Reliability analysis and lattice polynomial system representation

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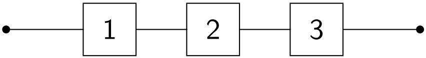
System

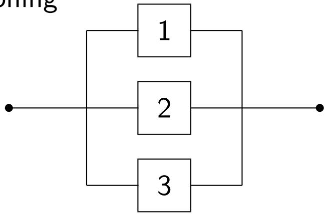
Definition. A *system* consists of several interconnected units

Assumptions:

1. The system and the units are of the crisply *on/off* kind
2. A serially connected segment of units is functioning if and only if every single unit is functioning



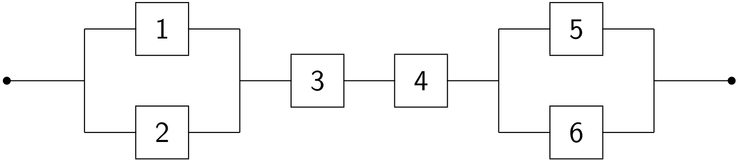
3. A system of parallel units is functioning if and only at least one unit is functioning



System

Example. Home video system

1. Blu-ray player
2. DVD player
3. LCD monitor
4. Amplifier
5. Speaker A
6. Speaker B



Structure function

Definition.
The *state of a component* $i \in [n] = \{1, \ldots, n\}$ can be represented by a Boolean variable

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is in a failed state} \end{cases}$$

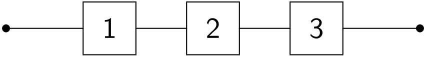
The *state of the system* is described from the component states through a Boolean function $\phi : \{0,1\}^n \to \{0,1\}$

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state} \end{cases}$$

This function is called the *structure function* of the system

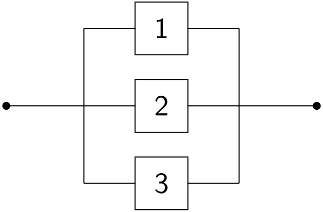
Structure function

Series structure



$$\phi(\mathbf{x}) = x_1 \ x_2 \ x_3 = \prod_{i=1}^{3} x_i$$

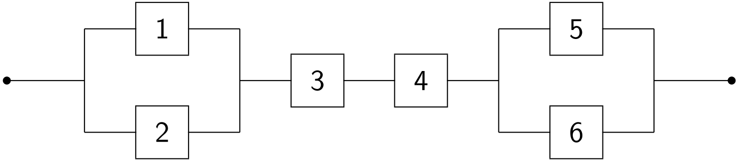
Parallel structure



$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2)(1 - x_3) = \coprod_{i=1}^{3} x_i$$

Structure function

Home video system



$$\phi(\mathbf{x}) = (x_1 \amalg x_2)\ x_3\ x_4\ (x_5 \amalg x_6)$$

Coherent and semicoherent systems

Definition.
Let $\phi : \{0,1\}^n \to \{0,1\}$ be a structure function on $[n] = \{1, ..., n\}$.

The system is said to be *semicoherent* if

- ϕ is nondecreasing : $\mathbf{x} \leqslant \mathbf{x}' \;\Rightarrow\; \phi(\mathbf{x}) \leqslant \phi(\mathbf{x}')$
- $\phi(\mathbf{0}) = 0$, $\phi(\mathbf{1}) = 1$

The system is said to be *coherent* if, in addition

- every component is relevant to ϕ:

$$\exists \mathbf{x} \in \{0,1\}^n \;:\; \phi(1_i, \mathbf{x}) \neq \phi(0_i, \mathbf{x})$$

where

$$\begin{aligned} (1_i, \mathbf{x}) &= (x_1, \ldots, \overset{(i)}{1}, \ldots, x_n) \\ (0_i, \mathbf{x}) &= (x_1, \ldots, \overset{(i)}{0}, \ldots, x_n) \end{aligned}$$

Representations of Boolean functions

Boolean function	$\longleftrightarrow$	set function
$\phi : \{0,1\}^n \to \{0,1\}$		$v : 2^{[n]} \to \{0,1\}$

$$v(A) = \phi(\mathbf{e}_A) \qquad A \subseteq [n]$$

$\to$ We write ϕ_v instead of ϕ

Representations of a Boolean function

$$\phi_v(\mathbf{x}) = \sum_{A \subseteq [n]} v(A) \prod_{i \in A} x_i \prod_{i \in [n] \setminus A} (1 - x_i)$$

