t) \in V^*$ for every $t \in [0, t_0]$.*
- (2) *$k^*(\lambda, \mu, 0) = k_0(\lambda, \mu)$.*
- (3) *For $t > 0$ $k^*(\lambda, \mu, t)$ is a fixed point for $E_i^{t\mu} \circ E_i^{t\lambda}$.*

Proof. Let $k^{(i)} : V_i \times [0, \epsilon_i) \rightarrow U_i \subset \mathcal{T}$ be the family of maps defined in (8) indexed by $i \in I$. The uniqueness property of the function $k^{(\cdot)}$ shows that, for i and j in I , the subset of $(V_i \times [0, \epsilon_i)) \cap (V_j \times [0, \epsilon_j))$ where $k^{(i)}$ and $k^{(j)}$ coincide is open. Since $k^{(i)} = k^{(j)}$ on $V_i \cap V_j \times \{0\}$ the corresponding maps $k^{(i)}$ and $k^{(j)}$ coincide on the whole $(V_i \times [0, \epsilon_i)) \cap (V_j \times [0, \epsilon_j))$. It follows that on the set

$$V^* = \bigcup_i V_i \times [0, \epsilon_i)$$

the map k^* can be defined by gluing the maps $k^{(i)}$. □

Let

$$\pi : \mathcal{FML}_1 \times (0, +\infty) \rightarrow \mathcal{FML}_S$$

be the homeomorphism defined by $\pi(\lambda, \mu, t) = (t\lambda, t\mu)$. Let us define $V = \pi(V^* \setminus (\mathcal{FML}_1 \times \{0\}))$ where V^* is the open subset defined in Lemma 3.6. On the set V we can consider the map $k = k^* \circ \pi^{-1}$. Notice that $k(\lambda, \mu)$ is a fixed point of $E_i^\mu \circ E_i^\lambda$. By means of this map, we can construct on V a right inverse of the map Φ' defined in (3).

Corollary 3.7. *The open set V verifies the following properties.*

- (1) *For every $(\lambda, \mu) \in \mathcal{FML}_S$ there is $t > 0$ such that $(t\lambda, t\mu) \in V$.*
- (2) *If $(\lambda, \mu) \in V$ then $(t\lambda, t\mu) \in V$ for every $t \in (0, 1)$.*
- (3) *There is a continuous map $\sigma : V \rightarrow \mathcal{T} \times \mathcal{T}$ that is a right inverse for the map Φ' . Moreover*

$$\lim_{t \rightarrow 0} \sigma(t\lambda, t\mu) = (k_0(\lambda, \mu), k_0(\lambda, \mu)).$$

Proof. The first two properties follow directly from the definition on V . Using the fact that k sends (λ, μ) to a fixed point of $E_i^\mu \circ E_i^\lambda$, the map σ can be defined by putting

$$(9) \quad \sigma(\lambda, \mu) = (k(\lambda, \mu), E_i^\lambda(k(\lambda, \mu))).$$

□

3.4. All metrics near the Fuchsian locus are obtained. Through this section we use the same notations as in the previous section. In particular we consider the map σ constructed in Corollary 3.7. This map is clearly injective, so by the Theorem of the Invariance of Domain the image of σ is an open set U in $\mathcal{T} \times \mathcal{T}$. Notice that the restriction map $\Phi'|_U$ is a homeomorphism of U onto V .

Proposition 3.8. *$\hat{U} = U \cup \Delta$ is an open neighborhood of the diagonal Δ in $\mathcal{T} \times \mathcal{T}$.*

To prove Proposition 3.8, we need two technical results about the behavior of the map Φ' near the diagonal.

Lemma 3.9. *Let $(u_k, v_k)_{k \in \mathbb{N}} \in \mathcal{T}^2$ be a sequence converging to $(u, u) \in \Delta \subset \mathcal{T}^2$ and $(t_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers such that, putting $\Phi'(u_k, v_k) = (t_k \lambda_k, t_k \mu_k)$, we have that (λ_k) converges to a measured lamination $\lambda \neq 0$. Then the sequence of measured laminations (μ_k) also converges to a measured lamination μ . Moreover λ and μ fill the surface and u is the point $k_0(\lambda, \mu)$ where e_i^λ and e_r^μ meet.*

Proof. We have

$$E_l^{t_k \lambda_k}(u_k) = E_r^{t_k \mu_k}(u_k)$$

so by considering the Taylor expansion we deduce that

$$e_l^{\lambda_k}(u_k) - e_r^{\mu_k}(u_k) = O(t_k).$$

Taking the limit for $k \rightarrow +\infty$ we have that

$$\lim_{k \rightarrow +\infty} e_r^{\mu_k}(u_k) = e_l^\lambda(u).$$

Since the map $\mathcal{ML} \times \mathcal{T} \ni (\mu, u) \mapsto e_r^\mu(u) \in T\mathcal{T}$ is a homeomorphism (see [23]) we deduce that (μ_k) converges to a measured lamination μ such that

$$e_l^\lambda(u) = e_r^\mu(u).$$

Thus, since we are assuming $\lambda \neq 0$, we have that λ and μ fill the surface and $u = k_0(\lambda, \mu)$. \square

Lemma 3.10. *If $(u_k, v_k) \in \mathcal{T}^2$ converges to (u, u) , then there exists a sequence of positive numbers $(t_k)_{k \in \mathbb{N}}$ such that if we put $\Phi'(u_k, v_k) = (t_k \lambda_k, t_k \mu_k)$ then $\{(\lambda_k, \mu_k)\}$ is precompact in \mathcal{FML}_S .*

This lemma states that the two factors of $\Phi'(u_k, v_k)$ approach zero with the same speed.

Proof. We fix a sequence $t_k \rightarrow 0$ such that if we put $\Phi'(u_k, v_k) = (t_k \lambda_k, t_k \mu_k)$ we have that the sequence λ_k is precompact in $\mathcal{ML} \setminus \{0\}$. By Lemma 3.9, for every subsequence k_j such that λ_{k_j} converges we have that $(\lambda_{k_j}, \mu_{k_j})$ converges to a point in \mathcal{FML}_S . This proves that the set $\{(\lambda_k, \mu_k)\}$ is precompact in \mathcal{FML}_S . \square

We can prove now Proposition 3.8.

Proof of Proposition 3.8. Suppose by contradiction that \hat{U} is not a neighborhood of Δ . There is a sequence $(u_k, v_k) \in \mathcal{T} \times \mathcal{T}$ converging to some point $(u, u) \in \Delta$ such that $(u_k, v_k) \notin \hat{U}$.

By Lemma 3.10, there is a sequence of positive numbers $t_k \rightarrow 0$ such that if we put $\Phi'(u_k, v_k) = (t_k \lambda_k, t_k \mu_k)$ then — up to passing to a subsequence — $(\lambda_k, \mu_k) \rightarrow (\lambda, \mu) \in \mathcal{FML}_S$. By property (1) of Corollary 3.7, there is $\epsilon > 0$ such that $(t\lambda, t\mu) \in V$ for $t < 2\epsilon$. Since V is open we have that $(\epsilon\lambda_k, \epsilon\mu_k)$ lies in V for k sufficiently large. On the other hand, for k large enough, $t_k < \epsilon$ and, by property (2) of Corollary 3.7 we deduce that $\Phi'(u_k, v_k) = (t_k \lambda_k, t_k \mu_k)$ lies in V . Thus $(u_k, v_k) = \sigma(t_k \lambda_k, t_k \mu_k)$ and this contradicts our assumption on the sequence (u_k, v_k) . \square

4. AN UPPER BOUND ON THE LENGTH OF LAMINATIONS

4.1. The main estimate. The goal of this section is to prove the following key estimate on pairs of laminations that fill a surface.

Proposition 4.1. *There exist constants $\epsilon_0, h_0 > 0$ (depending only on the genus of S) as follows. Let $(\lambda, \mu) \in \mathcal{FML}_S$, and let $g \in \mathcal{T}_S$ be such that $E_r^\lambda(g) = E_l^\mu(g)$. Then $i(\lambda, \mu) \geq \epsilon_0 l_g(\lambda) \min(l_g(\lambda), h_0)$.*

This proposition will be used twice below. In particular, note that the inequality it contains is qualitatively different, depending on whether $l_g(\lambda)$ is small or not. When it is small the inequality is quadratic in $l_g(\lambda)$ and that fact will be important in Section 6 where Proposition 4.1 will be a key tool for understanding the behavior of the pleating lamination in the neighborhood of the Fuchsian locus. On the other hand, it is only linear in $l_g(\lambda)$ when $l_g(\lambda)$ is large. In that form, it will play a key role in Section 5 in the proof of the main compactness statement, Proposition 5.1. In fact, in that section we will use the following simple consequence of Proposition 4.1.

