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Procedia
Social and Behavioral Sciences

Procedia - Social and Behavioral Sciences 80 (2013) 860 – 879

20th International Symposium on Transportation and Traffic Theory (ISTTT 2013)

Equilibrium in capacitated network models with queueing delays, queue-storage, blocking back and control

Mike Smith[a], Wei Huang[b] and Francesco Viti[b,c]

[a]Departement of Mathematics, University of York, United Kingdom

[b]Katholieke Universiteit Leuven, Belgium

[c]University of Luxembourg, Luxembourg

Abstract

This paper considers a steady-state, link-based, fixed (or inelastic) demand equilibrium model with explicit link-exit capacities, explicit bottleneck or queuing delays and explicit bounds on queue storage capacities. The spatial queueing link model at the heart of this equilibrium model takes account of the space taken up by queues both when there is no blocking back and also when there is blocking back. The paper shows in theorem 1 that a feasible traffic assignment model has an equilibrium solution provided prices are used to impose capacity restrictions and utilises this result to show that there is an equilibrium with the spatial queueing model, provided queue-storage capacities are sufficiently large. Other results are obtained by changing the variables and sets in theorem 1 suitably. These results include: (1) existence of equilibrium results (in both a steady state and a dynamic context) which allow signal green-times to respond to prices and (2) an existence of equilibrium result which allow signal green-times to respond to spatial queues; provided each of these responses follows the P_0 control policy described in Smith (1979a, 1987). These results show that under certain conditions the P_0 control policy maximises network capacity. The operation of the spatial queueing link model is illustrated on a simple network. Finally the paper includes elastic demand; this is necessary for long-run evaluations. Each of the steady state equilibria whose existence is shown here may be thought of as a stationary solution to the dynamic assignment problem either with or without blocking back; they are quasi-dynamic equilibria.

Keywords: Link-based traffic assignment, Capacity-constrained equilibrium, Elastic demand, Queueing, Blocking back.

1. Introduction

Equilibrium models dealing with congested networks require an explicit consideration of road capacities and queueing. These factors give rise to complex patterns of interacting demands, flows and queues (and prices if these are considered); and means that modelling (i) the spatial extent of queues within links and (ii) blocking back propagation of these queues from link to link are both of substantial importance. These phenomena are not well-represented within the simple link performance functions currently often utilised in steady state equilibrium models. On the other hand the current simple steady state cost / delay formulations do presently guarantee the existence, uniqueness and stability of equilibrium; and these properties are, for a number of reasons, good to have.

Attempts to relax some of the current modelling simplifications have been made by, e.g., Thompson and Payne (1975) and Smith (1987), who formulated a rigid demand or inelastic traffic equilibrium model with capacity constraints and considered queueing delays as independent variables (*not given by a cost-flow function*). However queue storage capacities and queue spillback were not addressed; the models created were point-queue models.

1877-0428 © 2013 The Authors. Published by Elsevier Ltd.
Selection and peer-review under responsibility of Delft University of Technology
doi:10.1016/j.sbspro.2013.05.047

This paper proposes an explicit steady state model of queueing which takes account of the space taken up by traffic queues and the effects of spillback (when queue storage capacities are exceeded). It is shown how this spatial queueing model fits within an equilibrium model very similar to that in Smith (1987); under certain conditions the model may be certainly solved by using a specific equilibration algorithm called algorithm (D) in Smith (1984a, b), however we do not give the details here.

A quasi-dynamic equilibrium model with realistic link performance functions that incorporate queues and blocking back effects is, at least at first sight, very appealing for helping to deal with short-term ITS strategies aimed at ameliorating both systematic and non-systematic incidents. For example, to deal with a non-systematic incident (such as a suddenly blocked link), the model (with that link blocked) might be run to yield, very quickly, estimates of a corresponding changed future state of the network on which routeing advice could be provided to road users. (Further developments would seem possible in this direction; we do not explore these here.)

Furthermore a transport model that enables planners to make both short-term assessments (using small or zero elasticity) and long run assessments (using a larger elasticity) in large congested networks is currently lacking. This study also aims to at least partially fill this gap, by utilising the new link performance function (which incorporates queuing and blocking back), and also retaining the option of modelling both elastic and fixed or inelastic demand. Long-run assessments require elastic demand equilibrium models. Such models have been much studied; see for example Beckmann et. al. (1956), Smith (1984c, 2009, 2010), Patriksson (1994) and Bar-Gera and Boyce (2006). Short run assessments require a much smaller or even zero elasticity. (See Goodwin (1992), Victoria Transport Institute and, for a review, Graham and Glaister (2004).)

1.1. A short steady state equilibrium modelling context

The route-choice principle adopted throughout this paper is the familiar Wardrop (1952) notion: for each origin-destination pair, more costly routes are not used. This Wardrop equilibrium traffic assignment principle may also be expressed using link flows and costs (as in for example Smith (1979b) and more recently He et al (2010)). *In this paper we utilise a link flow formulation of "Wardrop equilibrium"; this formulation is given in detail later.*

All the equilibrium models described in this paper utilise link flows, explicit link-exit capacities, explicit queues, and either elastic or inelastic demand. The Thompson-Payne (1975) queueing model is utilised to represent queues in steady state assignment models. The whole equilibrium model may then be expressed as a variational inequality. A multi-level framework (with one base network and several upper networks) building upon that suggested by Charnes and Cooper (1961) and Aashtiani and Magnanti (1983) is utilised. The base level network represents the real road network and each upper level contains flow to just a single destination. In the last part of the paper the model is expressed as a complementarity problem. (See Karamardian (1971).)

First we suppose that the queueing capacity of all links is very large so that blocking back does not occur and then we include storage capacities and blocking back in the model. The model in this paper may be thought of as being more or less exactly between the standard type of steady state equilibrium model and the standard dynamic equilibrium model. For example, steady state models usually have no explicit queues, whereas this model does and dynamic models usually have a demand which varies with time, which this model does not. The difficultly in guaranteeing existence, uniqueness, convergence and stability conditions, while relaxing the simplifying assumption of point queues makes these hybrid modelling approaches of great interest and importance in traffic management applications on congested networks. One may refer to Bliemer et al. (2012) to find arguments in support of the need for more realistic traffic propagation models in steady-state traffic assignment models.

The inelastic demand version of the model may be utilised to compare and generate short-run strategies (including ITS sstrategies such as switching signal plans in response to a capacity-reducing incident); and also (with more demand elasticity) to compare different proposed long term strategies (such as increasing the capacities of certain links). Thus the short and the long run models, differing mainly in just the elasticities utilised, would be likely to have more consistent outputs than is now sometimes the case. The elastic model uses the demand function directly, and not its inverse. It is this which allows the model described here to be applied to the same network with both inelastic and elastic demand.

1.2. Capacity constraints and blocking back in steady state models

Most equilibrium models currently do not represent queueing delays at bottlenecks. Payne and Thompson (1975) added queueing delays to an equilibrium model; many have followed: Smith (1984a,b; 1987), Larsson and Patriksson (1995), Smith et al. (1996), Logie et. al. (1998), Lam and Zhang (2000), Marcotte et al. (2003), Nie et.al. (2004) and Gentile et. al. (2005).

Examples of capacity-constrained equilibrium models with queuing that explicitly consider the impact of blocking-back are very much less numerous. Most models incorporating blocking-back have focussed on the impact on the dynamic network loading rather than investigating the effects on route choice (Daganzo (1995a, b), Yperman (2006), Bliemer (2007)). Notable exceptions are in Daganzo (1998), Gentile et al (2007) and Bliemer et. al. (2012) who have considered various issues which arise when blocking back occurs with route choice and have shown how important blocking back is. However as Bliemer et.

al. (2012) point out, there remains the need to integrate blocking back modelling and steady state traffic assignment modelling.

Thus, despite the above work, what is clearly still absent is a sound comprehensive modelling methodology for dealing with capacity constraints, queueing delays and blocking back within a route choice model, ideally with guaranteed convergence; and ideally retaining elastic demand as well as fixed or inelastic demand, so that short run and long run assessments may be performed by essentially the same model, with a great gain in consistency. One of the contributions of this paper is to make progress towards this unifying approach.

Certain of the steady state models outlined here may be thought of as quasi-dynamic models: spatial queues are represented but there is only one "time slice" or time period. These models may also be thought of as seeking to model the peak of a peak period where queues are at their maximum, without modelling the building up or the decay of those queues. The idea being that options which deal best with the worst queues are likely to be the best or close to the best if assessed over a complete peak. The idea is also that the model should run quickly so that many options may be tested within a given time/cost budget. The model may be extended, as indicated later in the paper, by including the P_0 control policy (see Smith, 1979a)) and so the model may also be used to provide signal timing suggestions suitable for different scenarios.

The first (pricing) model in this paper is a link-based steady state inelastic equilibrium model with queuing delays and spatial queueing but no blocking back. Being link-based, the model should be computationally fast. Then we consider a dynamic result involving prices and green-times, extend the model to take account of blocking back, and finally we introduce elastic demand following Smith (2013).

1.3. A dynamic state modelling context

The literature on dynamic network models can be divided into studies that focus mainly on developing dynamic network loading models, and studies that take special care with the (route and other) choice updating process. In this section we focus on the former models.

Following the conventional representation of the network into links and nodes (with the notable exception of the cell-based representation of Daganzo, 1995a, b), dynamic link and node models have been developed and used in traffic assignment processes.

In link models, often referred to as whole-link models, the propagation of flows onto a link is described by relationships between variables such as travel times, number of vehicles, inflow and outflow rates etc. See for example Carey and McCartney, 2002. These whole link models typically do not usually keep track of the physical extension of the queues (Viti and Tampere, 2010).

