# Explicit Descriptions of Associative Sugeno Integrals

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**Abstract.** The associativity property, usually defined for binary functions, can be generalized to functions of a given fixed arity $n \geq 1$ as well as to functions of multiple arities. In this paper, we investigate these two generalizations in the case of Sugeno integrals over bounded distributive lattices and present explicit descriptions of the corresponding associative functions. We also show that, in this case, both generalizations of associativity are essentially the same.

**Key words:** bounded distributive lattice, Sugeno integral, associativity, idempotency, functional equation

## 1 Introduction

Let $X$ be an arbitrary nonempty set. Throughout this paper, we regard vectors $\mathbf{x}$ in $X^n$ as $n$-strings over $X$. The 0-string or *empty* string is denoted by $\varepsilon$ so that $X^0 = \{\varepsilon\}$. We denote by $X^*$ the set of all strings over $X$, that is, $X^* = \bigcup_{n\in\mathbb{N}} X^n$. Moreover, we consider $X^*$ endowed with concatenation for which we adopt the juxtaposition notation. For instance, if $\mathbf{x} \in X^n$, $y \in X$, and $\mathbf{z} \in X^m$, then $\mathbf{x}y\mathbf{z} \in X^{n+1+m}$. Furthermore, for $\mathbf{x} \in X^m$, we use the short-hand notation $\mathbf{x}^n = \mathbf{x}\cdots\mathbf{x} \in X^{n\times m}$. In the sequel, we will be interested both in functions of a given fixed arity (i.e., functions $f\colon X^n \to X$) as well as in functions defined on $X^*$, that is, of the form $g\colon X^* \to X$. Given a function $g\colon X^* \to X$, we denote by $g_n$ the restriction of $g$ to $X^n$, i.e. $g_n := g|_{X^n}$. In this way, each function $g\colon X^* \to X$ can be regarded as a family $(g_n)_{n\in\mathbb{N}}$ of functions $g_n\colon X^n \to X$. We convey that $g_0$ is defined by $g_0(\varepsilon) = \varepsilon$.

In this paper, we are interested in the associativity property, traditionally considered on binary functions. Recall that a function $f\colon X^2 \to X$ is said to be *associative* if $f(f(xy)z) = f(xf(yz))$ for every $x, y, z \in X$. The importance of this notion is made clear by its natural interpretation. Essentially, it expresses the fact that the order in which variables are bracketed is not relevant. This algebraic property was extended to functions $f\colon X^n \to X$, $n \geq 1$, as well as to functions $g\colon X^* \to X$ in somewhat different ways.

A function $f\colon X^n \to X$ is said to be *associative* if, for every $\mathbf{x}\mathbf{z}, \mathbf{x}'\mathbf{z}' \in X^{n-1}$ and every $\mathbf{y}, \mathbf{y}' \in X^n$ such that $\mathbf{x}\mathbf{y}\mathbf{z} = \mathbf{x}'\mathbf{y}'\mathbf{z}'$, we have $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{x}'f(\mathbf{y}')\mathbf{z}')$.

This generalization of associativity to $n$-ary functions goes back to Dörnte [6] and led to the generalization of groups to $n$-groups (polyadic goups).[1] In a somewhat different context, this notion has been recently used to completely classify closed intervals made of equational classes of Boolean functions; see [2].

On a different setting, associativity can be generalized to functions on $X^*$ as follows. We say that a function $g\colon X^* \to X$ is *associative* if, for every $\mathbf{xyz}, \mathbf{x'y'z'} \in X^*$ such that $\mathbf{xyz} = \mathbf{x'y'z'}$, we have $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{x'}g(\mathbf{y'})\mathbf{z'})$. Alternative formulations of this definition appeared in the theory of aggregation functions, where the arity is not always fixed; see for instance [1, 14, 16, 17].

In general, the latter definition is more restrictive on the components $g_n$ of $g\colon X^* \to X$. For instance, the ternary real function $f(xyz) = x - y + z$ is associative but cannot be the ternary component of an associative function $g\colon \mathbb{R}^* \to \mathbb{R}$. Indeed, the equations

$$g_2(g_2(xy)z) = g_2(xg_2(yz)) = x - y + z \tag{1}$$

have no solution, for otherwise we would have $y = g_2(g_2(y0)0)$ and hence

$$g_2(xy) = g_2(xg_2(g_2(y0)0)) = g_2(g_2(xg_2(y0))0) = x - g_2(y0).$$

This would imply $g_2(xy) = x - y$, which contradicts (1).