Corollary 4.2. *For all $C_1 > 0$ there exists $C_2 > 0$ such that, if $i(\lambda, \mu) \leq C_1$ and if $E_l^\lambda(g) = E_r^\mu(g)$, then $l_g(\lambda) \leq C_2$ and $l_g(\mu) \leq C_2$.*

The proof of Proposition 4.1 is based on the geometry of 3-dimensional AdS manifolds. The fact that $E_r^\lambda(g) = E_l^\mu(g)$ means that g is the left representation of a globally hyperbolic AdS manifold for which the bending lamination on the upper boundary of the convex core is $\lambda_+ = \lambda/2$, while the bending lamination on the lower boundary of the convex core is $\lambda_- = \mu/2$. The left representation is then $E_l^\lambda(g)$.

Proposition 4.1 will be proved by bounding $i(\lambda_+, \lambda_-)$ in terms of the length of λ_+ for the induced metric on the upper boundary of the convex core. This will be done by first considering the case when λ_+ is a weighted simple closed geodesic. In this case, the estimate will come from the area of a timelike geodesic annulus which hits the top in this closed geodesic and hits the bottom in a curve transverse to λ_- . In particular we want to measure the “width” of the convex core of M in certain directions.

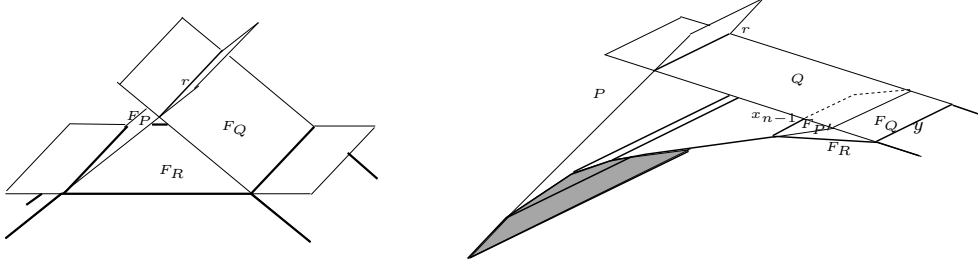


FIGURE 1. The proof of (11).

4.2. The width of the convex core of a MGHC AdS spacetime. Let M be a globally hyperbolic AdS spacetime homeomorphic to $S \times \mathbb{R}$ whose upper lamination λ_+ is a weighted curve (c, w) . We fix $x \in c$ and we consider a timelike geodesic τ through x that is orthogonal to a face containing x . In this section we find an estimate from below for the length of $\tau \cap C(M)$. This estimate will be a key point in the proof of Proposition 4.1.

It is based on the next two technical statements in AdS geometry. Recall (see Section 2.3) that space-like surfaces in AdS_3 have a hyperbolic induced metric, as do pleated surfaces in H^3 . However there are some differences between the geometry of pleated surfaces in the AdS and the hyperbolic case, as displayed in the following statement.

Sublemma 4.3. *Let Δ_+ be a convex surfaces obtained by bending a spacelike plane in AdS_3 along a locally finite measured lamination $\tilde{\lambda}_+$. Let σ be any geodesic path in Δ_+ joining a point $x \in \tilde{\lambda}_+$ to some point y . Let P be the space-like plane through x extending the face of Δ_+ that does not meet σ . Analogously let Q be the spacelike plane extending the face containing y (if y lies on a bending line, we choose the face that does not meet σ). Then P and Q meet along a spacelike line r . Moreover the following two properties hold:*

- If $\alpha(P, Q)$ denotes the angle between P and Q (as defined in Section 2.2),

$$(10) \quad \alpha(P, Q) \geq i(\sigma, \tilde{\lambda}_+).$$

- If $d_P(x, r)$ is the distance from x to r on the plane P and $d_Q(y, r)$ is the distance from y to r on Q , then

$$(11) \quad d_P(x, r) + d_Q(y, r) \leq l(\sigma),$$

where $l(\sigma)$ is the length of σ .

Proof. Suppose that P and Q are disjoint. Then, up to exchanging P and Q , P lies in the future of Q . On the other hand, Δ_+ is contained in $I^-(Q)$ and meets P at x (recall that $I^-(Q)$ is the past of Q , as defined in Section 2). This gives a contradiction.

In order to prove (10) first notice that we can easily reduce to the case where y lies on a bending line. Indeed let y' be the point on $\sigma \cap \tilde{\lambda}_+$ contained in the same face as y : if (10) holds for the segment of σ joining x to y' , then it can be easily checked that it holds for σ .

Assuming that y lies on some bending line, we proceed by induction on the number of bending lines between x and y . Let $x_1 = x, \dots, x_n = y$ be the intersection points of σ with $\tilde{\lambda}_+$. For the case $n = 2$, we consider the plane R extending the face through x containing σ . Lemma 6.15 of [5] states $\alpha(P, Q) \geq \alpha(P, R) + \alpha(R, Q) = i(\tilde{\lambda}_+, \sigma)$.

Consider now the inductive step. Let σ' be the segment of σ between x and x_{n-1} and R be the plane extending the face containing both x_{n-1} and y . It follows from the inductive hypothesis that $\alpha(P, R) \geq i(\sigma', \tilde{\lambda}_+)$. Still applying Lemma 6.15 of [5] to the planes P, R, Q , we obtain that $\alpha(P, Q) \geq i(\sigma', \tilde{\lambda}_+) + \alpha(R, Q)$. Notice that the last term is $i(\sigma, \tilde{\lambda}_+)$ so (10) is proved.

Let us prove now (11). Again in this case we can suppose that y lies on a bending line. We use again an induction on the number of points $x_1 = x, \dots, x_n = y$ of $\sigma \cap \tilde{\lambda}_+$.

When $n = 2$, let F_R be the face containing x and y and let R be the plane extending it. We consider the following surfaces with boundary: $F_P = P \cap I^-(Q) \cap I^+(R)$, $F_Q = Q \cap I^-(P) \cap I^+(R)$ (see Figure 1). Let Δ' be the surface obtained by replacing in Δ_+ the face F_R with $F_P \cup F_Q$.

Let χ be the geodesic segment connecting x to y in $F_P \cup F_Q$. Then σ and χ determine a time-like plane in AdS_3 , isometric to AdS_2 . In AdS_2 , space-like geodesics are maximizing length among space-like curves with the same endpoints (this can be checked easily by writing the AdS metric as a warped product, $-dt^2 + \cos^2(t)dx^2$,

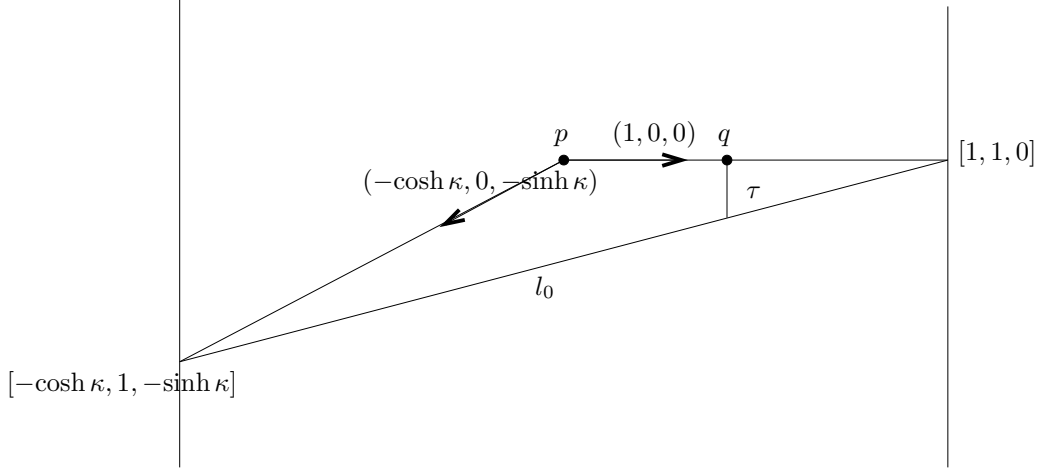


FIGURE 2. Sublemma 4.4.

with the geodesic segment contained in the line where $t = 0$ – see [4] for a more general viewpoint). So χ is shorter than σ . In other terms, the geodesic in Δ' connecting x to y is shorter than σ , and (11) easily follows.