The development of dynamic node models is more recent. These models have been proposed to keep track of the dynamic changes of node outflows as node inflows change and as downstream conditions change, allowing for blocking back (see e.g, Flötteröd and Nagel, 2005; Flötteröd, 2008; Gentile, 2010; Gibb, 2011; for an extensive review see Tampère et al., 2011) and (for a discussion of extra constraints due to intersection priorities) Flötteröd and Rohde, 2011; Corthout et al. 2012). To the best of our knowledge these models have not yet been integrated into a traffic assignment model with a convergence guarantee.

Nesterov and de Palma (2003) consider stationary solutions to the dynamic assignment problem and suggest solution algorithms. This paper fits very well with that viewpoint; adding a more realistic link model. This is the quasi-dynamic viewpoint.

2. A basic fixed demand capacitated traffic assignment model with queues or prices but no blocking back

This model follows Thompson and Payne (1975), extends Smith (1987, 2012) and is motivated by many papers including especially Daganzo (1998).

2.1. A basic link model with a spatial representation of queues (but without blocking back)

As is usual in traffic modelling, each real-life traffic lane is here represented by

1. a node which represents the entry point of the lane,
2. a node which represents the exit point or the stop line of the lane, and
3. a directed link joining these two nodes which represents the stretch of lane between the entry and the exit of the lane.

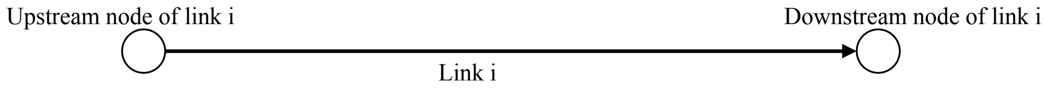

Figure 1. A single link in model 1 representing a single real life traffic lane.

A representation of link i is shown in figure 1. Representations of two lanes are connected in the model by short links when traffic may, in reality, pass from one to the other and are otherwise unconnected. These additional short links represent all possible movements at junctions, from one lane to another, and junctions are thus represented in an "expanded" form.

The flow along link i is to be v_i vehicles per minute; the saturation flow at the exit of link i is s_i vehicles per minute; the queue at the exit of link i is Q_i vehicles; the maximum possible value of Q_i is $MAXQ_i$ and the time to traverse the entire length of link i (when the queue $Q_i = 0$ and the flow is v_i) is $c_i(v_i)$. The link i "state" may be thought of as (v_i, Q_i) and this link state 2-vector is to be confined to a set of supply-feasible pairs (v_i, Q_i), as follows:

$$v_i \leq s_i \text{ and } Q_i \leq MAXQ_i.$$

Throughout this section the link cost function $c_i(.)$ is to be positive, continuous, non-decreasing and defined for all non-negative $v_i \leq s_i$. $c_i(v_i)$ constant is allowed.

Consider a link i with a feasible flow v_i and a feasible queue Q_i and no congestion on any downstream link; so that the saturation flow s_i at the link i exit is the only constraint on the link flow v_i. To calculate the queueing delay b_i (minutes per vehicle) at the link i exit it has often been proposed (see Thompson and Payne (1975) and Smith (1987) for example) that:

$$b_i = Q_i / s_i. \tag{1}$$

Then the whole time of travel along link i has often been written (see for example Thompson and Payne (1975) and Smith (1987)):

$$c_i(v_i) + b_i. \tag{2}$$

Here we are considering the steady state; however formula (1) is now also very common in the literature on DTA (appearing then as: $b_i(t) = Q_i(t) / s_i$ for all times t).

Returning to the steady state, suppose now that $Q_i = 0$. In this special case, where there is no queueing and so $b_i = 0$, formula (2) is entirely reasonable. Consider now the case where $0 < Q_i < MAXQ_i$. In this case, in reality, the queue on link i covers part of the length of link i and only the remainder has to be traversed (with no queue). The non-zero queue will take up space on link i and so formula (2) in general overestimates the travel time for a link; the overestimation is greater the larger the queue is. In real life there will be a non-queueing time incurred in traversing the part of link i which is not "queued", as well as the queueing time. The time for traversing link i will thus be the sum SUM_i of a non-queueing component and a queueing component. (We suppose that the head of the queue on link i is always adjacent to the link exit.)

We suppose here that the total travel time SUM_i spent in traversing link i depends on v_i and b_i. To reduce or eliminate the double-counting described above, here we choose to specify the sum SUM_i of the queueing and non-queueing travel-time components as follows:

$$\text{the total time of traversing link i} = SUM_i(v_i, b_i) = h_i c_i(v_i) + b_i. \tag{3}$$

The "shrinkage factors" h_i (≤ 1) here are to be chosen so as to take some careful but simple account of the fact that as the queueing delay grows the unqueued link length (and unqueued travel time) shrinks.

The h_i may here naturally be estimated by considering various values of Q_i when $v_i = s_i$ in (3), since in this section (where there is by assumption no blocking back) there is no queueing and no queueing delay when $v_i < s_i$, and in this case $h_i = 1$. Consider now the cases where $v_i = s_i$, then (3) becomes:

$$\text{the total time of traversing link i} = SUM_i(s_i, b_i) = h_i c_i(s_i) + b_i. \tag{4}$$

It is now natural to put

$$h_i = [MAXQ_i - Q_i] / MAXQ_i.$$

This is certainly correct if $Q_i = MAXQ_i$ for then $h_i = 0$ and all the time spent traversing link i (in (4)) becomes entirely queuing time b_i (which is as it should be) and is also correct if $Q_i = 0$ for then $h_i = 1$ and all the time spent traversing link i (in (4)) becomes entirely running time $c_i(s_i)$ (which again is as it should be).

Substituting $h_i = [MAXQ_i - Q_i] / MAXQ_i$ into (4) we obtain

$$\begin{aligned} SUM_i(s_i, b_i) &= [(MAXQ_i - Q_i) / MAXQ_i]c_i(s_i) + b_i \\ &= c_i(s_i) - [Q_i / MAXQ_i]c_i(s_i) + b_i \\ &= c_i(s_i) - [Q_i / (s_i MAXQ_i)]s_i c_i(s_i) + b_i \\ &= c_i(s_i) + [1 - s_i c_i(s_i)/MAXQ_i]b_i \end{aligned}$$

since $Q_i / s_i = b_i$. Thus this value of h_i in (4) leads to the following equation:

$$\text{the total time of traversing link } i = SUM_i(s_i, b_i) = c_i(s_i) + k_i b_i \tag{5a}$$

where

$$k_i = 1 - s_i c_i(s_i)/MAXQ_i. \tag{5b}$$

This shrinkage factor k_i in (5a) given in (5b) is independent of Q_i; unlike h_i. (This in turn helps in various ways with the design of solution algorithms.) In fact we may naturally generalise (5a) so that the formula applies also when the link is not saturated (as in this case, without blocking back, $v_i < s_i \Rightarrow b_i = 0$; this generalisation is:

$$\text{the total time of traversing link } i = SUM_i(v_i, b_i) = c_i(v_i) + k_i b_i \tag{5c}$$

A feasibility constraint arises here from the form and sense of (5a) and (5b). This is:

$$[1 - s_i c_i(s_i)/MAXQ_i] > 0 \text{ or } c_i(s_i) < MAXQ_i / s_i. \tag{6}$$

This condition (6) ensures that $k_i > 0$, which seems essential and this condition (6) is also itself reasonable: it says that the time taken to traverse link i when there is no queue is no larger than the maximum queueing delay which can be supported by link i.

2.2. The Upper Multi-level Network and Connection to the Base Network

The upper network comprises several levels; one level for each (destination, vehicle type) pair. Each level comprises a subnetwork of the single base network; for each level, each node corresponds to a node in the base network and each link corresponds to a link in the base network. Flow on each level of the multilevel network must have the same destination and be of vehicles of the same type. Base network links which are prohibited for certain destinations (or certain vehicle types) are omitted from the relevant upper level network.

For each base network link, flows on the corresponding upper multi-level network links add up to give the (total) link flow on the base network link. Also here we assume that the time needed to traverse each upper network link is easily derived from the time of travel along the corresponding base network link. (1) Link flows on the upper network (these will be called commodity flows) add to give link flows on the base network and (2) link travel times (including bottleneck delays) on the base network are felt on corresponding links in each level of the upper network. Flows of vehicles of different sizes are modelled here by assuming that all flows are in pcu terms. 1 pcu will be called "a vehicle".

The multi-level network structure is summarised by the matrices **L**, **A** and **B**. For a given network the entries $L_{i\,r}$, $A_{n\,r}$, and $B_{n\,r}$ in the matrices **L**, **A** and **B** are defined as follows.

$L_{i\,r} = 1$ if upper level link r corresponds to base link i and 0 otherwise;

$A_{n\,r} = 1$ if node n is the exit node of link r (n after r) *and not a destination*, and 0 otherwise; and

$B_{n\,r} = 1$ if node n is the entrance node of link r (n before r) and 0 otherwise.

NOTE: If n *is a destination node then (for later convenience) we have defined* $A_{n\,r} = 0$ *for all* r.

Main variables

X_r = flow along upper network link r;

v_i = flow along base network link i;

b_i = bottleneck or queueing delay at the bottleneck at the exit of base link i; and

Q_i = queue volume on base link i.

The queueing delays b_i are independent variables; *they are not given by a cost flow function but are instead determined by the equilibrium or fixed point conditions.*

Destination nodes

By definition, if n is a destination node, $A_{n\,r} = 0$ for all r.