Representations of Boolean functions

Alternative representations

$$\phi_v(\mathbf{x}) = \sum_{A\subseteq[n]} m_v(A) \prod_{i\in A} x_i$$

where

$$m_v(A) = \sum_{B\subseteq A} (-1)^{|A|-|B|}\, v(B)$$

If ϕ_v is nondecreasing and nonconstant:

$$\phi_v(\mathbf{x}) = \coprod_{\substack{A\subseteq[n]\\ v(A)=1}} \prod_{i\in A} x_i$$

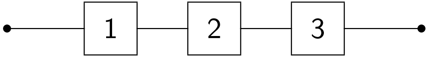
(Hammer and Rudeanu 1968)

System and component lifetimes

Any component $i \in [n]$ has a random lifetime : T_i
The system has a random lifetime : T_S

> The structure function induces a functional relationship between the variables $T_1, \ldots, T_n$ and the variable T_S

Example:

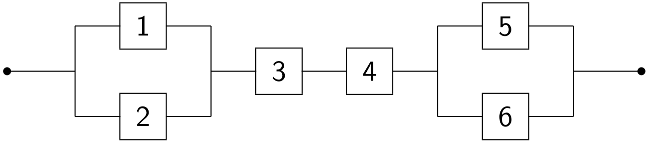


$$\phi(\mathbf{x}) = x_1 \; x_2 \; x_3 = \prod_{i=1}^{3} x_i$$

$$T_S = T_1 \wedge T_2 \wedge T_3 = \bigwedge_{i=1}^{3} T_i$$

System and component lifetimes

Home video system



$$\phi(\mathbf{x}) = (x_1 \amalg x_2)\ x_3\ x_4\ (x_5 \amalg x_6)$$

$$T_S = (T_1 \vee T_2) \wedge T_3 \wedge T_4 \wedge (T_5 \vee T_6)$$

In general,

$$T_S = p(T_1, \ldots, T_n)$$

where $p : \mathbb{R}_+^n \to \mathbb{R}_+$ is an n-ary lattice polynomial function

$\Rightarrow$ Formal parallelism between two representations of systems: *structure functions* and *lattice polynomial functions*

Lattice polynomial functions

Let $L \subseteq [-\infty, \infty]$ a totally ordered bounded lattice
$\Rightarrow \wedge = \min$ and $\vee = \max$

The class of n-ary lattice polynomial (l.p.) functions is defined as follows:

(i) For any $k \in [n]$, the projection $(t_1, \ldots, t_n) \mapsto t_k$ is an n-ary l.p. function

(ii) If p and q are n-ary l.p. functions then $p \wedge q$ and $p \vee q$ are n-ary l.p. functions

(iii) Every n-ary l.p. function is constructed by finitely many applications of the rules (i) and (ii).

Example:

$$p(t_1, t_2, t_3) = (t_1 \wedge t_2) \vee t_3$$

Lattice polynomial functions

Let $a = \inf L$ and $b = \sup L$

l.p. function	$\longleftrightarrow$	set function
$p : L^n \to L$		$w : 2^{[n]} \to \{a, b\}$

$$w(A) = p(\mathbf{e}_A^{a,b}) \qquad A \subseteq [n]$$

Example : $\mathbf{e}_{\{1,2\}}^{a,b} = (b, b, a, \ldots, a)$

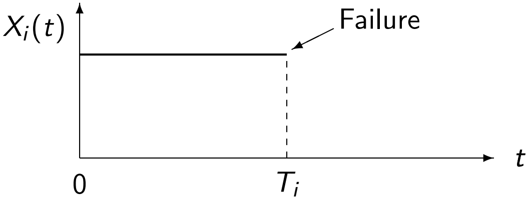
$\to$ We write p_w instead of p

Representations of an l.p. function (Birkhoff 1967)

$$p_w(\mathbf{t}) = \bigvee_{\substack{A \subseteq [n] \\ w(A) = b}} \bigwedge_{i \in A} t_i$$

Formal parallelism between the two representations

T_i = random lifetime of component $i \in [n]$
$X_i(t) = \mathrm{Ind}(T_i > t)$ = random state of i at time $t \geqslant 0$



$$X_i(t) = \begin{cases} 1 & \text{if } i \text{ is functioning at time } t \\ 0 & \text{if } i \text{ is in a failed state at time } t \end{cases}$$

