ON TWO GENERALIZATIONS OF ASSOCIATIVITY

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Let X be an arbitrary nonempty set. We regard vectors $\mathbf{x}$ in X^n as n-strings over X. We set $X^* = \bigcup_{n\in\mathbb{N}} X^n$ endowed with concatenation for which we adopt the juxtaposition notation. For instance, if $\mathbf{x} \in X^n$, $y \in X$, and $\mathbf{z} \in X^m$, then $\mathbf{x}y\mathbf{z} \in X^{n+1+m}$.

In the sequel, we will be interested both in functions of a given fixed arity (i.e., functions $f\colon X^n \to X$) as well as in functions defined on X^*, that is, of the form $g\colon X^* \to X$. Given a function $g\colon X^* \to X$, we denote by g_n the n-ary component of g, that is, the restriction of g to X^n. In this way, each function $g\colon X^* \to X$ can be regarded as a family $(g_n)_{n\in\mathbb{N}}$ of functions $g_n\colon X^n \to X$.

We are interested in the associativity property, traditionally considered on binary functions. Recall that a function $f\colon X^2 \to X$ is said to be *associative* if $f(f(xy)z) = f(xf(yz))$ for every $x, y, z \in X$. This algebraic property was extended to functions $f\colon X^n \to X$, $n \geqslant 1$, as well as to functions $g\colon X^* \to X$ in somewhat different ways.

A function $f\colon X^n \to X$ is said to be *associative* if, for every $\mathbf{xz}, \mathbf{x'z'} \in X^{n-1}$ and every $\mathbf{y}, \mathbf{y'} \in X^n$ such that $\mathbf{xyz} = \mathbf{x'y'z'}$, we have $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{x'}f(\mathbf{y'})\mathbf{z'})$. This generalization of associativity to n-ary functions goes back to Dörnte [4] and led to the generalization of groups to n-groups (polyadic groups).[1] In a somewhat different context, this notion has been recently used to completely classify closed intervals made of equational classes of Boolean functions; see [2].

On a different setting, associativity can be generalized to functions on X^* as follows. We say that a function $g\colon X^* \to X$ is *associative* if, for every $\mathbf{xyz}, \mathbf{x'y'z'} \in X^*$ such that $\mathbf{xyz} = \mathbf{x'y'z'}$, we have $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x'}g(\mathbf{y'})\mathbf{z'})$. Alternative formulations of this definition appeared in the theory of aggregation functions, where the arity is not always fixed; see for instance [1, 10, 12, 13].

In general, the latter definition is more restrictive on the components g_n of $g\colon X^* \to X$. For instance, the ternary real function $f(xyz) = x - y + z$ is associative but cannot be the ternary component of an associative function $g\colon \mathbb{R}^* \to \mathbb{R}$.

However, in the case of lattice polynomial functions (i.e., functions which can be obtained as combinations of projections and constant functions using the lattice operations $\wedge$ and $\vee$) the two notions of associativity are essentially the same. More precisely, given a bounded distributive lattice L, we have that a polynomial function $f\colon L^n \to L$ is associative if and only if it is the n-ary component of some associative function $g\colon L^* \to L$ (see [3]).

These two examples give rise to the following problem:

Problem. *For any fixed integer $n \geqslant 1$, determine the class of functions $f\colon X^n \to X$ which satisfy the following property: f is associative if and only if there exists an associative function $g\colon X^* \to X$ such that $g_n = f$.*

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[1]The first extensive study on polyadic groups was due to Post [15]. This study was followed by several contributions towards the classification and description of n-groups and similar "super-associative" structures; to mention a few, see [5, 6, 7, 8, 9, 11, 14, 16].

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