Consider now the case $n > 2$. Let R be the plane extending the face F_R containing y and x_{n-1} and P' be the plane extending the face containing x_{n-2} and x_{n-1} (see Figure 1). Let us set

$$F_{P'} = P' \cap I^-(Q) \cap I^+(R), \quad F_Q = Q \cap I^-(P') \cap I^+(R).$$

Consider the bent surface Δ' obtained by replacing in Δ_+ the face F_R by $(F_{P'} \cup F_Q)$. Let σ' be the geodesic path of Δ' joining x to y . We claim that σ' is shorter than σ . Indeed consider the path χ in Δ' that is the composition of the geodesic χ_1 joining x to x_{n-1} and the geodesic χ_2 joining x_{n-1} to y . Then χ_1 is the segment of σ with endpoints x and x_{n-1} , whereas, as before, χ_2 is shorter than the segment $[x_{n-1}, y]$. Thus χ is shorter than σ . Since σ' is shorter than χ , the claim is proved.

Let y' be the intersection point of σ' with $P' \cap Q$ and σ'' be the segment on σ' between x and y' . By the inductive hypothesis we have that $d_P(x, r) + d_Q(y', r) \leq l(\sigma'')$ so we deduce that $d_P(x, r) + d_Q(y, r) \leq l(\sigma') \leq l(\sigma)$. \square

Sublemma 4.4. *Let T be a triangle in AdS_2 formed by two spacelike rays l_1, l_2 starting from a point p and a complete spacelike geodesic l_0 joining the ideal end-points of l_1 and l_2 . Let q be the point on l_1 whose distance from p is 1 and τ be the time-like ray through q orthogonal to l_1 . If we put $\theta_0 = l(\tau \cap T)$, then*

$$\theta_0 = \arctan \left(e \frac{\sinh \kappa}{1 + \cosh \kappa} \right),$$

where κ is the angle between l_1 and l_2 .

Proof. We identify AdS_2 with the quadric in \mathbb{R}^3 given by the equation $x^2 - y^2 - z^2 = -1$ and equipped with the metric induced by the form $dx^2 - dy^2 - dz^2$. Under such identification the boundary of AdS_2 is identified with the set of light-like vectors up to multiplication by a positive factor (see Section 2.2).

Under this identification we can suppose that $p = (0, 1, 0)$ and that the ideal end-point of l_1 is the class of $x_1 = (1, 1, 0)$. Imposing that the angle between l_1 and l_2 is κ we deduce that the ideal end-point of l_2 is the class of $x_2 = (-\cosh \kappa, 1, -\sinh \kappa)$ (see Figure 2). In particular the line l_0 is the intersection of AdS_2 with the linear plane generated by x_1 and x_2 . So its equation is

$$(12) \quad (x - y)\sinh \kappa - z(1 + \cosh \kappa) = 0.$$

On the other hand, the coordinates of q are $(\sinh 1, \cosh 1, 0)$, so a parameterization of τ is given by $(\cos(\theta)\sinh(1), \cos(\theta)\cosh(1), -\sin \theta)$. Imposing that $\tau(\theta_0) \in l_0$ we get the result. \square

We can now state the estimate we need.

Lemma 4.5. *There exist $\kappa_0, \epsilon_1 > 0$ as follows. Let M be an AdS globally hyperbolic spacetime homeomorphic to $S \times \mathbb{R}$ such that the upper lamination λ_+ is a weighted curve (c, w) . Let $x \in c$ and let τ be the past-directed geodesic starting from x and orthogonal to some face F .*

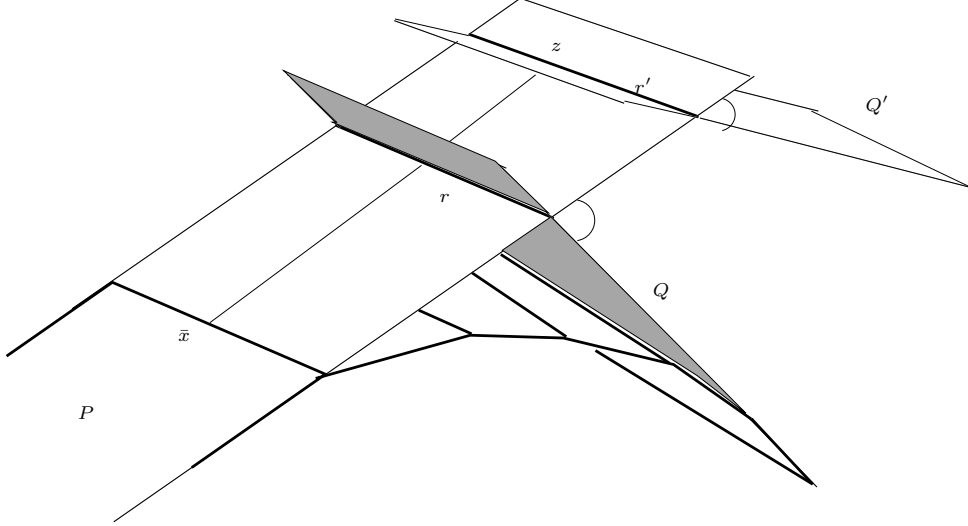


FIGURE 3. Proof of Lemma 4.5.

Given any geodesic σ of $\partial_+ C(M)$ that intersects F only at its end-point x and such that its length is equal 1 we have

$$(13) \quad l(\tau \cap C(M)) \geq \epsilon_1 \min(\kappa_0, i(\sigma, \lambda_+)).$$

Proof of Lemma 4.5. We lift the problem to the universal covering. Let Δ_+ be the upper boundary of the lift of $C(\tilde{M})$ in AdS_3 , and let \bar{x} be a lift of x . We consider the lifts $\bar{\tau}$ and $\bar{\sigma}$ of τ and σ from \bar{x} . Inequality (13) is equivalent to $l(\bar{\tau} \cap C(\tilde{M})) \geq \epsilon_0 \min(\kappa_0, i(\bar{\sigma}, \tilde{\lambda}_+))$

Let y be the end-point of $\bar{\sigma}$ and consider the plane P through \bar{x} orthogonal to $\bar{\tau}$ (that is, the support plane of Δ_+ which extends the face that does not meet $\bar{\sigma}$) and the plane Q extending the face of Δ_+ that contains y (if y lies on a bending line we choose the face that does not meet $\bar{\sigma}$). By Sublemma 4.3, the distance on P between \bar{x} and the line $r = P \cap Q$ is less than 1. So there exists a geodesic $r' \subset P \cap I^+(Q)$ at distance 1 from x . Let us consider the plane Q' containing r' and such that the angle between P and Q' is $\kappa = i(\bar{\sigma}, \tilde{\lambda}_+)$ (see Figure 3).

By Sublemma 4.3, the angle between P and Q is larger than κ . This implies that Q' cannot meet the half-plane $Q \cap I^-(P)$. Otherwise the surface obtained as the union of $P \cap I^-(Q)$, $Q \cap I^-(P) \cap I^-(Q')$ and $Q' \cap I^-(Q)$ would be a space-like surface bent along the geodesics $P \cap Q$ and $Q \cap Q'$, contradicting Equation (10) in Sublemma 4.3.

In particular the surface $\Delta = (P \cap I^-(Q')) \cup (Q' \cap I^-(P))$ is contained in the future of Δ_+ . Let $C(\Delta)$ be the convex hull of Δ . Clearly the past boundary of $C(\Delta)$ is contained in the future of the past boundary of $C(\tilde{M})$. It follows that $l(\bar{\tau} \cap C(\tilde{M}))$ is larger than $l(\bar{\tau} \cap C(\Delta))$.

Now consider the timelike plane Π through \bar{x} that is orthogonal to the bending line r' of $C(\Delta)$. Notice that $C(\Delta) \cap \Pi$ is a convex set whose upper boundary is made of two geodesic rays l_P and $l_{Q'}$ meeting at some point z . Let T be the triangle in Π bounded by l_P , $l_{Q'}$ and by the geodesic joining the ideal end-point of l_P and $l_{Q'}$. T is contained in $C(\Delta)$. The vertex z is the point on r' realizing the distance from \bar{x} , so the distance between \bar{x} and z is 1. Moreover, the angle between l_P and $l_{Q'}$ is κ . By Sublemma 4.4 the length $l(\bar{\tau} \cap T)$ is larger than some constant depending only on κ and proportional to κ as $\kappa \rightarrow 0$. Since $l(\bar{\tau} \cap C(\tilde{M})) \geq l(\bar{\tau} \cap C(\Delta)) \geq l(\bar{\tau} \cap T)$ the estimate is proved. \square

Remark 4.6. Choosing κ_0 sufficiently small, we can suppose that $\epsilon_1 \kappa_0 \leq \pi/3$. We make this hypothesis in the rest of this section.