For the system:
T_S = system lifetime
$X_S(t) = \mathrm{Ind}(T_S > t)$ = random state of the system at time $t \geqslant 0$

Formal parallelism between the two representations

Home video system

$$p_w(\mathbf{T}) = (T_1 \vee T_2) \wedge T_3 \wedge T_4 \wedge (T_5 \vee T_6)$$
$$\phi_v(\mathbf{X}(t)) = \big(X_1(t) \amalg X_2(t)\big)\; X_3(t)\; X_4(t)\; \big(X_5(t) \amalg X_6(t)\big)$$

ϕ_v is also an l.p. function that has just the same max-min form as p_w but applied to binary arguments

$$\phi_v \qquad \longleftrightarrow \qquad p_w$$
$$w = \gamma \circ v$$
$$\gamma\colon \{0,1\} \to \{a,b\},\; \gamma(0) = a,\, \gamma(1) = b$$

As the lifetimes are $[0,\infty]$-valued, we now assume that $a = 0$ and $b = \infty$

Formal parallelism between the two representations

Theorem. (Dukhovny and M. 2008)

Consider a system whose structure function $\phi_v : \{0,1\}^n \to \{0,1\}$ is nondecreasing and nonconstant. Then we have

$$T_S = p_w(T_1, \ldots, T_n) \tag{1}$$

where $w = \gamma \circ v$. Conversely, any system fulfilling (1) for some l.p. function $p_w : L^n \to L$ has the nondecreasing and nonconstant structure function ϕ_v, where $v = \gamma^{-1} \circ w$

The proof mainly lies on the immediate identities

$$\begin{aligned} \mathrm{Ind}(E \wedge E') &= \mathrm{Ind}(E) \wedge \mathrm{Ind}(E') \\ \mathrm{Ind}(E \vee E') &= \mathrm{Ind}(E) \vee \mathrm{Ind}(E') \end{aligned}$$

valid for all events E and E'

Formal parallelism between the two representations

Proof. For every $t \geqslant 0$ we have

$$\begin{aligned}
\phi_v(\mathbf{X}(t)) &= \coprod_{\substack{A\subseteq[n]\\ v(A)=1}} \prod_{i\in A} X_i(t) \\
&= \bigvee_{\substack{A\subseteq[n]\\ v(A)=1}} \bigwedge_{i\in A} \mathrm{Ind}(T_i > t) = \mathrm{Ind}\Big(\bigvee_{\substack{A\subseteq[n]\\ v(A)=1}} \bigwedge_{i\in A} T_i > t\Big) \\
&= \mathrm{Ind}(p_w(\mathbf{T}) > t)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& X_S(t) = \phi_v(\mathbf{X}(t)) \qquad \forall t \geqslant 0 \\
\Leftrightarrow\quad & \mathrm{Ind}(T_S > t) = \mathrm{Ind}(p_w(\mathbf{T}) > t) \qquad \forall t \geqslant 0 \\
\Leftrightarrow\quad & T_S = p_w(\mathbf{T})
\end{aligned}$$

□

Advantages of the lattice polynomial language

Properties of l.p. functions reveal properties of structure functions

Example. Any l.p. function $p\colon L^n \to L$ satisfies trivially the functional equations

$$\begin{aligned} p(u \wedge t_1, \ldots, u \wedge t_n) &= u \wedge p(t_1, \ldots, t_n) \qquad \forall u \in L \\ p(u \vee t_1, \ldots, u \vee t_n) &= u \vee p(t_1, \ldots, t_n) \qquad \forall u \in L \end{aligned}$$

The corresponding equations for the structure functions are

$$\begin{aligned} \phi(y\, x_1, \ldots, y\, x_n) &= y\ \phi(x_1, \ldots, x_n) \qquad \forall y \in \{0,1\} \\ \phi(y \amalg x_1, \ldots, y \amalg x_n) &= y \amalg \phi(x_1, \ldots, x_n) \qquad \forall y \in \{0,1\} \end{aligned}$$

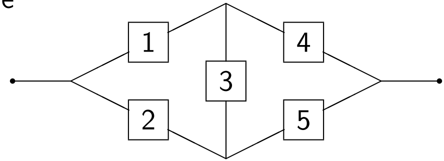
Advantages of the lattice polynomial language