In this paper we show that, in the case of Sugeno integrals on bounded distributive lattices, the two notions of associativity are essentially the same. More precisely, given a bounded distributive lattice $L$, we have that a Sugeno integral $f\colon L^n \to L$ is associative if and only if it is the $n$-ary component of some associative function $g\colon L^* \to L$; see Corollary 7. This paper is organized as follows: in Sect. 2 we provide some preliminary results, which are then used in Sect. 3 to obtain explicit descriptions of those associative Sugeno integrals; see Theorems 4 and 6.

## 2 Preliminary Results

The following proposition provides useful reformulations of associativity of functions $g\colon X^* \to X$.

**Proposition 1.** *Let $g\colon X^* \to X$ be a function. The following assertions are equivalent:*

*(i) $g$ is associative.*
*(ii) For every $\mathbf{xyz} \in X^*$, we have $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{xyz})$.*
*(iii) For every $\mathbf{xy} \in X^*$, we have $g(g(\mathbf{x})g(\mathbf{y})) = g(\mathbf{xy})$.*

[1] The first extensive study on polyadic groups was due to Post [20]. This study was followed by several contributions towards the classification and description of $n$-groups and similar "super-associative" structures; to mention a few, see [7–9, 11, 12, 15, 19].

*Remark 2.* (i) Associativity of functions $g\colon X^* \to X$ was defined in [16] and [17] as in assertions (*iii*) and (*ii*) of Proposition 1, respectively. For a recent reference, see [14].

(ii) As observed in [1], associative functions $g\colon X^* \to X$ are completely determined by their unary and binary components. Indeed, for every $n \in \mathbb{N}$, $n > 2$, and every $x_1, \ldots, x_n \in X$, we have

$$g(x_1 \cdots x_n) = g_2(g_2(\cdots g_2(g_2(x_1 x_2)x_3)\cdots)x_n).$$

## 3 Associative Sugeno Integrals

Let $L$ be a bounded distributive lattice, with 0 and 1 as bottom and top elements. A (*lattice*) *polynomial function* is any mapping $f\colon L^n \to L$ which can be obtained as combinations of projections and constant functions using the lattice operations $\wedge$ and $\vee$. Our interest in these lattice polynomial functions comes from the fact that, as observed in [18], (*discrete*) *Sugeno integrals* can be regarded as idempotent lattice polynomial functions, that is, polynomial functions satisfying $f(x^n) = x$ for every $x \in X$. This view has several appealing aspects, in particular, concerning normal form representations of Sugeno integrals. Indeed, as shown by Goodstein [13], polynomial functions on bounded distributive lattices coincide exactly with those functions representable in disjunctive normal form (DNF).

More precisely, for $I \subseteq [n] = \{1, \ldots, n\}$, let $\mathbf{e}_I \in \{0,1\}^n$ be the characteristic vector of $I$ and let $\alpha_f\colon 2^{[n]} \to L$ be the function given by $\alpha_f(I) = f(\mathbf{e}_I)$. Then

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \big(\alpha_f(I) \wedge \bigwedge_{i \in I} x_i\big). \tag{2}$$

Thus, a function $f\colon L^n \to L$ is a Sugeno integral integral if and only if $f$ fulfills (2) with $\alpha_f(\emptyset) = 0$ and $\alpha_f([n]) = 1$. For further background, see [3, 4].

**Theorem 3 ([4]).** *A function $f\colon L^n \to L$ is a Sugeno integral if and only if it is idempotent and satisfies*

$$f(\mathbf{x}y\mathbf{z}) = \mathrm{med}\big(f(\mathbf{x}0\mathbf{z}), y, f(\mathbf{x}1\mathbf{z})\big), \quad \textit{for every } \mathbf{x}y\mathbf{z} \in L^n. \tag{3}$$

The following theorem is an immediate consequence of Theorem 6 in [5] and it restricts the disjunctive normal form of $n$-ary Sugeno integrals.

**Theorem 4.** *Let $f\colon L^n \to L$ be a Sugeno integral. If $f$ is associative, then*

$$f(\mathbf{x}) = (b_n \wedge x_1) \vee \Big(\bigvee_{i=1}^{n} (b_n \wedge c_n \wedge x_i)\Big) \vee (c_n \wedge x_n) \vee \bigwedge_{i=1}^{n} x_i, \tag{4}$$

*where $b_n = f(10^{n-1})$ and $c_n = f(0^{n-1}1)$.*

*Remark 5.* (i) We observe that equation (4) can be rewritten in a more symmetric way as

$$f(\mathbf{x}) = (b_n \wedge x_1) \vee \mathrm{med}\Big(\bigwedge_{i=1}^{n} x_i, b_n \wedge c_n, \bigvee_{i=1}^{n} x_i\Big) \vee (c_n \wedge x_n).$$

This formula reduces to $f(\mathbf{x}) = \mathrm{med}\big(\bigwedge_{i=1}^{n} x_i, b_n, \bigvee_{i=1}^{n} x_i\big)$ as soon as $f$ is a symmetric function (i.e., invariant under permutation of its variables).