4.3. Pleated surfaces in AdS_3 . We will be using a technical statement from 2-dimensional hyperbolic geometry. The proof can be found in Appendix A. Consider any hyperbolic metric g on S , and a closed oriented geodesic c for g . Given a point $x \in c$, we consider the compact geodesic segment σ_x^l (resp. σ_x^r) of length 1 starting from x in the direction orthogonal to c towards the left (resp. right). We denote by $n^l(x)$ (resp. $n^r(x)$) the number of intersections of σ_x^l (resp. σ_x^r) with c , including x . Let λ be the measured geodesic lamination with support c and weight w .

Lemma 4.7. *There exists $\beta_0 > 0$ (depending only on the genus of S) such that*

$$l_g(\{x \in c \mid i(\sigma_x^r, \lambda) \leq \beta_0 l_g(\lambda)\}) \leq l_g(c)/2 .$$

In other terms, if the length of c is much larger than $\frac{1}{\beta_0}$ then, in at least half of c , the orthogonal segment of length 1 on either side of c intersects c many times.

Lemma 4.7 and Lemma 4.5, taken together, lead to a lower bound on the area of a totally geodesic time-like strip in the convex core of a globally hyperbolic AdS manifold. Here we fix a globally hyperbolic manifold M and we call $C(M)$ its convex core, λ_+ and λ_- the upper and lower measured bending laminations, and m_+, m_- the upper and lower induced metrics on the boundary of $C(M)$. We use the constants κ_0 and ϵ_1 appearing in Lemma 4.5.

Lemma 4.8. *Suppose that the support of λ_+ is a closed geodesic curve c , on which we choose an orientation. Let A be the time-like totally geodesic annulus in $C(M)$, with boundary contained in $\partial C(M)$, such that $c = A \cap \partial_+ C(M)$, and which is orthogonal to the face on the left of c . Then the area of A is bounded from below:*

$$\text{Area}(A) \geq \frac{l_{m_+}(c)}{4} \epsilon_1 \min(\kappa_0, \beta_0 l_{m_+}(\lambda_+)) .$$

Proof. Let $w > 0$ be the weight of c for λ_+ , and let $c_i \subset c$ be the subset of c defined as

$$c_i = \{x \in c \mid i(\sigma_x^r, \lambda_+) \geq \beta_0 l_{m_+}(\lambda_+)\} .$$

Lemma 4.7 indicates that the length of c_i is at least

$$(14) \quad l_{m_+}(c_i) \geq l_{m_+}(c)/2 .$$

Let $x \in c_i$, and let τ_x be the past-directed geodesic segment on A starting from x in the direction orthogonal to c . Because $x \in c_i$ the intersection of σ_x^r with the lamination λ_+ is larger than $\beta_0 l_{m_+}(\lambda_+)$. Applying Lemma 4.5, we deduce that $l(\tau_x) \geq L_1$ where

$$L_1 := \epsilon_1 \min(\kappa_0, \beta_0 l_{m_+}(\lambda_+)) .$$

Note that, by the choice made in Remark 4.6,

$$(15) \quad L_1 \leq \pi/3 .$$

Consider the map

$$\begin{aligned} \tau : c_i \times [0, L_1] &\rightarrow A \\ (x, s) &\mapsto \tau_x(s) . \end{aligned}$$

The Jacobian of τ at (x, s) is equal to the norm at $\tau_x(s)$ of the Jacobi field along τ_x which is orthogonal to τ_x and of unit norm at $s = 0$. Recall that orthogonal Jacobi fields along time-like geodesics in AdS_3 behave as $\cos(s)$ (it follows from the fact that the curvature is -1 , see e.g. [19, 4.4]). So the Jacobian of τ at (x, s) is equal to $\cos(s)$, and therefore at least $1/2$ by Equation (15).

As a consequence,

$$\text{Area}(A) \geq \text{Area}(\tau(c_i \times [0, L_1])) \geq \frac{l_{m_+}(c_i) L_1}{2} .$$

The result therefore follows from the definition of L_1 and from Equation (14). \square

4.4. Proof of the main estimate. The proof of Proposition 4.1 follows from Lemma 4.8 and from the following basic statement on Lorentz geometry.

Lemma 4.9. *Let Π be a time-like plane in AdS_3 , and let P, Q be two space-like planes, such that Π, P and Q meet exactly at one point. Then the angle in Π between $\Pi \cap P$ and $\Pi \cap Q$ is smaller than the angle between P and Q .*

Proof. Let x be the intersection point of Π, P and Q , and let $H_x \subset T_x AdS_3$ be the surface containing the future-oriented unit timelike vectors. Clearly H_x is isometric to the hyperbolic plane. The unit future-pointing vectors N_P and N_Q that are orthogonal to P and Q respectively lie in H_x . Equation (4) shows that the angle between P and Q is equal to the hyperbolic distance between N_P and N_Q in H_x .

On the other hand the set of future-pointing unit vectors tangent to Π at x is the geodesic $l = H_x \cap T_x \Pi$ of H_x . The future-pointing unit vectors in $T_x \Pi$ that are orthogonal to $P \cap \Pi$ and $Q \cap \Pi$ are the orthogonal projections of N_P and N_Q on l .

The result therefore follows from the fact that the orthogonal projection on a hyperbolic geodesic is distance-decreasing. \square

Let c_- be the lower boundary of A . Although c_- is not geodesic, it is not difficult to see that the geometric intersection of λ_- with c_- is equal to $i(c, \lambda_-)$. The previous lemma, along with an approximation of c_- by a sequence of polygonal curves, shows that the geometric intersection of λ_- with c_- is at least equal to the geodesic curvature of c_- as boundary of A , which by the Gauss-Bonnet theorem is equal to the area of A (see [3, 13]). So we obtain the following estimate.

Corollary 4.10. *Under the hypothesis of Lemma 4.8, if λ_- is the measured bending lamination on the lower boundary of $C(M)$, then $i(c, \lambda_-) \geq \text{Area}(A)$, so that*

$$i(\lambda_+, \lambda_-) \geq \frac{l_{m_+}(\lambda_+)}{4} \epsilon_1 \min(\kappa_0, \beta_0 l_{m_+}(\lambda_+)) .$$

We can prove now Proposition 4.1.

Proof of Proposition 4.1. We first find constants ϵ_0, h_0 that work assuming that λ is a weighted curve (c, w) . By a density argument we then conclude that those constants work for every measured geodesic lamination.

According to Theorem 1.5, there is a unique GHMC AdS manifold M for which the left representation is g , the upper bending lamination of the convex core is $\lambda_+ := \lambda/2$, and the lower bending lamination of the convex core is $\lambda_- := \mu/2$. Then the upper and lower induced metrics on the boundary of the convex core are

$$m_+ = E_r^{\lambda/2}(g) , \quad m_- = E_l^{\mu/2}(g) .$$

We can now apply Corollary 4.10, which shows that

$$i(\lambda_+, \lambda_-) \geq \frac{l_{m_+}(\lambda_+)}{4} \epsilon_1 \min(\kappa_0, \beta_0 l_{m_+}(\lambda_+)) .$$

But $g = E_l^{\lambda_+}(m_+)$, so that $l_g(\lambda_+) = l_{m_+}(\lambda_+)$, and it follows that

$$i(\lambda, \mu) \geq \frac{l_g(\lambda)}{2} \epsilon_1 \min(\kappa_0, \beta_0 l_g(\lambda)/2) .$$

So the constants

$$\epsilon_0 = \beta_0 \epsilon_1 / 4 , \quad h_0 = 2\kappa_0 / \beta_0$$

work.

Consider now the general case. We can consider a sequence of weighted curves λ_n converging to λ such that $\lambda_n = (c_n, w_n)$.

Let μ_n be the lamination such that $E_l^{\lambda_n}(g) = E_r^{\mu_n}(g)$. Notice that (μ_n) converges to μ as $n \rightarrow +\infty$. Indeed, μ_n is the right factor of $\Phi'(g, E^{t\lambda_n}(g))$.

Since the inequality

$$i(\lambda_n, \mu_n) > \epsilon_0 l_g(\lambda_n) \min(h_0, l_g(\lambda_n))$$

holds for every n and $i(\cdot, \cdot)$ is a continuous function of \mathcal{ML}_S^2 , passing to the limit we get the estimate for λ and μ . \square

5. COMPACTNESS

5.1. Limits of fixed points of compositions of earthquakes. The rather technical result in the previous section can be used to prove the following compactness statement, which is a key point in the proof of the main result.