2.2.1. Supply side conditions and supply feasibility

Suppose that the network has m base links. Recall that $c_i(v_i)$ is defined for all v_i such that $0 \leq v_i \leq s_i$. Then the vector

$$\mathbf{c(v)} = [c_1(v_1), c_2(v_2), \ldots, c_i(v_i), \ldots c_m(v_m)]$$

is defined for all **v** belonging to the set [**0**, **s**] where

$$[\mathbf{0}, \mathbf{s}] = \{\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots v_m];\ 0 \leq v_i \leq s_i \text{ for } 1 \leq i \leq m\}.$$

We define

$$S_i = \{v_i;\ v_i \leq s_i\} \tag{7a}$$

and

$$S = \{\mathbf{v} = [v_1, v_2, \ldots, v_i, \ldots v_m];\ v_i \leq s_i \text{ for } 1 \leq i \leq m\} = \{\mathbf{v};\ \mathbf{v} \leq \mathbf{s}\}. \tag{7b}$$

The set of supply-feasible base-link flows is the set of non-negative vectors belonging to S. (The lower bounds of zero are omitted here for technical reasons; they are not actually needed as non-negativity is assured by the specification of the demand feasible set D.)

Commodity flows (here these are pcu flows to specific destinations along links in the multi-level network) will be represented by upper-level-link flows X_r; here X_r is defined to be the (pcu) flow along link r in the upper network and *must be non-negative*. The column vector of all the X_r yields the vector **X**; this is the column vector of all upper-level-link flows, so that the r^{th} co-ordinate of **X** is X_r (the (pcu) flow along upper level link r). Base link flow vector **v** arises from the upper link flow vector **X** by:

$$v_i = \Sigma_r L_{i\,r} X_r \text{ for } 1 \leq i \leq m \quad \text{or} \quad \mathbf{v = LX}.$$

The above definition of supply-feasible base-link flow vectors now leads to the following definition of the set of supply-feasible upper-link flow vectors:

$$SU = \{\mathbf{X};\ \Sigma_r L_{ir} X_r \leq s_i \text{ for all } i\} = \{\mathbf{X};\ \mathbf{LX} \leq \mathbf{s}\}.$$

The $c_i(v_i)$ in turn yield, for $\mathbf{X} \in SU$ and $\mathbf{X} \geq \mathbf{0}$, the upper level link traversal times $C_r(\mathbf{X})$:

$$C_r(\mathbf{X}) = \Sigma_i L^T_{r\,i}\, c_i(v_i) \quad \text{or } \mathbf{C(X) = L^T c(v)}$$

(In the first of these equations, for each r there is just one term in the sum over i.) Plainly **C(X)** satisfies:

S1. **C(X)** is a continuous function of **X**, defined for all $\mathbf{X} \in SU$ and $\mathbf{X} \geq \mathbf{0}$;

S2. $\mathbf{C}(\mathbf{X}) > \mathbf{0}$ (since we are assuming that $\mathbf{c(v) > 0}$);

S3. **C(X)** is a monotone function of **X** defined for all $\mathbf{X} \in SU$ and $\mathbf{X} \geq \mathbf{0}$.

Travellers also feel bottleneck delays b_i as discussed above. So here:

$$SUM_r(\mathbf{X}, \mathbf{b}) = \Sigma_i L^T_{r\,i}\,[c_i(v_i) + k_i b_i] = \Sigma_i L^T_{r\,i}\,\{c_i(v_i) + [1 - s_i(c_i(s_i))/MAXQ_i] b_i\}$$

for $\mathbf{X} \in SU$ and $\mathbf{X} \geq \mathbf{0}$.

2.2.2. Demand feasibility

We suppose initially here that for each upper level non-destination node n there is a given positive number W_n representing traffic originating at n; and that the set of demand feasible flow vectors **X** satisfies:

$\mathbf{X} \geq \mathbf{0}$ *and for each upper level non-destination node* n *the total flow along all links leaving node* n *equals the total flow into node* n *plus the fixed flow* W_n *generated at node* n.

This may be written:

$$\mathbf{X} \geq \mathbf{0} \text{ and for each upper level non-destination node n: } W_n + \Sigma_r A_{n\,r} X_r - \Sigma_r B_{n\,r} X_r = 0. \tag{8a}$$

The set of vectors **X** satisfying this condition (8a) is closed and convex. The (closed and convex) demand-feasible set D of base link flows is then (provisionally):

$$D(\text{prov}) = \{\mathbf{v = LX};\ \mathbf{X \geq 0} \text{ and } W_n + \Sigma_r A_{n\,r}X_r - \Sigma_r B_{n\,r}X_r = 0 \text{ for each upper level non-destination node n}\}. \quad (8b)$$

This allows loops and so is an unbounded set; so we add the condition that **X** is loop free (or cannot be reduced and still satisfy (8a)); obtaining:

$$D = \{\mathbf{v = LX};\ \mathbf{X \geq 0, X} \text{ is loop-free and } W_n + \Sigma_r A_{n\,r}X_r - \Sigma_r B_{n\,r}X_r = 0 \text{ for each upper level non-destination node n}\}. \quad (8c)$$

Then D is closed, convex and bounded.

2.2.3. Feasibility

Here the set of feasible base link flows is then defined to be D∩S where D is given by (8c) and S is given by (7). Throughout this paper all feasible variables will be non-negative. Thus:

$X_r \geq 0$ for all r, $b_i \geq 0$, $v_i \geq 0$ and $Q_i \geq 0$ for all i.

2.3. Definition of a Wardrop / queueing equilibrium (*without blocking back*)

In this section we suppose there is no blocking back. We will use the base network to define "Wardrop equilibrium". This is so as to involve queues (and, later, blocking back) in as simple a way as possible.

Two pieces of notation.

1. If **k** and **b** are m-vectors then **k•b** is the m-vector whose i^{th} co-ordinate is k_ib_i.
2. The vector **u** is normal at **v** to D if and only if $\mathbf{u \cdot (w\text{-}v)} \leq 0$ for all $\mathbf{w} \in$ D. (This condition is illustrated in figure 2.)

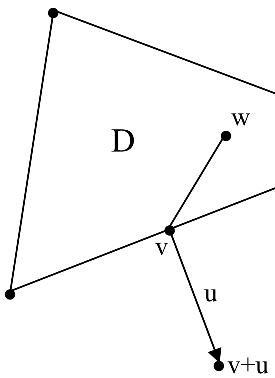

Figure 2. This diagram illustrates a point **v** *in the convex set D (the triangle) and a vector* **u** *normal at* **v** *to the set D. This means that, for all* **w** *in D,* **u · (w-v)** *≤ 0; as illustrated for a particular* **w** *in D.*

All the models discussed here use the standard Wardrop (1952) notion that for each OD pair more costly routes are unused. This user-equilibrium condition, using the (base level flow vector, base level bottleneck delay vector) pair (**v**, **b**), may be written using the normality condition illustrated above, following Smith (1979b, 1987) but using link performance function (5), as follows:

v belongs to D∩S and
- (c(v) + k•b) is normal at **v** to D. (9)

The top line of (9) says that the vector of link flows **v** is both demand and supply feasible; the second line is illustrated in figure 2, where in this case **u = - (c(v) + k•b)**.

Clearly not all queueing delay vectors **b** can realistically arise; it is reasonable to impose the condition that if a link is unsaturated then there is no queueing delay (in a steady state model). To restrict attention to only feasible bottleneck delay vectors we impose the following condition which may be thought of as a "queuing equilibrium" condition on v_i and b_i:

for each base network link i, $b_i \geq 0$ and the flow $v_i = \Sigma_r L_{i\,r} X_r$ along link i equals s_i or $v_i = \Sigma_r L_{i\,r} X_r < s_i$ and $b_i = 0$ (so there is no queueing delay at the exit of link i).

This condition may be written as a complementarity condition:

$$[b_i \geq 0 \text{ and } v_i - s_i = 0] \text{ or } [v_i - s_i < 0 \text{ and } b_i = 0]. \tag{10}$$

Condition (10) may be simply stated as follows.
Queue equilibrium condition: the flow v_i is supply-feasible ($\leq s_i$) and the non-negative queueing delay must be zero if $v_i < s_i$.

Thus queues only occur on saturated links. It may be argued that some queueing delay occurs (usually) when a link exit is nearly saturated, This can be allowed for by thinking of s_i as a non-decreasing function of b_i (the inverse of a strictly increasing bottleneck delay function) and then writing $s_i(b_i)$ in place of s_i in (10). We do not do this here because we wish to keep things as uncomplicated as possible.

Condition (10) may be also written shortly as follows:

$$b_i \text{ is normal at } v_i \text{ to } S_i. \tag{11a}$$

By virtue of the definition of S_i in (7a) above, condition (10) or (11a) ensures that $b_i \geq 0$. (11a) holds for all links i if and only if

$$\mathbf{b} \text{ is normal at } \mathbf{v} \text{ to } S \tag{11b}$$

and also if and only if

$$\mathbf{k \bullet b} \text{ is normal at } \mathbf{v} \text{ to } S \tag{11c}$$

where S is given in (7b) above. (Again, this condition implies that $\mathbf{b \geq 0}$.)