(ii) A *term function* $f\colon L^n \to L$ is a Sugeno integral satisfying $\alpha_f(I) \in \{0,1\}$ for every $I \subseteq [n]$. By Theorem 4, the only associative term functions $f\colon L^n \to L$ are $\mathbf{x} \mapsto x_1$, $\mathbf{x} \mapsto x_n$, $\mathbf{x} \mapsto \bigwedge_{i=1}^{n} x_i$, and $\mathbf{x} \mapsto \bigvee_{i=1}^{n} x_i$.

We say that a function $g\colon L^* \to L$ is a *Sugeno integral* if every $g_n$, $n \geq 1$, is a Sugeno integral. The following theorem yields a description of associative Sugeno integrals $g\colon L^* \to L$. For a generalization to polynomial functions $g\colon L^* \to L$, see Theorem 7 in [5].

**Theorem 6.** *A Sugeno integral $g\colon L^* \to L$ is associative if and only if $g_1(x) = x$ and, for $n \geq 2$,*

$$g_n(\mathbf{x}) = (b_2 \wedge x_1) \vee \Big(\bigvee_{i=1}^{n} (b_2 \wedge c_2 \wedge x_i)\Big) \vee (c_2 \wedge x_n) \vee \bigwedge_{i=1}^{n} x_i, \qquad (5)$$

*where $b_2 = g_2(10)$ and $c_2 = g_2(01)$.*

*Proof.* Sufficiency can be verified by making use of Proposition 1.

To verify that the conditions are necessary, note that since each $g_n$ is associative, by Theorem 4, each $g_n$ has the form (4) with $b_n = g_n(10^{n-1})$ and $c_n = g_n(0^{n-1}1)$. By associativity and Theorem 3, for every $n \geq 3$,

$$g_n(10^{n-1}) = g_2(g_{n-1}(10^{n-2})0) = \mathrm{med}(0, g_{n-1}(10^{n-2}), g_2(10)).$$

By reasoning recursively, one can see that $b_n = b_2$. Similarly, one can verify that $c_n = c_2$, for every $n \geq 3$. □

Even though associativity for functions $g\colon L^* \to L$ seems more restrictive on their components $g_n$ than associativity for functions of a given fixed arity, from Theorems 4 and 6 it follows that associativity for Sugeno integrals $f\colon L^n \to L$ naturally extends componentwise to Sugeno integrals $g\colon L^* \to L$.

**Corollary 7.** *Let $f\colon L^n \to L$ be a Sugeno integral. Then $f$ is associative if and only if there is an associative Sugeno integral $g\colon L^* \to L$ such that $g_n = f$.*

*Proof.* Clearly, the condition is sufficient. Conversely, if $f$ is associative, then by Theorem 4

$$f(\mathbf{x}) = (b_n \wedge x_1) \vee \Big(\bigvee_{i=1}^{n} (b_n \wedge c_n \wedge x_i)\Big) \vee (c_n \wedge x_n) \vee \bigwedge_{i=1}^{n} x_i,$$

where $b_n = f(10^{n-1})$ and $c_n = f(0^{n-1}1)$. Let $g\colon L^* \to L$ be the Sugeno integral such that $g_1(x) = x$ and, for $m \geq 2$,

$$g_m(\mathbf{x}) = (b \wedge x_1) \vee \Big(\bigvee_{i=1}^{m}(b \wedge c \wedge x_i)\Big) \vee (c \wedge x_n) \vee \bigwedge_{i=1}^{m} x_i,$$

where $b = f(10^{n-1})$ and $c = f(0^{n-1}1)$. Clearly, $g_n = f$ and by Theorem 6 we have that $g$ is associative. □

*Remark 8.* (i) The case when $L$ is a connected order topological space was considered by Fodor [10] who obtained an explicit description of those nondecreasing binary functions which are idempotent, continuous, and associative.

(ii) Many associative functions $g\colon L^* \to L$ have been investigated in aggregation theory in the special case when $L$ is the real unit interval $[0,1]$; see, e.g., [14]. To give an example, there are only four such associative functions whose $n$-ary restrictions are nondecreasing and stable under interval scale transformations (i.e., $g_n$ commutes with unary positive affine functions), namely, $\mathbf{x} \mapsto x_1$, $\mathbf{x} \mapsto x_n$, $\mathbf{x} \mapsto \bigwedge_{i=1}^{n} x_i$, and $\mathbf{x} \mapsto \bigvee_{i=1}^{n} x_i$. In particular, an associative function $g\colon [0,1]^* \to [0,1]$ whose $n$-ary restrictions are *discrete Choquet integrals* necessarily reduces to one of these four functions.

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