Proposition 5.1. *Let $(\lambda_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}}$ be two sequences of measured laminations on S , converging respectively to λ, μ . Suppose that λ and μ fill S . Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of hyperbolic metrics on S such that, for all $n \in \mathbb{N}$, $E_l^{\lambda_n}(g_n) = E_r^{\mu_n}(g_n)$. Then, after extracting a subsequence, (g_n) converges to a limit $g \in \mathcal{T}_S$, and $E_l^\lambda(g) = E_r^\mu(g)$.*

We recall a well-known result needed in the proof.

Lemma 5.2 (Kerckhoff [22]). *Let $\lambda, \mu \in \mathcal{ML}_S$ be two measured laminations that fill a surface. The function*

$$\begin{aligned} L_{\lambda, \mu} : \mathcal{T}_S &\rightarrow \mathbb{R} \\ g &\mapsto l_g(\lambda) + l_g(\mu) \end{aligned}$$

is proper and convex along earthquakes paths.

Proof of Proposition 5.1. Since $i(\cdot, \cdot)$ is continuous, there exists a constant $C_1 > 0$ such that for all $n \in \mathbb{N}$, $i(\lambda_n, \mu_n) \leq C_1$. So it follows from Corollary 4.2 that both $l_{g_n}(\lambda_n)$ and $l_{g_n}(\mu_n)$ are bounded by a constant C_2 . It follows from Lemma 5.2 that, for all $n \in \mathbb{N}$,

$$g_n \in K_n := L_{\lambda_n, \mu_n}^{-1}([0, 2C_2]) .$$

Since by Lemma 5.2 the subsets K_n are compact and convex for earthquake paths, $K_n \rightarrow K := L_{\lambda, \mu}^{-1}([0, 2C_2])$. So g_n remains in a compact subset of \mathcal{T}_S and, after taking a subsequence, (g_n) converges to a limit g .

The fact that $E_l^\lambda(g) = E_r^\mu(g)$ is clear since E_l and E_r are continuous functions of both arguments. \square

6. PROOFS OF THE MAIN RESULTS

In this section we combine results of Sections 3 and 5 to prove Theorems 1.1, 1.4 and 1.6.

Consider the map Φ' described at the beginning of Section 2. Proposition 5.1 precisely states that this map is proper. In particular the degree of Φ' can be defined. The main idea of the argument is to show that the degree of Φ' is 1. As a consequence, we will deduce that Φ' is surjective, and this will prove Theorem 1.1 and Theorem 1.4.

6.1. Proof of Theorem 1.1 and Theorem 1.4. Let us consider the set

$$X = \{(u, v) \in \mathcal{T}^2 \mid \Phi'^{-1}\Phi'(u, v) = \{(u, v)\}\} .$$

Proposition 6.1. $\hat{X} = \Delta \cup X$ is a neighbourhood of Δ in \mathcal{T}^2 .

Proof. By contradiction, suppose there exists a sequence $(u_k, v_k) \notin X$ converging to $(u, u) \in \Delta$. By Proposition 3.8, there exists another sequence $(u'_k, v'_k) \in \mathcal{T}$ such that

$$\Phi'(u'_k, v'_k) = \Phi'(u_k, v_k) .$$

By Lemma 3.10 there is an infinitesimal sequence of positive numbers t_k such that

$$\Phi'(u'_k, v'_k) = \Phi'(u_k, v_k) = (t_k \lambda_k, t_k \mu_k)$$

with $\{(\lambda_k, \mu_k)\}$ running in some compact set of \mathcal{FML}_S . Taking a subsequence we can suppose that $(\lambda_k, \mu_k) \rightarrow (\lambda, \mu)$ and u is the point $k_0(\lambda, \mu)$ where e_l^λ and e_r^μ meet.

By Proposition 4.1 we have

$$t_k^2 i(\lambda_k, \mu_k) \geq \epsilon_0 t_k^2 l_{u'_k}(\lambda_k)^2$$

so we get that $l_{u'_k}(\lambda_k)$ is bounded by some constant independent of k . Analogously we deduce that $l_{u'_k}(\mu_k)$ is bounded. Since (λ_k, μ_k) runs in a compact set of \mathcal{FML}_S , the functions

$$l(\lambda_k) + l(\mu_k)$$

are uniformly proper by Lemma 5.2.

So we deduce that (u'_k) runs in some compact set of \mathcal{T} (and analogously for (v'_k)). Moreover, by Lemmas 3.9 and 3.10, any convergent subsequence of (u'_k) (resp. (v'_k)) must converge to $k_0(\lambda, \mu) = u$. Thus we deduce that $(u'_k, v'_k) \rightarrow (u, u)$.

By Corollary 3.7, there is a neighbourhood U of $(u, u) \in \mathcal{T}^2$ such that the map $\Phi'|_{U \setminus \Delta}$ is a homeomorphism onto an open set in \mathcal{FML}_S . But for $k \gg 0$ both (u_k, v_k) and (u'_k, v'_k) lie in U and this is a contradiction. \square

Corollary 6.2. *There is an open set $V \in \mathcal{FML}_S$ such that the restriction of Φ' to $\Phi'^{-1}(V)$ is a homeomorphism onto V .*

Proof. There is an open neighbourhood U of Δ contained in X . Let us put $V = \Phi'(U)$. Clearly the restriction of Φ' on U is injective so V is an open set. By definition of X the inverse image of any point $x \in V$ consists only of one point. Thus $\Phi'^{-1}(V) = U$. \square

Corollary 6.3. *The degree of the map Φ' is 1. In particular, Φ' is surjective.*

Proof. This follows from Corollary 6.2, since a continuous (proper) map which restricts to a homeomorphism from an open subset to its image has degree one, see [20]. \square

Since Φ' has degree one, it is surjective, and this proves Theorem 1.1. We have already mentioned that Theorem 1.4 is equivalent to Theorem 1.1.

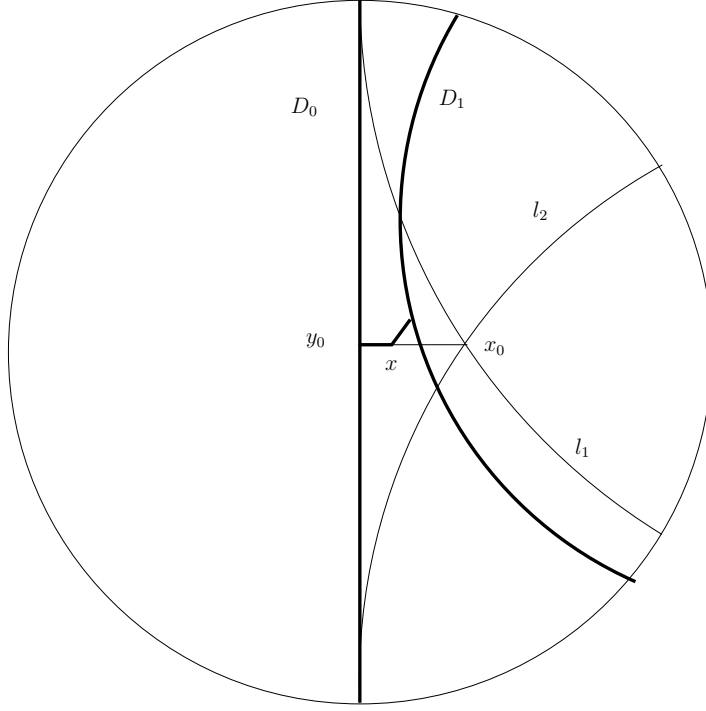


FIGURE 4. Sublemma A.2

6.2. Proof of Theorem 1.6. Let $(\lambda, \mu) \in \mathcal{FML}_S$. We have to prove that for t sufficiently small $(t\lambda, t\mu)$ are uniquely realized as the bending laminations of the convex core of a GHMC AdS spacetime. By Equation (5), this is equivalent to showing that $\Phi'^{-1}(t\lambda, t\mu)$ contains exactly one point for t small.

We consider the right inverse of Φ' , $\sigma : V \rightarrow \mathcal{T} \times \mathcal{T}$, defined in Corollary 3.7. There is $\epsilon > 0$ such that $(t\lambda, t\mu) \in V$ for $t < \epsilon$. In particular $\sigma(t\lambda, t\mu)$ is in $\Phi'^{-1}(t\lambda, t\mu)$. On the other hand, $\sigma(t\lambda, t\mu)$ approaches the diagonal Δ as $t \rightarrow 0$. By Proposition 6.1, there is $\epsilon' \leq \epsilon$ such that $\Phi'^{-1}(t\lambda, t\mu) = \{\sigma(t\lambda, t\mu)\}$.