In this section, to ensure that blocking back does not occur we add the sufficient queue storage condition: $\mathbf{s \bullet b < MAXQ}$ where $\mathbf{MAXQ} = [MAXQ_1, MAXQ_2, \ldots, MAXQ_m]$. Combining this "sufficient queue storage" condition, the feasible Wardrop equilibrium condition (9) and (11c), we obtain the following condition on $(\mathbf{v, b})$:

$$\begin{aligned} &\mathbf{v} \text{ belongs to } D \cap S,\ \mathbf{s \bullet b < MAXQ}, \\ &\mathbf{-(c(v) + k \bullet b)} \text{ is normal at } \mathbf{v} \text{ to } D \text{ and also} \\ &\mathbf{k \bullet b} \text{ is normal at } \mathbf{v} \text{ to } S, \end{aligned} \tag{12a}$$

where $b_i = Q_i / s_i$ for $1 \leq i \leq m$, and k_i is given by (6).

This condition (12a) is our basic definition of a Wardrop equilibrium with spatial queueing delays in a steady state model. Condition (12a) does not seek to address blocking back: the assumption here is that **MAXQ** is a large vector. Here, in this section 2, the only effect of the upper limit on Q_i ($= MAXQ_i$) is in the definition of the constants k_i (given in (6)) in the link cost functions

$$SUM_i(v_i, b_i) = c_i(.) + k_i b_i$$

given in (5).

2.4. Proof of existence of an equilibrium, firstly with prices (or point queues) at link exits, and then with spatial queueing

Suppose given the sets D and S defined in (7b) and (8c), and the continuous positive function $\mathbf{c(.)}$ defined throughout $D \cap S$. Let $\mathbf{v}$ be a non-negative m-vector of link flow and $\mathbf{p}$ be a non-negative m-vector of prices (or point-queue delays) at the exits of all m links and consider the following condition on $(\mathbf{v, p})$:

$$\begin{aligned} &\mathbf{v} \text{ belongs to } D \cap S, \\ &\mathbf{-(c(v) + p)} \text{ is normal at } \mathbf{v} \text{ to } D \text{ and also} \\ &\mathbf{p} \text{ is normal at } \mathbf{v} \text{ to } S. \end{aligned} \tag{12b}$$

This condition is exactly condition (12a) if, in (12b), (i) $\mathbf{s \bullet b < MAXQ}$ is added and (ii) $\mathbf{p}$ is replaced by $\mathbf{k \bullet b}$. In fact if we know that (12b) has a solution $(\mathbf{v^*, p^*})$ then if we put

$b^*_i = p^*_i / k_i$ for $i = 1, 2, 3, \ldots, m$;

then $(\mathbf{v, b})$ solves (12a) if **MAXQ** is large enough. (We need this "large enough" condition now as we are here extending the pricing result to a spatial queueing model and not just a point queue model.) Plainly, showing that (12b) has a solution is a long way toward showing that (12a) has a solution. Theorem 1 below shows that under natural conditions (12b) has a solution.

Theorem 1: Suppose given the (convex) sets D and S defined above in (7b) and (8c). Suppose that $D\cap S$ is non-empty and that c(.) is continuous, positive and defined throughout S. Then there is a pair (**v**, **p**) *satisfying (12b).*

Proof: The cost function **c(.)** is continuous and defined throughout $D\cap S$; which is non-empty, closed, bounded and convex. So by Brouwer's fixed point theorem there is at least one vector $\mathbf{v}^* \in D\cap S$ such that

$-\,\mathbf{c}(\mathbf{v}^*)$ is normal at $\mathbf{v}^*$ to $D\cap S$.

Consider any one such $\mathbf{v}^* \in D\cap S$. D, defined in 7b, is polyhedral and so is the intersection of a finite number of closed half spaces; suppose that there are N(D) of these. S, defined in (8c), also is the intersection of a finite number of closed half spaces (since there is one closed half space for each link in the network and there are m of these in this case); suppose that there are N(S) = m of these. Then $D\cap S$ is the intersection of all N(D) + N(S) closed half spaces; let these half-spaces be: $H_1, H_2, H_3, \ldots, H_{N(D)+N(S)}$.

Consider any one of these closed half spaces, H_k say. Then H_k contains $D\cap S$ and so contains $\mathbf{v}^*$ as $\mathbf{v}^*$ belongs to $D\cap S$. Given v*, if v* is on the boundary of H_k there is a unique unit normal to the closed half space H_k at v*; for such an H_k let $\mathbf{n}_k$ be this unit normal. On the other hand if H_k is such that v* is interior to H_k then the only normal at v* to H_k is the zero vector; for such H_k let $\mathbf{n}_k$.be this zero vector. However each H_k contains $D\cap S$ (which is the intersection of all the H_j), so the normal $\mathbf{n}_k$ is not only normal at $\mathbf{v}^*$ to H_k but is also normal at $\mathbf{v}^*$ to $D\cap S$. This argument applies to each of the closed half-spaces H_1, H_2, $H_3, \ldots, H_{N(D)+N(S)}$; so $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \ldots, \mathbf{n}_{N(D)+N(S)}\}$ is a set of N(D) + N(S) vectors each of which is normal at $\mathbf{v}^*$ to $D\cap S$.

Now $D\cap S$ equals the intersection of the N(D) + N(S) closed half spaces H_k; so *any* normal at $\mathbf{v}^*$ to $D\cap S$ must be a non-negative combination of these N(D) + N(S) special normals $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \ldots, \mathbf{n}_{N(D)+N(S)}\}$. Therefore $-\,\mathbf{c}(\mathbf{v}^*)$, being normal at $\mathbf{v}^*$ to $D\cap S$, must be a non-negative combination of these N(D) + N(S) normals at $\mathbf{v}^*$ to $D\cap S$. Each $\mathbf{n}_k$ is normal, at $\mathbf{v}^*$ to either D or S. So we may split this non-negative combination into two parts. Hence $-\,\mathbf{c}(\mathbf{v}^*)$ is the sum of

[a non-negative combination (= $\mathbf{n}_D$ say) of N(D) normals at $\mathbf{v}^*$ to D] and

[a non-negative combination (= $\mathbf{n}_S$ say) of N(S) normals at $\mathbf{v}^*$ to S].

The vector $\mathbf{n}_D$, being a non-negative combination of N(D) unit normals at $\mathbf{v}^*$ to D, must itself be normal at $\mathbf{v}^*$ to D and the vector $\mathbf{n}_S$, being a non-negative combination of N(S) unit normals at $\mathbf{v}^*$ to S, must itself be normal at $\mathbf{v}^*$ to S. So

$$-\,\mathbf{c}(\mathbf{v}^*) = \mathbf{n}_D + \mathbf{n}_S$$

where $\mathbf{n}_D$ is normal at $\mathbf{v}^*$ to D and $\mathbf{n}_S$ is normal at $\mathbf{v}^*$ to S. Subtracting the vector $\mathbf{n}_S$ from both sides:

$$-\,(\mathbf{c}(\mathbf{v}^*) + \mathbf{n}_S) = \mathbf{n}_D$$

and so is normal at $\mathbf{v}^*$ to D.

Now choose $\mathbf{p}^* = \mathbf{n}_S$ (which is, as shown above, normal at $\mathbf{v}^*$ to S). Then:

$\mathbf{v}^*$ belongs to $D\cap S$,
$-\,(\mathbf{c}(\mathbf{v}^*) + \mathbf{p}^*) = \mathbf{n}_D$ is normal at $\mathbf{v}^*$ to D and also
$\mathbf{p}^* = \mathbf{n}_S$ is normal at $\mathbf{v}^*$ to S.
And so $(\mathbf{v}^*, \mathbf{p}^*)$ satisfies (12b). Note that, by the specification of S, $\mathbf{p}^* \geq \mathbf{0}$. □

Theorem 1 may be utilised to prove a number of "similar" results. We give several of these results here as corollaries of theorem 1 since they are obtained fairly simply by re-interpreting the variables and constraint sets of theorem 1 appropriately. The last of these results is a dynamic equilibrium-pricing-control result using the P_0 control policy.

Corollary 1: (Existence of an equilibrium with spatial queues when **MAXQ** *is large.) Under the conditions of theorem 1, there is a pair* (**v**, **b**) *satisfying (12a) provided* **MAXQ** *is large enough.*

Proof: Under these conditions (12b) has a solution $(\mathbf{v}^*, \mathbf{p}^*)$, by theorem 1. So if we put
$b^*_i = p^*_i/k_i$ for i = 1, 2, 3, . . . , m;
then $(\mathbf{v}^*, \mathbf{b}^*)$ satisfies (12a) aside from the constraint $\mathbf{s}\cdot\mathbf{b}^* \leq \mathbf{MAXQ}$. (If this last constraint is not satisfied then the bottleneck delay vector $\mathbf{b}^*$ may not be supportable as the implied queue vector $\mathbf{s}\bullet\mathbf{b}^*$ may have some co-ordinates which are too large). To ensure that this is not the case here; we utilise the vector $\mathbf{b}^*$ and impose the following obvious constraint on the queueing capacity vector **MAXQ**:

$$\mathbf{MAXQ} > \mathbf{s}\bullet\mathbf{b}^*.$$

For any **MAXQ** satisfying this inequality it follows that (12a) holds with feasible queue lengths and the corollary is proved. □

2.5. Existence of equilibrium when there are green-times and a pricing variant of control policy P_0 is utilised (steady state)

In this section we suppose that there is a price and a green-time proportion corresponding to each link exit and these prices and green-times may be controlled responsively as flows vary. So now a closed bounded convex set F of feasible green-time vectors **g** is supposed given; and we consider the new demand set D×F and the new set S^ of supply-feasible (link flow, link green time) vectors defined as follows:

$S\hat{} = \{(\mathbf{v}, \mathbf{g});\ \mathbf{v} - \mathbf{s}\bullet\mathbf{g} \leq 0\}$.