Properties of structure functions reveal properties of l.p. functions

Example. Pivotal decomposition of the structure function

$$\phi(\mathbf{x}) = x_i\,\phi(1_i,\mathbf{x}) + (1-x_i)\phi(0_i,\mathbf{x})$$

Bridge structure



$$\phi(\mathbf{x}) = x_3\,\phi(1_3,\mathbf{x}) + (1-x_3)\phi(0_3,\mathbf{x})$$

$$\begin{aligned} \phi(1_3,\mathbf{x}) &= (x_1 \amalg x_2)(x_4 \amalg x_5) \\ \phi(0_3,\mathbf{x}) &= (x_1\ x_4) \amalg (x_2\ x_5) \end{aligned}$$

Advantages of the lattice polynomial language

Corresponding property of the l.p. functions ?

$$p(\mathbf{t}) = \mathrm{median}\big(p(a_i,\mathbf{t}),\ t_i,\ p(b_i,\mathbf{t})\big)$$

where

$$\mathrm{median}(x_1,x_2,x_3) = (x_1\wedge x_2)\vee(x_2\wedge x_3)\vee(x_3\wedge x_1)$$

Proof.

$$\begin{aligned}
\phi(\mathbf{x}) &= \mathrm{median}\big(\phi(0_i,\mathbf{x}),\ x_i,\ \phi(1_i,\mathbf{x})\big)\\
&= \underbrace{\big(\phi(0_i,\mathbf{x})\wedge x_i\big)}_{\textit{redundant}}\vee\big(x_i\wedge\phi(1_i,\mathbf{x})\big)\vee\underbrace{\big(\phi(1_i,\mathbf{x})\wedge\phi(0_i,\mathbf{x})\big)}_{=\phi(0_i,\mathbf{x})}\\
&= x_i\,\phi(1_i,\mathbf{x})\ \amalg\ \phi(0_i,\mathbf{x})\\
&= x_i\,\phi(1_i,\mathbf{x}) + \phi(0_i,\mathbf{x}) - x_i\ \underbrace{\phi(1_i,\mathbf{x})}_{\textit{redundant}}\,\phi(0_i,\mathbf{x})\\
&= x_i\,\phi(1_i,\mathbf{x}) + (1-x_i)\phi(0_i,\mathbf{x})
\end{aligned}$$

□

Reliability analysis

Reliability function of component $i \in [n]$

$$R_i(t) = \Pr(T_i > t) = \Pr(X_i(t) = 1) = \mathrm{E}[X_i(t)]$$

$=$ probability that component i does not fail in the interval $[0, t]$

System reliability function

$$R_S(t) = \Pr(T_S > t) = \Pr(X_S(t) = 1) = \mathrm{E}[X_S(t)]$$

$=$ probability that the system does not fail in the interval $[0, t]$

Reliability analysis

Theorem. (Dukhovny 2007)

$$R_S(t) = \sum_{A\subseteq[n]} v(A)\,\Pr(\mathbf{X}(t) = \mathbf{e}_A)$$

Remarks.

(i) All the needed information is the distribution of $\mathbf{X}(t)$ (the knowledge of the joint distribution of $\mathbf{T}$ is not necessary)

(ii) The distribution of $\mathbf{X}(t)$ can be easily expressed in terms of the *joint probability generating function* of $\mathbf{X}(t)$

$$G(\mathbf{z},t) = \mathrm{E}\Big[\prod_{i=1}^{n} z_i^{X_i(t)}\Big] \qquad (|z_i|\leqslant 1).$$

We have

$$\Pr(\mathbf{X}(t) = \mathbf{e}_A) = \sum_{B\subseteq A} (-1)^{|A|-|B|}\, G(\mathbf{e}_B,t)$$

Reliability analysis

When $T_1, \ldots, T_n$ are independent, we have

$$R_S(t) = \sum_{A \subseteq [n]} v(A) \prod_{i \in A} R_i(t) \prod_{i \in [n] \setminus A} (1 - R_i(t))$$

Alternative expression (Dukhovny and M. 2008)

$$R_S(t) = \sum_{A \subseteq [n]} m_v(A) \Pr(T_i > t \ \forall i \in A)$$

In case of independence

$$R_S(t) = \sum_{A \subseteq [n]} m_v(A) \prod_{i \in A} R_i(t)$$

Mean time-to-failure of the system

The *mean time-to-failure* of the system is defined as

$$\mathrm{MTTF}_S = \mathrm{E}[T_S]$$

It is easy to show that

$$\mathrm{MTTF}_S = \int_0^\infty R_S(t)\, dt$$

(Rausand and Høyland 2004)