6.3. Surfaces with cone singularities. As pointed out in the introduction, the arguments given for the proof of Theorem 1.1 and of Theorem 1.4 can be extended basically as they are to surfaces with cone singularities of angle $\theta_i \in (0, \pi)$, and to GHMC AdS manifolds with “particles” of the same angles. Thurston’s Earthquake Theorem is then replaced by its version with “particles” as described in [11], where the geometry of GHMC AdS manifolds with particles was also studied. This leads directly to the proof of Theorem 1.10 or, equivalently, Theorem 1.11.

APPENDIX A. PROOF OF LEMMA 4.7

Recall that we consider a hyperbolic metric g on S , and a closed oriented geodesic c for g . Given $x \in c$, σ_x^l (resp. σ_x^r) is the compact geodesic segment of length 1 starting from x in the direction orthogonal to c towards the left (resp. right), and $n^l(x)$ (resp. $n^r(x)$) is the number of intersections of σ_x^l (resp. σ_x^r) with c , including x . Lemma 4.7 is a direct consequence of the following statement.

Lemma A.1. *There exists $\beta_0 > 0$ (depending only on the genus of S) such that*

$$l_g(\{x \in c \mid n^r(x) \leq \beta_0 l_g(c)\}) \leq l_g(c)/2$$

for every simple closed geodesic c .

The proof uses an elementary statement from plane hyperbolic geometry.

Sublemma A.2. *There exists $\gamma_0 > 0$ as follows. Let D_0, D_1 be two disjoint lines in \mathbb{H}^2 , and let $x \in \mathbb{H}^2$ be in the connected component of $\mathbb{H}^2 \setminus (D_0 \cup D_1)$ having both D_0 and D_1 in its boundary. Suppose that $d(x, D_0) \leq \gamma_0$, $d(x, D_1) \leq \gamma_0$. Then the geodesic segment of length 1 starting orthogonally from D_0 and containing x intersects D_1 .*

Proof. It is sufficient to find γ_0 assuming that D_0 is a fixed geodesic and that the end-point on D_0 of the minimizing geodesic segment going from x to D_0 is a fixed point y_0 . Finally we can assume that x is contained in a fixed half-plane P_0 bounded by D_0 .

Let $x_0 \in P_0$ be the end-point of the geodesic segment of length 1 starting from y_0 and orthogonal to D_0 , and let l_1, l_2 be the two complete geodesics that share an ideal end-point with D_0 and pass through x_0 .

Notice that either D_1 meets the segment $[y_0, x_0]$ or its distance from y_0 is larger than the distance δ of the lines l_i from y_0 . Now, taking $\gamma_0 = \delta/2$, the distance between D_1 and y_0 is less than $d(D_1, x) + d(x, y_0) \leq 2\gamma_0 = \delta$, so D_1 intersects $[y_0, x_0]$. \square

Proof of Lemma A.1. We take here some $\beta_0 > 0$, and will later see how it has to be chosen so as to obtain the desired result. Let

$$c_i := \{x \in c \mid n^r(x) \leq \beta_0 l_g(c)\} .$$

Fix γ_0 as in Sublemma A.2 and consider the normal exponential map:

$$\begin{aligned} \exp : c_i \times [0, \gamma_0] &\rightarrow S \\ (s, r) &\mapsto \sigma_s^r(r) . \end{aligned}$$

This map is dilating, so it increases the area.

Moreover, Sublemma A.2 shows that each point $x \in S$ has at most n_0 inverse images in $c_i \times [0, \gamma_0]$, where n_0 is the integer part of $\beta_0 l_g(c)$. Indeed, suppose that x is the image of $(y_1, r_1), \dots, (y_n, r_n)$. Let $\bar{x}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_n$ be lifts of x, y_1, \dots, y_n to the universal cover \mathbb{H}^2 of (S, g) chosen so that some lift of $\exp(y_j, [0, \gamma_0])$ contains both \bar{x} and \bar{y}_j . Finally let D_i be the lift of c passing through y_i .

For $i \neq j$, D_i and D_j are disjoint. Indeed, since the segment $[\bar{x}, \bar{y}_i]$ is orthogonal to D_i , the lines D_i and D_j cannot coincide.

Up to changing the indices, we can suppose that there are half-planes P_1 and P_2 bounded by D_1 and D_2 respectively that do not meet any other leaf D_i . Up to exchanging D_1 and D_2 we can in particular suppose that $\bar{x} \notin P_1$. For $i \geq 2$ either D_i disconnects D_1 from \bar{x} or \bar{x} is contained in the region bounded by D_1 and D_i . In the latter case Sublemma A.2 can be applied since the distance of \bar{x} from D_1 and D_i is less than γ_0 . In both cases, the segment of length 1 starting from \bar{y}_1 and passing through \bar{x} — which is a lift of $\sigma_{y_1}^r$ — meets D_i . Since $\sigma_{y_1}^r$ meets c at most n_0 times (including y_1), we conclude that $n \leq n_0$.

Since the area of (S, g) is $2\pi|\chi(S)|$, it follows that

$$\gamma_0 l_g(c_i) \leq 2\pi n_0 |\chi(S)| \leq 2\pi \beta_0 l_g(c) |\chi(S)| .$$

The result clearly follows, with a choice of $\beta_0 = \gamma_0 / (4\pi|\chi(S)|)$. \square

APPENDIX B. FLAT CASE

Let $\mathbb{R}^{2,1}$ be the standard 3-dimensional Minkowski space, that is \mathbb{R}^3 equipped with the flat Lorentzian metric $dx_1^2 + dx_2^2 - dx_3^2$. The isometry group of $\mathbb{R}^{2,1}$ is the affine group of transformations whose linear part preserves the Minkowski product. Thus we have

$$Isom(\mathbb{R}^{2,1}) = \mathbb{R}^3 \rtimes O(2, 1)$$

where $O(2, 1)$ acts on \mathbb{R}^3 by multiplication.

In this section we will consider the hyperboloid model of \mathbb{H}^2 , that is the set of future-pointing unit time-like vectors in $\mathbb{R}^{2,1}$. Using this model, we identify the orientation-preserving isometry group of \mathbb{H}^2 with the connected component to the identity of $O(2, 1)$, which we denote by $SO^+(2, 1)$.

Finally we will identify the Teichmüller space of S with the set of conjugacy classes of faithful and discrete representations $h : \pi_1(S) \rightarrow SO^+(2, 1)$.

For any Fuchsian representation

$$h : \pi_1(S) \rightarrow SO^+(2, 1)$$

the cones $I^+(0)$ and $I^-(0)$ are invariant subsets of $\mathbb{R}^{2,1}$ and the quotients $I^+(0)/h$ and $I^-(0)/h$ are globally hyperbolic flat spacetimes homeomorphic to $S \times \mathbb{R}$.

Now consider an affine deformation of h , that is, a representation $\rho : \pi_1(S) \rightarrow \mathbb{R}^3 \rtimes SO^+(2, 1)$ which is of the form

$$\rho(\alpha) = h(\alpha) + \tau(\alpha)$$

where $\tau(\alpha) \in \mathbb{R}^3$ is the translation part. In [25], Mess showed that there are two maximal convex domains in $\mathbb{R}^{2,1}$ — say $\Omega^+(\rho)$ and $\Omega^-(\rho)$ — such that:

- They are invariant for the action of $\rho(\pi_1(S))$. The restriction of the action of $\rho(\pi_1(S))$ on them is free and properly discontinuous.

- $M^+(\rho) = \Omega^+(\rho)/\rho$ and $M^-(\rho) = \Omega^-(\rho)/\rho$ are globally hyperbolic spacetimes homeomorphic to $S \times \mathbb{R}$.
- $\Omega^+(\rho)$ is complete in the future: if a point x lies in $\Omega^+(\rho)$ then every future-directed timelike path starting at x is contained in $\Omega^+(\rho)$. Analogously $\Omega^-(\rho)$ is complete in the past.

Mess proved that all globally hyperbolic flat spacetimes homeomorphic to $S \times \mathbb{R}$ are contained in a spacetime of this form.