With these changes (12b) becomes:

$$\begin{aligned}&(\mathbf{v}, \mathbf{g}) \text{ belongs to } (D\times F)\cap S\hat{},\\ &-[(\mathbf{c}(\mathbf{v}, \mathbf{g}), \mathbf{0}) + (\mathbf{p}, -\mathbf{s}\bullet\mathbf{p})] = [-(\mathbf{c}(\mathbf{v}, \mathbf{g}) + \mathbf{p}), \mathbf{s}\bullet\mathbf{p})] \text{ is normal at } (\mathbf{v}, \mathbf{g}) \text{ to } D\times F \text{ and also}\\ &(\mathbf{p}, -\mathbf{s}\bullet\mathbf{p}) \text{ is normal at } (\mathbf{v}, \mathbf{g}) \text{ to } S\hat{}.\end{aligned} \tag{12c}$$

This incorporates a pricing variant of the P0 control policy. This variant here pushes the green time vector **g**, *within F*, in the direction **s●p**. Such actions are here thought of as being within the control of a network manager / controller. This pricing variant of the P0 control policy may be expressed more fully and accurately as follows:

for given **p**, **g** is to be chosen so that the "force" **s●p** is normal at **g** to F.

This condition is part of the second line of condition (12c). A more dynamic statement of this policy is as follows:

continually move **g** in the direction of the "force" **s●p** projected onto F.

(Of course it is also imagined that the network controller will control prices as well; probably increasing prices where there are queues, aiming to reduce queues.)

Following this P0 policy, a link i with a high saturation flow and a high price will push up the link i green-time. Eventually this will tend to increase the total of the prices paid (subject to the condition that prices on unsaturated links must be zero); under suitable conditions the signals become profit-maximising green-time allocators. To show that there is an equilibrium consistent with this pricing variant of P_0 (and with some suitable prices) we need to show that (12c) has a solution, and this is our corollary 2 below.

Corollary 2. Suppose that D×F∩S^ *is non-empty and* **c = c(v, g)** *is a positive continuous function defined throughout S^. Then there is a pair* **((v, g), p)** *satisfying (12c)*

Proof: This follows from theorem 1 above by replacing the variables and sets mentioned in theorem 1 as follows:

(v, g) replaces **v** (so the new "flow" vector now includes green times);
S^ replaces S (so the supply feasible set involves both link green times and link flows);
D×F replaces D (so the new demand set now includes the constraints on link green times);
s●g replaces **s** (so the new capacities now depend on green times and so are not constant);
c(v, g) replaces **c(v)** (so now costs may depend on both flows and green times);
- ((c(v, g), 0) replaces **- c(v)** (so –(the new cost vector) is formally felt by both **v** in D and **g** in F, although the force on **g** here is **0**); and
(p, - (s●p)) replaces **p** (so as to ensure that we have a normal to the new constraint set S^).

It is critical here to observe that any normal $\mathbf{n}_{S\hat{}}$ to the new supply-feasible S^ must be of the form **(p, - (s●p))** where **p ≥ 0**. This is central to the proof outlined in theorem 1 when applied in this new setting with the new supply-feasible set S^. □

2.6. Existence of equilibrium when there are green-times and a spatial queueing variant of control policy P0 is utilised (steady state)

It is natural to now replace the price vector **p** above by the spatial queueing delay vector **k·b** just as we did above in corollary 1. Then (12c) becomes:

$$\begin{aligned}&(\mathbf{v}, \mathbf{g}) \text{ belongs to } D\times F\cap S\hat{},\\ &(-[\mathbf{c}(\mathbf{v}, \mathbf{g}) + \mathbf{k}\bullet\mathbf{b}], \mathbf{s}\bullet(\mathbf{k}\bullet\mathbf{b})) \text{ is normal at } (\mathbf{v}, \mathbf{g}) \text{ to } D\times F \text{ and also}\\ &((\mathbf{k}\bullet\mathbf{b}), -\mathbf{s}\bullet(\mathbf{k}\bullet\mathbf{b})) \text{ is normal at } (\mathbf{v}, \mathbf{g}) \text{ to } S\hat{}.\end{aligned} \tag{12d}$$

Provided **MAXQ** is large enough this is an equilibrium condition with spatial queueing delays consistent with a spatial queueing variant the P_0 policy. The spatial queueing variant of P_0 is:

choose feasible green times so that the "force" **s●(k●b)** is normal at **g** to F.

This condition is part of the second line of condition (12d). A more dynamic statement of this policy is as follows:

continually move **g** in the direction of the force **s·(k·b)** projected onto F.

Here the drivers determine **k●b**, and not the network controller, since queueing delays rise naturally on saturated links. Corollary 3 below shows that there is an equilibrium consistent with this spatial queueing variant of P_0.

Corollary 3. Suppose that D×F∩S^ *is non-empty and* **c** *is a positive continuous function defined throughout S^. Then, provided* **MAXQ** *is sufficiently large, there is a pair* **((v, g), k●b)** satisfying (12d). *Proof.* This is omitted, but follows by again using natural replacements in theorem 1; like in the proof of corollary 2. [Here **k** depends on **g**, however this may be allowed for by very slightly changing the proof of theorem 1.] □

Let **k·Q** / **s·g** be the vector whose i^{th} co-ordinate is $k_i q_i$ / $s_i g_i$ (and so on). Then **k●Q / g = s●(k●Q/s●g) = s●(k●b)**. This spatial queueing P_0 variant may then be written more explicitly as follows: for any given queue size vector **Q**

choose a green time vector in F so that the "force" **k●Q** / **g** is normal at **g** to F.

This condition is still essentially part of the second line of condition (12d). A more dynamic statement of this policy is as follows:

continually move **g** in F in the direction of the force **k●Q** / **g** projected onto F.

Of course in this case on the real network the bottleneck delays arise from drivers decisions and (unlike prices) are not directly controllable by a network manager; only the green-time vector is directly controllable here. It is notable that here the saturation flows have disappeared from the policy statement. Thus this queueing version of the P_0 policy applies even if saturation flows change; a sudden snow fall leaves the policy unchanged, or certain approaches to a junction being blocked by an accident leaves the policy unchanged. This may allow the model to generate, quickly, signal timing suggestions that take some (anticipatory) account of such changes.

Calculation of equilibria subject to the pricing variant of the P_0 responsive control policy. Suppose now that **c(v) = c** is a constant m-vector; so that all congestion delays occur at the bottlenecks in the form of queueing delays. Now

" - c(v*) is normal at **v*** to D∩S" becomes the linear program: **"-c** is normal at **v*** to D∩S".

Solve this linear program to yield **v***. Then a suitable $\mathbf{n}_S$ may be located by choosing **p** to minimise the norm of the projection

$\mathrm{Proj}_D(- \mathbf{c} - \mathbf{p})$

of – **c** - **p** onto D. Such a **p** must have non-negative co-ordinates by virtue of the design of S. Finally, using the k_i given by (6), find **b*** such that **k●b* = p**. Using the notation introduced above, **b* = p/k.** (The norm of the projection here must be minimisable to zero.)

2.7. Existence of equilibrium when there are green-times and a pricing variant of control policy P_0 is utilised (dynamic case)

2.7.1. Demand feasibility

Time will now be explicitly considered and we let t ≥ 0 throughout this section 2.7. At each time t ≥ 0 the outflow rate from upper level link r will be denoted by $X_r(t) \geq 0$. We suppose that these outflow rate functions X_r are always measurable and bounded. We also suppose that each lower level link traversal time c_i is constant. Then as before we suppose that the upper level link (uncongested) traversal times C_r satisfy:

$C_r = \Sigma_i L^T_{ri} c_i$ or $\mathbf{C} = \mathbf{L}^T\mathbf{c}$.

(In the first of these equations, for each r there is just one term in the sum over i; so all upper level links r corresponding to base link i have the same travel time c_i.) Given any outflow rate function X_r defined for all t ≥ 0 we define a corresponding upper link r inflow rate function X^+_r by putting $X^+_r(t) = X_r(t + C_r)$ for all t ≥ 0, where C_r is the constant link r traversal time.

We suppose given, for each upper level non-destination node n, a non-negative bounded measurable function W_n defined for all t ≥ 0 satisfying: $W_n(t) = 0$ if t ≥ 1. This function W_n specifies inelastic or rigid time-varying flow rates leaving n headed for dest(n). These departure flows all occur between times 0 and 1. We also suppose that for each t ≥ 0,

for each upper level non-destination node n *the total inflow rate along all links leaving node* n *equals the total of the link outflow rates into node* n *plus the fixed flow rate* $W_n(t)$ *generated at node* n *headed for destination* dest(n).

This may be written: for all non-destination nodes n,

$W_n(t) + \Sigma_r A_{nr}X_r(t) - \Sigma_r B_{nr}X_r^+(t) = 0$ for all t ≥ 0.

Finally, given the time T > 1 already specified above, we insist that the network is empty after time T or that all journeys are completed by time T. So we impose the following condition:

$X_r(t) = 0$ for all $t \geq T$.

Thus we are supposing (in an idealized morning peak period say) that everyone leaves home between time 0 and time 1 and that everyone has reached their place of work by time T > 1. Then the upper level demand set is

$D_0 = \{[\mathbf{X}, \mathbf{X}^+];$ (i) $X_r(t) \geq 0$ and $X^+_r(t) = X_r(t + C_r)$ for all links r and all $t \geq 0$,
(ii) $W_n(t) + \Sigma_r A_{n\,r}X_r(t) - \Sigma_r B_{n\,r}X_r^+(t) = 0$ for all non-destination nodes n and all $t \geq 0$,
(iii) $X_r(t) = 0$ for all links r and all $t \geq T$ and
(iv) each $X_r = X_r(.)$ is a bounded measurable function of time $t \geq 0.\}$

By following the argument in Smith and Wisten (1995), there is a metric on D_0 such that D_0 becomes a compact convex set of bounded measurable functions. Here we consider just the set of demand feasible link outflows arising from D_0. So we let

$D\hat{} = \{\mathbf{v} = \mathbf{LX}; [\mathbf{X}, \mathbf{X}^+] \in D_0$ for some $\mathbf{X}^+\}$.