In case of independence

$$\mathrm{MTTF}_S = \sum_{A \subseteq [n]} v(A) \int_0^\infty \prod_{i \in A} R_i(t) \prod_{i \in [n] \setminus A} (1 - R_i(t))\, dt$$

$$\mathrm{MTTF}_S = \sum_{A \subseteq [n]} m_v(A) \int_0^\infty \prod_{i \in A} R_i(t)\, dt$$

Mean time-to-failure of the system

Example. Assume $R_i(t) = e^{-\lambda_i t}$, $i = 1, \ldots, n$

$$\begin{aligned}
\mathrm{MTTF}_S &= \sum_{A \subseteq [n]} m_v(A) \int_0^\infty \prod_{i \in A} e^{-\lambda_i t}\, dt \\
&= \sum_{A \subseteq [n]} m_v(A) \int_0^\infty e^{-\lambda_A t}\, dt \qquad \Big(\lambda_A = \sum_{i \in A} \lambda_i\Big) \\
&= \sum_{\substack{A \subseteq [n] \\ A \neq \varnothing}} m_v(A) \, \frac{1}{\lambda_A}
\end{aligned}$$

Series structure: $\mathrm{MTTF}_S = \frac{1}{\lambda_{[n]}}$

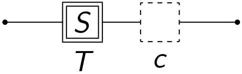
Parallel structure: $\mathrm{MTTF}_S = \sum_{\substack{A \subseteq [n] \\ A \neq \varnothing}} (-1)^{|A|-1} \frac{1}{\lambda_A}$

Another advantage of the lattice polynomial language

Generalization to weighted lattice polynomial functions

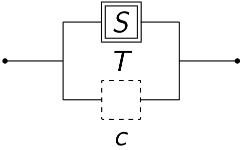
Suppose there are

(i) collective upper bounds on lifetimes of certain subsets of units (imposed by the physical properties of the assembly)



subset lifetime $= T \wedge c$

(ii) collective lower bounds (imposed by back-up blocks with constant lifetimes)



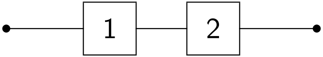
subset lifetime $= T \vee c$

Another advantage of the lattice polynomial language

The lifetime of a general system with upper and/or lower bounds can be described through a weighted lattice polynomial function

$$T_S = p(T_1, \ldots, T_n)$$

Example.



Suppose that the lifetime of component #2 must lies in the time interval $[c, d]$

$$\begin{aligned} T_S &= T_1 \wedge \mathrm{median}(c, T_2, d) \\ &= T_1 \wedge \big(c \vee (T_2 \wedge d)\big) \\ &= (c \wedge T_1) \vee (d \wedge T_1 \wedge T_2) \end{aligned}$$

Weighted lattice polynomial functions

The class of n-ary weighted lattice polynomial (w.l.p.) functions is defined as follows:

(i) For any $k \in [n]$ and any $c \in L$, the projection $(t_1, \ldots, t_n) \mapsto t_k$ and the constant function $(t_1, \ldots, t_n) \mapsto c$ are n-ary w.l.p. function

(ii) If p and q are n-ary w.l.p. functions then $p \wedge q$ and $p \vee q$ are n-ary w.l.p. functions

(iii) Every n-ary w.l.p. function is constructed by finitely many applications of the rules (i) and (ii).

Weighted lattice polynomial functions

w.l.p. function	$\longleftrightarrow$	set function
$p : L^n \to L$		$w : 2^{[n]} \to L$

$$w(A) = p(\mathbf{e}_A^{a,b}) \qquad A \subseteq [n]$$

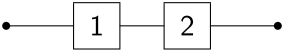
$\to$ We write p_w instead of p

Representations of a w.l.p. function (Goodstein 1967)

$$p_w(\mathbf{t}) = \bigvee_{A \subseteq [n]} \Big(w(A) \wedge \bigwedge_{i \in A} t_i \Big)$$

Weighted lattice polynomial functions

Example (cont'd)



$$T_S = (c \wedge T_1) \vee (d \wedge T_1 \wedge T_2)$$

$$p_w(t_1, t_2) = (c \wedge t_1) \vee (d \wedge t_1 \wedge t_2)$$

A	$w(A)$
$\varnothing$	0
$\{1\}$	c
$\{2\}$	0
$\{1,2\}$	d

We can show that

$$X_S(t) = \big(\mathrm{Ind}(c > t)\ X_1(t)\big)\ \amalg\ \big(\mathrm{Ind}(d > t)\ X_1(t)\ X_2(t)\big)$$