Proposition B.1. [25] *Let M be a time-oriented globally hyperbolic flat spacetime homeomorphic to $S \times \mathbb{R}$. There is an affine deformation of some Fuchsian representation*

$$\rho : \pi_1(S) \rightarrow \mathbb{R}^3 \rtimes SO^+(2, 1)$$

such that M isometrically embeds either in $M^+(\rho)$ or in $M^-(\rho)$.

We denote by \mathcal{H} the set of conjugacy classes of representations $\rho : \pi_1(S) \rightarrow \mathbb{R}^3 \rtimes SO^+(2, 1)$ whose linear part is faithful and discrete. The Fuchsian locus of \mathcal{H} — that we will denote by \mathcal{H}_0 — corresponds to representations that are conjugate to Fuchsian representations.

We have a projection map $\pi_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{T}$, sending ρ to its linear part. The fiber over $h \in \mathcal{T}$ consists of all the elements of the form

$$\rho = h + \tau$$

where $\tau : \pi_1(S) \rightarrow \mathbb{R}^3$ represents the translation part. Then ρ is a representation if and only if τ satisfies the cocycle condition

$$\tau(\alpha\beta) = \tau(\alpha) + h(\alpha)\tau(\beta).$$

Thus τ is an element of $Z_h^1(\pi_1(S), \mathbb{R}^3)$, which is the group of 1-cocycles of $\pi_1(S)$ with values in \mathbb{R}^3 , where the action of $\pi_1(S)$ on \mathbb{R}^3 is induced by h .

Two representations obtained by two cocycles τ, τ' are conjugate if and only if $\tau - \tau'$ is a coboundary. So the fiber of h is identified to the cohomology group $H_h^1(\pi_1(S), \mathbb{R}^3)$.

It turns out that the fibers of $\pi_{\mathcal{H}}$ have a natural structure of vector spaces. In fact, the map $\pi_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{T}$ is a vector bundle of rank $6g - 6$ (see [25]). Notice that \mathcal{H}_0 is the image of the zero section.

B.1. Laminations associated to an affine deformation. Given some $\rho = h + \tau \in \mathcal{H}$, we consider the distance of points in $\Omega^+(\rho)$ from the boundary, that is, the function defined by the formula

$$\tilde{t}(x) = \sup_{y \in \partial\Omega^+ \cap I^-(x)} \left(- \langle (x - y), (x - y) \rangle \right)^{1/2}.$$

This function is $C^{1,1}$ and its level sets are achronal [25, 10]. It induces a function t on $M^+(\rho)$ and we consider the level surface $S^+ = t^{-1}(1)$.

Notice that when ρ is a Fuchsian representation, then we simply have $S^+ = \mathbb{H}^2/\rho$. In the general case, Mess showed that S^+ is obtained by *grafting* the hyperbolic surface corresponding to the linear part of ρ along a measured geodesic lamination λ_+ . More precisely, the induced metric on S_+ is isometric to the “grafted metric” which, if λ_+ is a weighted multicurve, is obtained by replacing in the hyperbolic metric associated to ρ each leave of λ_+ by a flat cylinder of width equal to the weight. Through this section λ_+ is called the *upper lamination* of ρ .

In the same way, the surface $S^- \subset M^-(\rho)$ of points at distance 1 from the boundary is obtained by grafting the hyperbolic surface corresponding to the linear part of ρ along a measured geodesic lamination λ_- , which is called the *lower lamination* of ρ .

We consider the map

$$\Phi_0 : \mathcal{H} \rightarrow \mathcal{ML}_S^2$$

defined by $\Phi_0(\rho) = (\lambda_+, \lambda_-)$. Notice that if ρ is a Fuchsian representation then $\Phi_0(\rho) = (0, 0)$. In this section we study the map Φ_0 outside the Fuchsian locus. In particular we prove the following theorem.

Theorem B.2. *The map*

$$\Phi_0 : \mathcal{H} \setminus \mathcal{H}_0 \rightarrow \mathcal{ML}_S^2$$

is a homeomorphism onto \mathcal{FML}_S .

In [25], it is proved that given a measured geodesic λ and a Fuchsian representation h , there is a unique cocycle $\tau_+ \in H_h^1(\pi_1(S), \mathbb{R}^3)$ such that λ is the upper lamination of $h + \tau_+$. Analogously there is a unique cocycle $\tau_- \in H_h^1(\pi_1(S), \mathbb{R}^3)$ such that the lower lamination of $h + \tau_-$ is λ .

In this way, for a fixed λ , we determine two sections $\tau_+^\lambda, \tau_-^\lambda : \mathcal{T} \rightarrow \mathcal{H}$ by requiring that λ is the upper (resp. lower) lamination of $h + \tau_+^\lambda(h)$ (resp. $h + \tau_-^\lambda(h)$).

These sections are explicitly described in [25]. For the reader's convenience we describe the simple case where the lamination λ is a weighted multicurve, referring to [25, 5] for details. Given a Fuchsian representation $h : \pi_1(S) \rightarrow SO^+(2, 1)$, we realize λ as a weighted geodesic multicurve in \mathbb{H}^2/h . Let $\tilde{\lambda}$ be its lift to \mathbb{H}^2 . Now let us fix a point $x_0 \in \mathbb{H}^2 \setminus \tilde{\lambda}$. For any $\alpha \in \pi_1(S)$, we consider the intersection points — say p_1, \dots, p_N — of the geodesic segment $[x_0, h(\alpha)(x_0)]$ with $\tilde{\lambda}$. Then we define the following vector in \mathbb{R}^3 :

$$(16) \quad \tau(\alpha) = \sum_{i=1}^N m_i w_i,$$

where m_i is the weight of the leaf through p_i and $w_i \in \mathbb{R}^3$ is the unit vector orthogonal to the leaf through p_i pointing towards $\alpha(x_0)$. It turns out that τ is a cocycle and (up to a coboundary) we have

$$(17) \quad \tau_+^\lambda(h)(\alpha) = \tau \quad \tau_-^\lambda(h)(\alpha) = -\tau.$$

Notice that if $\Phi_0(h + \tau) = (\lambda_+, \lambda_-)$ then we clearly have $\tau = \tau_+^{\lambda_+}(h) = \tau_-^{\lambda_-}(h)$. In particular, the sections $\tau_+^{\lambda_+}$ and $\tau_-^{\lambda_-}$ meet over h . Conversely if for some given $\lambda, \mu \in \mathcal{ML}_S$ there is $h \in \mathcal{T}$ such that $\tau_+^\lambda(h) = \tau_-^\mu(h)$ then $\Phi(h + \tau^\lambda(h)) = (\lambda, \mu)$. Thus there is a 1-to-1 correspondence between $\Phi_0^{-1}(\lambda, \mu)$ and the intersection of τ_+^λ and τ_-^μ .

The following proposition combined with Proposition 3.2 shows that τ_+^λ and τ_-^μ are disjoint if λ and μ does not fill, whereas they intersect at exactly one point otherwise. This shows that the image of Φ_0 is \mathcal{FML}_S and that Φ_0 is injective.

Proposition B.3. *There is a vector bundle isomorphism*

$$\xi_* : \mathcal{H} \rightarrow T\mathcal{T}_S$$

such that $\xi_* \circ \tau_+^\lambda = e_r^\lambda$ and $\xi_* \circ \tau_-^\lambda = e_l^\lambda$.

Proof. We consider on \mathbb{R}^3 the Minkowski vector product: given $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ it is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, -x_1 y_2 + x_2 y_1).$$

We refer to [14] for the details. Given $x \in \mathbb{R}^3$, the linear operator $\xi(x)$ defined by $\xi(x)(y) = x \times y$ is skew-symmetric with respect to the Minkowski product $\langle \cdot, \cdot \rangle$, so it lies in the Lie algebra $\mathfrak{o}(2, 1)$ of $SO^+(2, 1)$. The induced map $\xi : \mathbb{R}^3 \rightarrow \mathfrak{o}(2, 1)$ is an isomorphism. Moreover we have

$$\xi(Ax) = Ad(A)\xi(x).$$

Given $h \in \mathcal{T}$ and an element $\tau \in Z_h^1(\pi_1(S), \mathbb{R}^3)$ we have that $\tau^* = \xi \circ \tau : \pi_1(S) \rightarrow \mathfrak{o}(2, 1)$ satisfies the cocycle rule

$$\tau^*(\alpha\beta) = \tau^*(\alpha) + Ad(h(\alpha))\tau^*(\beta).$$

In particular, τ^* represents an infinitesimal deformation of the representation h . Moreover τ^* is a trivial deformation if and only if τ is a coboundary. Using the canonical identification between $T_h\mathcal{T}$ and $H_{Ad \circ h}^1(\pi_1(S), \mathfrak{o}(2, 1))$ (see [18]) we obtain an isomorphism

$$\xi_{*,h} : H_h^1(\pi_1(S), \mathbb{R}^3) \rightarrow T_h\mathcal{T}$$

defined by $\xi_{*,h}([\tau]) = [\tau^*]$.