This is also a compact convex set (it is a section of the compact set D_0), and of course $\mathbf{v}(t) = \mathbf{0}$ if $t \geq T$.

2.7.2. Supply-feasibility

Here also we suppose that there is a green-time corresponding to each link and that now these may vary with time. We suppose given a closed bounded convex set F of feasible link green-time vectors $\mathbf{g}(t)$ and define the set F^ as follows:

$F\hat{} = \{\mathbf{g}; \mathbf{g}$ is a measurable function of time $t \geq 0$, $\mathbf{g}(t)$ belongs to F for all $t \geq 0$ and $\mathbf{g}(t) = 0$ if $t > T\}$.

The new set S* of supply-feasible (link outflow, link green time) vector functions of time is then defined as follows:

$S^* = \{(\mathbf{v}, \mathbf{g}); \mathbf{v(t)} - \mathbf{s} \bullet \mathbf{g(t)} \leq 0$ for all $t \geq 0\}$

[It follows from 2.7.1 that for $\mathbf{v} \in D\hat{}$, $\mathbf{v}(t) = \mathbf{0}$ if $t \geq T$ so that making $\mathbf{g}(t) = 0$ for $t > T$ in the definition of F^ makes no real difference as all relevant link outflow vectors $\mathbf{v}$ are zero if $t > T$.]

The set of feasible base (link outflow, link green time) functions is then defined to be $(D\hat{}\times F\hat{})\cap S^*$ where D^ is specified above in section 2.7.1. The sets D^, F^ and S* are all compact convex sets.

2.7.3. Result

Suppose that $(D\hat{}\times F\hat{})\cap S^*$ *is non-empty. Then there is a pair* $[(\mathbf{v}, \mathbf{g}), \mathbf{p}]$ satisfying:

$(\mathbf{v}, \mathbf{g})$ belongs to $D\hat{}\times F\hat{}\cap S^*$,
$-((\mathbf{c}, \mathbf{0}) + (\mathbf{p}, -\mathbf{s}\bullet\mathbf{p})) = (-[\mathbf{c}+\mathbf{p}], \mathbf{s}\bullet\mathbf{p})$ is normal at $(\mathbf{v}, \mathbf{g})$ to $D\hat{}\times F\hat{}$ and also (12e)
$(\mathbf{p}, -\mathbf{s}\bullet\mathbf{p})$ is normal at $(\mathbf{v}, \mathbf{g})$ to S*.

Proof. This is omitted; this follows the proof of theorem 1, but that proof must be changed to deal with the compact convex sets D^, F^ and S*, which are here functions of time.

(12e) is a changed version of (12b) allowing for the new context. (12e) now includes, in the middle line, a dynamic pricing variant of the P_0 control policy, which may be written more explicitly as follows: for any given $\mathbf{p}$,

choose a feasible green time vector function $\mathbf{g}$ so that the "force" $\mathbf{s}\bullet\mathbf{p}$ is normal at $\mathbf{g}$ to F^.

This condition is still essentially part of the second line of the condition (12e) above. A more dynamic statement of this policy is:

continually move the green-time vector function $\mathbf{g}$ in the direction of the "force" $\mathbf{s}\bullet\mathbf{p}$ projected onto F.

The equilibrium condition given in the central line of (12e) above may be extended to read:

$[\,(-(\mathbf{c}+\mathbf{p}), \mathbf{s}\bullet\mathbf{p}), \mathbf{v} - \mathbf{s}\bullet\mathbf{g}]$ is normal at $[(\mathbf{v}, \mathbf{g}), \mathbf{p}]$ to $(D\hat{}\times F\hat{})\times M(R_+^m)$

where $M(R_+^m)$ is the set of bounded measurable functions from R_+^m to R_+^m. This condition may be soluble by suitably amending algorithm (D) given in Smith (1984a,b); we leave this for later study.

3. A fixed demand capacitated traffic assignment model with blocking back

3.1. The link model with blocking back

In section 2 we introduced a condition (9) for equilibrium with queues, taking some account of the extent of queues but assuming that these do not overflow. This condition makes sense so long as the equilibrium queue lengths are within the queueing capacities of the links where the queue starts. We proved that a solution of the model certainly existed provided these queueing capacities $MAXQ_i$ are large enough.

Now we consider what happens if the bounds $MAXQ_i$ on the queue lengths are exceeded. In this case we must change (9) to allow for the possibility that a link outflow is restricted by downstream queues filling a downstream link and overflowing onto the link. This means that the link model (5) introduced above must be changed.

Suppose now that flow along link i, v_i, is constrained not by the exit saturation flow of link i but by a queue on a downstream link exceeding the queue capacity of that downstream link and spilling back into link i. The maximum possible number of queueing vehicles on link i is unchanged at $MAXQ_i$ and so now the maximum bottleneck delay on link i, *when the flow is* v_i, is $MAXQ_i / v_i$. This is the *maximum* possible bottleneck or queueing delay; so that if v_i is very small the *maximum* queueing delay can be very large with a given queue storage capacity.

We utilise equation 3 (as we did in section 2) and argue just like we did in section 2 above *but now the flow is constrained to be less than s_i by an overspill queue and we do not yet know what it is;* so we assume the flow is v_i (as yet unknown). In this case we obtain

$$k_i = 1 - v_i c_i(v_i)/MAXQ_i,$$

instead of (6a). Thus k_i depends on v_i; and so is no longer constant. Now, with blocking back, a link model which may prove useful in estimating the sum of the travelling time and the queueing time on link i is:

$$\begin{aligned} SUM_i(v_i, b_i) &= c_i(v_i) + k_i(v_i)b_i \\ &= c_i(v_i) + [1 - v_i c_i(v_i)/MAXQ_i]b_i \end{aligned} \quad (13)$$

Link model (13) in a sense includes (5), (6a) as a special case; of course in section 2 above the only queues arise when $v_i = s_i$, whereas here queues may arise when $v_i < s_i$ due to blocking back. *Equation (13) gives the set of all those triples* $\{v_i, b_i, SUM_i(v_i, b_i)\}$ *which can occur at a solution of the model.* (13) may perhaps be regarded as a rudimentary flow – density – cost relation.

Travel time function (13) here is again (like (5)) a *two dimensional* travel time function: the travel time for link i depends on both the flow v_i and the queueing delay b_i. The total link travel time function here, $SUM_i(v_i, b_i)$ in (13), is defined ONLY if

$$0 \le v_i \le s_i \text{ and } 0 \le b_i \le MAXQ_i/v_i$$

or

$$0 \le v_i \le s_i \text{ and } 0 \le v_i b_i \le MAXQ_i. \quad (14)$$

(Constraint (14) is non-convex; this suggests that even if equilibrium solutions exist there may be many of them.)

A further feasibility constraint apparently arises here from the form and sense of (13). This is:

$$MAXQ_i - v_i c_i(v_i) \ge 0 \text{ for all } v_i \le s_i$$

or:

$$v_i c_i(v_i) - MAXQ_i \le 0 \text{ for all } v_i \le s_i.$$

Assuming that $c_i(v_i)$ is non-decreasing, this is true if

$$s_i c_i(s_i) - MAXQ_i \le 0$$

which is our section 2 condition (6b); and so this apparently new condition is implied by (6b), which, as remarked above in section 2, is entirely reasonable.

3.2. Supply-feasibility with blocking back

The link i flow v_i may be constrained by

(i) the link i exit capacity or

(ii) the queue Q_j on link j immediately upper-level downstream being = $MAXQ_j$.

It is necessary now to take account of *both* possible constraining mechanisms within the definition of "link i supply feasibility" and the consequent constraints imposed on **v** and the bottleneck delay vector **b**. In reading what follows it may be suitable to think, to be specific, that *just the two base links* j *and* k *are upper-network-downstream from base link* i.

So we are here thinking, to be specific, of a simple diverge. The set of feasible base link flows is now, with possible blocking back, to be defined as follows. First for each such base (diverge) link i we put

$BBS_i = \{(\mathbf{v}, \mathbf{Q}); v_i - s_i \leq 0$ and $Q_j - MAXQ_j \leq 0$ for all links j upper-level-downstream of link i$\}$.

BBS_i is the set of blocking-back supply-feasible $(\mathbf{v}, \mathbf{Q})$ pairs associated with the link i; of course, this supply-feasible set now involves flows and bottleneck delays on downstream links j as well as on link i. Then the set of blocking-back supply-feasible $(\mathbf{v}, \mathbf{Q})$ pairs is to be

$BBS = \{(\mathbf{v}, \mathbf{Q}); (\mathbf{v}, \mathbf{Q}) \in BBS_i \text{ for all } i\} = \cap_i BBS_i = \{(\mathbf{v}, \mathbf{Q}); \mathbf{Q} - MAX\mathbf{Q} \leq \mathbf{0} \text{ and } \mathbf{v} - \mathbf{s} \leq \mathbf{0}\}$.

This may be compared with (7b) which lacks the constraint $\mathbf{Q} - MAX\mathbf{Q} \leq \mathbf{0}$.