Weighted lattice polynomial functions

Representation of w.l.p. functions (DNF)

$$p_w(\mathbf{t}) = \bigvee_{A\subseteq[n]} w(A) \wedge \bigwedge_{i\in A} t_i$$

Theorem. (Dukhovny and M. 2008)

If $T_S = p_w(T_1, \ldots, T_n)$ then

$$X_S(t) = \coprod_{A\subseteq[n]} v_t(A) \prod_{i\in A} X_i(t) \qquad (t \geqslant 0)$$

where $v_t(A) = \mathrm{Ind}(w(A) > t)$

Reliability analysis

Exact reliability formulas (Dukhovny and M. 2008)

$$R_S(t) = \sum_{A\subseteq[n]} v_t(A)\,\Pr(\mathbf{X}(t) = \mathbf{e}_A)$$

$$R_S(t) = \sum_{A\subseteq[n]} m_{v_t}(A)\,\Pr(T_i > t\ \forall i \in A)$$

In case of independence

$$R_S(t) = \sum_{A\subseteq[n]} v_t(A) \prod_{i\in A} R_i(t) \prod_{i\in[n]\setminus A} (1 - R_i(t))$$

$$R_S(t) = \sum_{A\subseteq[n]} m_{v_t}(A) \prod_{i\in A} R_i(t)$$

Mean time-to-failure of the system

$$
\begin{aligned}
\mathrm{MTTF}_S &= \int_0^\infty R_S(t)\,dt \\
&= \sum_{A\subseteq[n]} \int_0^\infty m_{v_t}(A) \prod_{i\in A} R_i(t)\,dt \\
&= \sum_{A\subseteq[n]} \int_0^\infty \left(\sum_{B\subseteq A} (-1)^{|A|-|B|} v_t(B) \right) \prod_{i\in A} R_i(t)\,dt \\
&= \sum_{A\subseteq[n]} \sum_{B\subseteq A} (-1)^{|A|-|B|} \int_0^\infty \mathrm{Ind}(w(B) > t) \prod_{i\in A} R_i(t)\,dt \\
&= \sum_{A\subseteq[n]} \sum_{B\subseteq A} (-1)^{|A|-|B|} \int_0^{w(B)} \prod_{i\in A} R_i(t)\,dt
\end{aligned}
$$

Mean time-to-failure of the system

Example. Assume $R_i(t) = e^{-\lambda_i t}$, $i = 1, \ldots, n$

$$\begin{aligned}
\mathrm{MTTF}_S &= \sum_{A \subseteq [n]} \sum_{B \subseteq A} (-1)^{|A|-|B|} \int_0^{w(B)} \prod_{i \in A} e^{-\lambda_i t}\, dt \\
&= \sum_{A \subseteq [n]} \sum_{B \subseteq A} (-1)^{|A|-|B|} \int_0^{w(B)} e^{-\lambda_A t}\, dt \qquad \Big(\lambda_A = \sum_{i \in A} \lambda_i\Big) \\
&= w(\varnothing) + \sum_{\substack{A \subseteq [n] \\ A \neq \varnothing}} \sum_{B \subseteq A} (-1)^{|A|-|B|} \frac{1 - e^{-\lambda_A w(B)}}{\lambda_A}
\end{aligned}$$

Conclusion

We have discussed the formal parallelism between two representations of systems

- Structure functions
- Lattice polynomial functions

$\rightarrow$ Their languages are equivalent in many ways

Advantages

- Generalization to w.l.p. functions + exact reliability formulas
- Exact formulas for the distribution functions of w.l.p. functions of random variables

$$Y = p_w(X_1, \ldots, X_n)$$

- Several special cases can be investigated
  - Symmetric w.l.p. functions : $w(A) = f(|A|)$
  - The reliability of any subsystem depends only on the number of units in the subsystem $\Pr(\mathbf{X}(t) = \mathbf{e}_A) = g_t(|A|)$
  - $\cdots$

Thank you for your attention !

A. Dukhovny and J.-L. Marichal, *Reliability analysis of semicoherent systems through their lattice polynomial descriptions*

arXiv : 0809.1332