Since $H_h^1(\pi_1(S), \mathbb{R}^3)$ is the fiber of the projection $\pi_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{T}$ over h , the maps $\xi_{*,h}$ produce a fiber bundle isomorphism

$$\xi_* : \mathcal{H} \rightarrow T\mathcal{T}.$$

To conclude the proof, we have to prove that $\xi_* \circ \tau_+^\lambda(h) = e_r^\lambda(h)$ and $\xi_* \circ \tau_-^\lambda(h) = e_l^\lambda(h)$. Since $\tau_-^\lambda = -\tau_+^\lambda$ and $e_l^\lambda = -e_r^\lambda$ it is sufficient to prove only the first equality.

We will prove that the equality holds when λ is a weighted curve. The general case follows by a simple approximation argument.

We denote by $\tilde{\lambda}$ the measured geodesic lamination on \mathbb{H}^2 that projects on the geodesic lamination on \mathbb{H}^2/h that realizes λ . We fix a point $x_0 \in \mathbb{H}^2 \setminus \tilde{\lambda}$. Given $\alpha \in \pi_1(S)$, let l_1, \dots, l_N be the leaves of $\tilde{\lambda}$ meeting the segment $[x_0, \alpha(x_0)]$ at points x_1, \dots, x_N respectively. We denote by m_i the weight of l_i and by $w_i \in \mathbb{R}^3$ the unit spacelike vector orthogonal to l_i and pointing towards $\alpha(x_0)$.

It can be easily checked that the transformation $\xi(w_i)$ is an infinitesimal generator of the hyperbolic group of transformations with axis l_i . More precisely, for $m > 0$, a simple computation shows that $\exp(m\xi(w_i))$ is the hyperbolic transformation of axis l_i and translation length equal to m moving $\alpha(x_0)$ to the right of x_0 .

Let us put $h_t = E_r^{t\lambda}(h)$. It follows from computations in [16] that

$$h_t(\alpha) = \exp(tm_1\xi(w_1)) \circ \cdots \circ \exp(tm_N\xi(w_N))h(\gamma).$$

Taking the derivative at $t = 0$,

$$\frac{dh_t(\gamma)}{dt}\bigg|_0 = (R_{h(\gamma)})_* \left(\sum m_i \xi(w_i) \right).$$

So the cocycle representing the derivative at 0 of such a deformation (that is, the cocycle corresponding to $e_r^\mu(h)$ via the identification of $T_h\mathcal{T}$ with $H_{\text{Adp}}^1(\pi_1(S), \mathbb{R}^3)$) is

$$e_r^\lambda(h)(\alpha) = \sum m_i \xi(w_i) = \xi \left(\sum m_i w_i \right).$$

Comparing this formula with Equations (16) and (17) we deduce that $e_r^\lambda(h) = \xi_*(\tau_-^\lambda(h))$. \square

REFERENCES

1. Lars Ahlfors and Lipman Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404. MR MR0115006 (22 #5813)
2. Lars Andersson, Thierry Barbot, Riccardo Benedetti, Francesco Bonsante, William M. Goldman, François Labourie, Kevin P. Scannell, and Jean-Marc Schlenker, *Notes on: "Lorentz spacetimes of constant curvature"* [Geom. Dedicata **126** (2007), 3–45; mr2328921] by G. Mess, Geom. Dedicata **126** (2007), 47–70. MR MR2328922
3. André Avez, *Formule de Gauss-Bonnet-Chern en métrique de signature quelconque.*, C. R. Acad. Sci. Paris **255** (1962), 2049–2051. MR MR0145462 (26 #2993)
4. John K. Beem, Paul E. Ehrlich, and Kevin L. Easley, *Global Lorentzian geometry*, second ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 202, Marcel Dekker Inc., New York, 1996. MR MR1384756 (97f:53100)
5. Riccardo Benedetti and Francesco Bonsante, *Canonical Wick rotations in 3-dimensional gravity*, Memoirs of the American Mathematical Society **198** (2009), 164pp, math.DG/0508485.
6. Lipman Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), 94–97. MR MR0111834 (22 #2694)
7. Francis Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. (6) **5** (1996), no. 2, 233–297.
8. ———, *Kleinian groups which are almost Fuchsian*, J. Reine Angew. Math. **587** (2005), 1–15. MR MR2186972
9. Francis Bonahon and Jean-Pierre Otal, *Laminations mesurées de plissage des variétés hyperboliques de dimension 3*, Ann. Math. **160** (2004), 1013–1055.
10. Francesco Bonsante, *Flat spacetimes with compact hyperbolic Cauchy surfaces*, J. Differential Geom. **69** (2005), no. 3, 441–521. MR MR2170277 (2006h:53068)
11. Francesco Bonsante and Jean-Marc Schlenker, *AdS manifolds with particles and earthquakes on singular surfaces*, Geom. Funct. Anal. **19** (2009), no. 1, 41–82. MR MR2507219
12. Francesco Bonsante and Jean-Marc Schlenker, *Maximal surfaces and the universal Teichmüller space*, arXiv:0911.4124. To appear, *Inventiones Mathematicae*, 2009.
13. Shiing-shen Chern, *Pseudo-Riemannian geometry and the Gauss-Bonnet formula*, An. Acad. Brasil. Ci. **35** (1963), 17–26. MR MR0155261 (27 #5196)
14. Todd A. Drumm and William M. Goldman, *Complete flat Lorentz 3-manifolds with free fundamental group*, Internat. J. Math. **1** (1990), no. 2, 149–161. MR MR1060633 (91f:57022)
15. Emily B. Dryden and Hugo Parlier, *Collars and partitions of hyperbolic cone-surfaces*, Geom. Dedicata **127** (2007), 139–149. MR MR2338522
16. D. B. A. Epstein and A. Marden, *Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces*, Analytical and geometric aspects of hyperbolic space (D. B. A. Epstein, ed.), L.M.S. Lecture Note Series, vol. 111, Cambridge University Press, 1986.
17. Robert Geroch, *Domain of dependence*, J. Mathematical Phys. **11** (1970), 437–449. MR MR0270697 (42 #5585)
18. W. M. Goldman, *Geometric structures on manifolds and varieties of representations*, Contemporary Math. **74** (1988), 168–198.
19. S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, London, 1973, Cambridge Monographs on Mathematical Physics, No. 1. MR MR0424186 (54 #12154)
20. Morris W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original. MR MR1336822 (96c:57001)
21. Craig D. Hodgson and Steven P. Kerckhoff, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Differential Geom. **48** (1998), 1–60.
22. Steven P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), no. 2, 235–265. MR MR690845 (85e:32029)
23. ———, *Lines of minima in Teichmüller space*, Duke Math. J. **65** (1992), no. 2, 187–213. MR MR1150583 (93b:32027)
24. F. Labourie, *Métriques prescrites sur le bord des variétés hyperboliques de dimension 3*, J. Differential Geom. **35** (1992), 609–626.
25. Geoffrey Mess, *Lorentz spacetimes of constant curvature*, Geom. Dedicata **126** (2007), 3–45. MR MR2328921
26. Igor Rivin and Craig D. Hodgson, *A characterization of compact convex polyhedra in hyperbolic 3-space*, Invent. Math. **111** (1993), 77–111.
27. Jean-Marc Schlenker, *Convex polyhedra in Lorentzian space-forms*, Asian J. Math. **5** (2001), no. 2, 327–363. MR MR1868937 (2003f:53127)
28. Caroline Series, *On Kerckhoff minima and pleating loci for quasi-Fuchsian groups*, Geom. Dedicata **88** (2001), no. 1–3, 211–237. MR MR1877217 (2002j:30066)

- 29. ———, *Limits of quasi-Fuchsian groups with small bending*, Duke Math. J. **128** (2005), no. 2, 285–329. MR MR2140265 (2006a:30043)
- 30. William P. Thurston, *Three-dimensional geometry and topology.*, Originally notes of lectures at Princeton University, 1979. Recent version available on <http://www.msri.org/publications/books/gt3m/>, 1980.
- 31. ———, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431. MR MR956596 (89k:57023)
- 32. Marc Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. **324** (1991), no. 2, 793–821.
- 33. Scott A. Wolpert, *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math. (2) **117** (1983), no. 2, 207–234. MR MR690844 (85e:32028)

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