3.3. The bottleneck delay equilibrium constraint (with blocking back)

Taking the two constraining effects in (i) and (ii) in 3.2 together (instead of just (i) in section 2) we now insist that the following queue-equilibrium condition holds for our link i:

$b_i \geq 0$ and
$v_i = s_i$ or
$Q_j = MAXQ_j$ for some link j upper-level-downstream of link i, or (15)
$[(v_i < s_i)$ and $(Q_j < MAXQ_j$ for all links upper-level-downstream of link i) and $(b_i = 0)]$.

This is a more extensive condition than the more simple complementarity condition (10) above. Both (10) and (15) are saying that the queueing delay may be non-zero if there is a cause; but that if there is no cause then the bottleneck delay = 0. (Non-negativity constraints apply to all variables including the bottleneck delays b_i.)

It is now natural to give a possible queue for each possible cause as follows. Let

$$w_{ii} = v_i - s_i,\quad w_{ij} = Q_j - MAXQ_j, \text{ and}\quad w_{ik} = Q_k - MAXQ_k. \tag{16}$$

Then the three-level queueing delay equilibrium condition (15) may be written in this case with just links k and j downstream:

$$w_{ii} = 0, \text{ or } [w_{ii} < 0 \text{ and } b_{ii} = 0],\quad w_{ij} = 0, \text{ or } [w_{ij} < 0 \text{ and } b_{ij} = 0], \text{ and}\quad w_{ik} = 0, \text{ or } [w_{ik} < 0 \text{ and } b_{ik} = 0] \tag{17}$$

This condition (17) in this case comprises three "simple" complementarity conditions; but there can be only one physical queue on lane i so now let

$$b_i = b_{ii} + b_{ij} + b_{ik}.$$

Here the w_{ih} take on and extend the role of $v_i - s_i$ in (10) but w_{ij} depends on the downstream Q_j. w_{ij} may here be thought of as the excess demand for travel along link i (relative to the "downstream" constraint imposed by link j). Then condition (17) says that the excess demands w_{ih} for travel along link i is either zero (when b_{ih} may be positive) or negative and then b_{ih} must be zero.

3.4. Definition of a Wardrop / queueing equilibrium (now with blocking back)

The Wardrop / queueing equilibrium condition is now, with blocking back:

$(\mathbf{v}, \mathbf{b})$ belongs to $[D \times R_+^m] \cap BBS$,
$-(\mathbf{c}(\mathbf{v}) + \mathbf{k} \bullet \mathbf{b})$ is normal at $\mathbf{v}$ to D, (18)
$b_i = \sum_h b_{ih}$ and $w_{ih} \leq 0$ and $w_{ih} < 0$ implies $b_{ih} = 0$, for all (i, h) with h = i or link h downstream of link i.

As is shown by Daganzo (1998), we cannot hope to obtain the kind of existence results previously stated. So in this paper we do not prove that an equilibrium exists in this case. Finding controls (including prices and signal green-times) and geometries which ensure existence of equilibrium in this framework is likely to prove a profitable, if possibly arduous, research direction.

4. An illustrative example: using link model (13) with spatial queueing, blocking back and merges in a simple network

In this section, we illustrate how link model (13) together with simple equilibrium analysis leads, in a simple network model, to different equilibrium steady states. These include: (1) spatial queueing without blocking back, (2) spatial queueing with blocking back and (3) spatial queueing with blocking back and a simple merge.

The example network

Consider the network in figure 3; two OD pairs are joined by three routes as follows.

Route 1 (the bypass) joins the OD pair [A,B], and has just link 1.

Route 2 joins OD pair [A,B], and has links 2, 3 and 0.

Route 3 joins OD pair [C,B], and has links 4, 3 and 0.

Link 2 has a saturation flow at the exit of s_2 vehicles per minute and link 3 has a saturation flow at the exit of s_3 vehicles per minute. All other links will have very large saturation flows and $s_2 < s_3$ so the exit of link 3 is a bottleneck. Links 2 and 4 merge at M. The steady OD flow rate from A to B is fixed at T_{AB} vehicles per minute where $T_{AB} > 0$. The steady OD flow rate from C to B is fixed at T_{CB} vehicles per minute where T_{CB} may be zero. If v_i is the flow rate along link i, we suppose that with no queueing the time taken to traverse link i (i = 0, 1, 2, 3, 4) will be $c_i(v_i)$ where $c_i(v_i) > 0$ unless i = 0 and $c_0(v_0) = 0$.

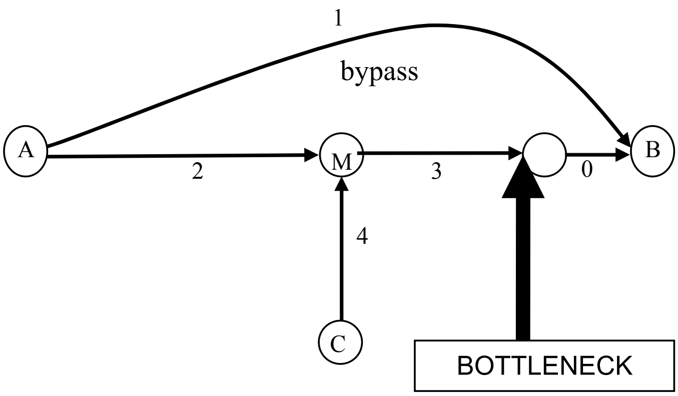

Figure 3. Simple network with one bottleneck, two origins and one destination

4.1. Equilibrium with spatial queueing on just link 3 and with no blocking back and no merging traffic

We first consider a case with no merging traffic, no blocking back and just spatial queueing on link 3. In this first case we assume that there is no merging traffic flow; so we assume that $T_{CB} = 0$. We also assume that $T_{AB} > s_3$; then not all AB traffic can traverse link 3 and so some AB traffic must at a steady state equilibrium use route 1. We also suppose that

$$c_2(v_2) + c_3(v_3) < c_1(v_1) \leq c_2(v_2) + \mathrm{MAX}b_3$$

Then route 1 is used and is more costly than route 2 (when there is no queueing) and so at equilibrium there must be some queueing delay along route 2 to equilibrate the network. Queueing can only occur on a saturated link, and $s_3 < s_2$, so link 3 must be the saturated link and $v_2 = v_3 = s_3$. To determine the bottleneck delay on link 3 at equilibrium we use link performance model (13). Then

$$c_1(v_1) = c_2(s_3) + c_3(s_3) + k_3b_3$$

and hence

$$b_3 = [c_1(T_{AB}\text{-}s_3) - (c_2(s_3) + c_3(s_3))]/k_3$$

where $k_3 = [1 - s_3c_3(s_3)/\mathrm{MAXQ}_3]$ by (13) and so

$$b_3 = [c_1(T_{AB}\text{-}s_3) - (c_2(s_3) + c_3(s_3))]/[1 - s_3c_3(s_3)/\mathrm{MAXQ}_3]$$

and

$$Q_3=b_3s_3.$$

It is easy to check that $Q_3 \leq MAXQ_3$ when the conditions imposed in this section 4.1 hold.

Thus under the conditions specified at the start of this section, the steady-state equilibrium queueing delay (and the equilibrium queue) depend on the uncongested travel times, the outflow capacity of link 3 and the storage-capacity of link 3. The above formulae demonstrate that there exists an equilibrium with queueing delay on just link 3 (and no blocking back) if the specified conditions hold.

4.2. Equilibrium with spatial queueing on links 2 and 3 and blocking back but no merging traffic

In this second case we also assume that there is no merging traffic flow. Suppose also now that

$$c_2(v_2) + MAXb_3 < c_1(v_1) \leq MAXb_2 + MAXb_3.$$

The link 3 queue must then spillback onto link 2 (since $c_2(v_2) + MAXb_3 < c_1(v_1)$). The equilibrium queues will be such that the total delay incurred on the two queued links 2 and 3 equals the uncongested travel time difference between two alternative routes joining A and B. So, now,

$$c_1(v_1) = c_2(s_3) + k_2b_2 + MAXb_3 = c_2(s_3)+ c_3(s_3) + k_2b_2 + (MAXQ_3/s_3 – c_3(s_3)),$$

using (13). So in this case

$$b_2 = [(c_1(T_{AB} - s_3) - (c_2(s_3) + c_3(s_3))) – (MAXQ_3/s_3 – c_3(s_3))] / k_2$$

Again queueing delay becomes a function of the link travel times the link saturation flows and the storage-capacity of the saturated link 3, and the parameter k_2 arising in (13). Then

$$b_2 = [(c_1(T_{AB} - s_3) - (c_2(s_3) + c_3(s_3))) – (MAXQ_3/s_3 – c_3(s_3))] / [1 – s_3c_2(s_3) / MAXQ_2]$$

and

$$Q_2 = b_2s_3.$$

It is easy to check that now $Q_2 \leq MAXQ_2$ when the conditions imposed in this section 4.2 hold. These equations show the dependence of the steady-state queue on a link upstream of the block back node M and on the queue storage-capacity of the blocking-back link 3.

4.3. No equilibrium due to blocking back

In this third case we also assume that there is no merging traffic flow. But now suppose that link 1 is even longer than before; here we suppose that always

$$c_1(v_1) > MAXb_3 + MAXb_2.$$

It is clear that in this case no queueing within the specified queue storage limits can balance the network. This situation is the similar to that described by Daganzo (1998).

4.4. Equilibrium with spatial queueing on links 2 and 3, blocking back and a simple merge

In this fourth case we make the same assumptions as in 4.2, including the assumption that always

$$c_2(v_2) + MAXb_3 < c_1(v_1) \leq MAXb_2 + MAXb_3,$$

except that now we let $T_{CB} > 0$ so that $v_4 = T_{CB} > 0$. In this case, at equilibrium, the flow v_2 on link 2 at equilibrium must assume a smaller value (than the s_3 in 4.2). Here, as we are assuming that all OD flows are met, $v_2 = s_3 - v_4$ and a corresponding new equilibrium solution, again using (13), will be

$$c_1(v_1) = c_2(v_2) + k_2b_2 + MAXb_3 = c_2(v_2) + c_3(s_3) + (1 – v_2c_2(v_2) / MAXQ_2)b_2 + (MAXQ_3/s_3 – c_3(s_3)).$$

So in this case

$$b_2 = [(c_1(T_{AB} – v_2) - (c_2(v_2) + c_3(s_3))) – (MAXQ_3/s_3 – c_3(s_3))] / [1 – v_2c_2(v_2) / MAXQ_2]$$

and

$$Q_2 = b_2v_2.$$

Since $v_2 = s_3 – T_{CB}$, the solution will therefore depend again on the travel time difference, the storage capacity of queueing links and the capacity of the most restrictive saturated link 3, and also on the amount of this capacity that is taken from the flows merging onto this same saturated link; the flow T_{CB} in this case.

4.5. Equilibrium with spatial queueing on links 2 and 3, blocking back and two merge models

The conditions in this section are those in section 4.4 above *with an added merge model*. It is natural to extend the example being considered here to show how the equilibrium assignment model may be extended to allow for these and other merge models.

Various merge models are suitable for different merge layouts, different junction geometries and different "controls"; for example: the Daganzo "fixed ratio" merge model (Daganzo, 1995), "Fair shares" merge models introduced by Jin and Zhang (2003) and Nie and Leonard (2005), and so on. Here we briefly consider for illustrative purposes just two simple merge models:

(a) the flow rates admitted to link 3 from links 2 and 4 experience equal queueing delays and
(b) the flow rates admitted to link 3 from links 2 and 4 are equal.

With merge model (a) the flows at equilibrium in section 4.4 remain the same but the queueing delay on link 4 becomes equal to the queueing delay on link 2. (Note that this delay on link 4 is not calculated in section 4.4.)

With merge model (b) there will now usually be no equilibrium solution. As shown in section 4.4, at an equilibrium the flow on link 2 must satisfy: $v_2 = s_3 - v_4 = s_3 – T_{CB}$, Now merge model (b) insists that $v_2 = v_4 = T_{CB}$. Thus if merge model (b) holds at equilibrium:

$$T_{CB} = s_3 – T_{CB} \text{ or } T_{CB} = s_3/2.$$

This shows that an equilibrium with queueing in this network is unlikely to be consistent with merge model (b). Further research on the impact of various merge models within this overall framework would be natural. One interesting research topic is to find a 'good' merge model permitting efficient assignment techniques to be utilised with a convergence-to-equilibrium guarantee.

5. An elastic demand capacitated traffic assignment model with spatial queueing

Here we add in an elastic demand (but still with spatial queues), following Smith (2013), instead of the rigid demand constraint in section 2.2.2 above. Fixed demand is modelled here by making the demand function constant. We utilise the network setup given in section 2 above. We need the time to traverse each upper link r which here we define as before to be

$$\Sigma_i L^T_{r\,i}\,(c_i(v_i) + k_i b_i)$$

using (13). There is (for each r) only one term in this sum. Here v_i is the flow along base link i arising from **X**, by adding all the X_r for those upper level links r which correspond to lower level link i and k_i is defined in (13).

Now the model has a variable Y_n at each node n in the upper level network. Let n be an upper level node and let dest(n) be the single destination in the upper level network containing n; Y_n is to be an estimate of the average cost of travel between n and dest(n). At equilibrium Y_n is to be exactly the average cost of travel between n and dest(n). These node-cost variables Y_n (for all n in the upper level network) will form the column vector **Y**. $Y_n = 0$ if n is a destination.

Suppose now that a (non-negative) upper level node-vector **Y** is given; then, thinking of each Y_m as the average cost to dest(m), we suppose given for each upper level node n the demand $W_n(\mathbf{Y})$ for travel from n to dest(n) when the node average cost-to-destination vector is **Y**. The demand function **W**(.) is supposed given: $W_n(\mathbf{Y}) = 0$ if n is a destination and satisfies the following conditions throughout.

D1. **W(Y)** is a continuous function of **Y**.

D2. There is a constant vector **maxW** such that $\mathbf{0} \le \mathbf{W}(\mathbf{Y}) \le \mathbf{maxW}$ for all upper level node cost vectors **Y**.

5.1. Routeing or Wardrop equilibrium condition

As above the model uses the standard Wardrop (1952) notion that more costly routes are unused. This user-equilibrium condition is here written:

for each upper link r the average cost $\Sigma_n B^T{}_{rn}Y_n$ to the destination from the node upstream of link r equals the cost

$$\Sigma_i L^T{}_{ri}(c_i(v_i) + k_i b_i) = C_r(\mathbf{X}) + \Sigma_i L^T{}_{ri} k_i b_i$$

of traversing link r plus the average cost $\Sigma_n A^T{}_{rn}Y_n$ to the destination from the node downstream of link r; or it is less and also no flow enters link r. Using inequalities this may be written as follows:

$$\begin{aligned} &\Sigma_n B^T{}_{rn}Y_n - \Sigma_n A^T{}_{rn}Y_n - [C_r(\mathbf{X}) + \Sigma_i L^T{}_{ri} k_i b_i] = 0, \text{ or} \\ &\Sigma_n B^T{}_{rn}Y_n - \Sigma_n A^T{}_{rn}Y_n - [C_r(\mathbf{X}) + \Sigma_i L^T{}_{ri} k_i b_i] < 0 \text{ and } X_r = 0. \end{aligned} \tag{21}$$

This is a well-known link flow formulation of the standard Wardrop condition.

5.2. Demand equilibrium condition

It is natural to insist that flows balance at each node n in the upper level network; taking account of flow generated. Thus we suppose that:

for each upper level node n, the total flow along all links leaving node n equals the total flow into node n plus the flow generated at node n. We slightly weaken this natural demand equilibrium condition here to obtain the following condition. For each node n in the upper level network:

$$\begin{aligned} &W_n(\mathbf{Y}) + \Sigma_r A_{nr}X_r - \Sigma_r B_{nr}X_r = 0, \text{ or} \\ &W_n(\mathbf{Y}) + \Sigma_r A_{nr}X_r - \Sigma_r B_{nr}X_r < 0 \text{ and } Y_n = 0. \end{aligned} \tag{22}$$

This is equivalent to the standard demand constraint under very weak conditions (mainly $\mathbf{c} > \mathbf{0}$):

$$W_n(\mathbf{Y}) + \Sigma_r A_{nr}X_r - \Sigma_r B_{nr}X_r = 0. \tag{23}$$

5.3. Queueing delay equilibrium condition

The following is a natural capacity-constraint "equilibrium" condition:

for each base network link i the flow $v_i = \Sigma_r L_{ir}X_r$ along link i equals s_i or the flow $v_i = \Sigma_r L_{ir}X_r < s_i$ and also there is no queueing delay at the exit of link i. This condition may be written:

$$\begin{aligned} &\Sigma_r L_{ir}X_r - s_i = 0, \text{ or} \\ &\Sigma_r L_{ir}X_r - s_i < 0 \text{ and } b_i = 0. \end{aligned} \tag{24}$$

5.4. Combined Wardrop-demand-queueing equilibrium condition

The totality of the three routeing, demand and queueing equilibrium conditions, (22, 23, 24) above is our elastic equilibrium model which now involves a more realistic link model than in Smith (2012).

6. Conclusions

The central model considered in this paper is a steady-state, link-based, fixed (or inelastic) demand equilibrium model with explicit link-exit capacities, explicit bottleneck or queuing delays and explicit bounds on queue storage capacities. The link model at the heart of this equilibrium model takes account of the space taken up by queues both when there is no blocking back and also when there is blocking back. (We have called this link model a "spatial queueing" model.) In theorem 1 we have shown that a feasible traffic assignment model has an equilibrium solution provided prices are used to impose capacity restrictions. The paper has utilised this result to show that there is an equilibrium with the spatial queueing model, provided queue-storage capacities are sufficiently large. Other results have been given; obtained by changing the variables and sets in theorem 1 suitably. These results included: (1) existence of equilibrium results (in both a steady state and a dynamic context) which allow signal green-times to respond to prices and (2) an existence of equilibrium result which allows signal green-times to respond to spatial queues; provided these control responses follow the P_0 control policy in Smith (1979a, 1987). These results show that under certain conditions the P_0 control policy maximises network capacity. We have illustrated how the spatial queueing model works on a simple network both without and with blocking back and with a simple merge. Finally the paper included elastic demand; this is necessary for long-run evaluations. Each of the steady state models here may be thought of as a stationary solution to the dynamic assignment problem either with or without blocking back. Stationary solutions of dynamic equilibrium assignment models have been previously considered by Nesterov and de Palma (2003) and Bliemer et al

(2012); these stationary solutions have been called "quasi-dynamic" equilibria. Directions for further research include (a) solution algorithms in the steady state and the dynamic state, (b) merge constraints (we have just scratched the surface of this here) and (c) the ramifications of extensive blocking back; how to control this and how to solve the equilibrium conditions given in this paper in this case. This paper has outlined several methods of seeking to allow correctly for spatial queueing without going via dynamic assignment and so the approach is complementary to the approach adopted by Bliemer et al (2012).

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