Behavioral analysis of aggregation functions

J.-L. Marichal[1], M. Grabisch[2], R. Mesiar[3], E. Pap[4]

[1]University of Luxembourg, Luxembourg
[2]University of Panthéon-Sorbonne, France
[3]Slovak University of Technology, Slovakia
[4]University of Novi Sad, Serbia

Aggregation functions over $[0, 1]$

Definition. An *aggregation function* in $[0, 1]^n$ is a function $A : [0, 1]^n \to [0, 1]$ satisfying

(i) nondecreasing monotonicity (in each variable)

(ii) $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$

Consider a problem that requires the use of an aggregation function

Question: Given an aggregation function A, how can we justify its use in the problem under consideration?

- Analyze the expression of A
- Examine its main properties or, better, find axiomatizations
- Explore its *behavioral properties*

Behavioral analysis of aggregation functions

$\longrightarrow$ offers a better understanding of the general behavior of A with respect to its variables

This is typically done through *indices* or *parameters*

Arithmetic mean — ID Card

$$\mathsf{A}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Properties	**Axiomatizations**	**Behavioral indices**
Continuous	1. Continuity + ...	$\phi_i(\mathsf{A}) = 1/n$
Symmetric	...	$AC(\mathsf{A}) = 1$
Idempotent	2. ...	$\mathrm{orness}(\mathsf{A}) = 1/2$
...	...	...

$\longrightarrow$ Possibility of comparing various aggregation functions

Behavioral analysis of aggregation functions

For instance, let P be a target property (reference) for aggregation

$P =$ "A is symmetric"
$P =$ "$\mathsf{A}(x, \ldots, x) = x \ \forall x$"
$P =$ "$\mathsf{A}(\mathbf{x}) \leqslant x_k \ \forall \mathbf{x}$"

Define the degree to which A satisfies property P:

$$\theta_P(\mathsf{A}) \in [0, 1]$$

$\theta_P(\mathsf{A}) \approx 1$: A almost satisfies property P
$\theta_P(\mathsf{A}) \approx 0$: A barely satisfies property P

Behavioral analysis of aggregation functions

Other indices:

- Typical value of $A(\mathbf{x})$
- Influence degree of each input variable over $A(\mathbf{x})$
- Interaction degree of each subset of variables in the computation of $A(\mathbf{x})$
- ...

Importance indices

Idea: Given an aggregation function A, what is the degree of influence of variable x_k over A? Denote it by $\phi_k(\mathrm{A})$

Example: Weighted arithmetic mean

$$\mathrm{WAM}_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$$

where $\sum_i w_i = 1$, and $w_i \geqslant 0$

Intuitively,

$$\boxed{\phi_k(\mathrm{A}) = w_k}$$

Measure of the *contribution* of x_k in the computation of $\mathrm{WAM}_{\mathbf{w}}(\mathbf{x})$

Importance indices

. . . and in general?

Proposal: Assume A is differentiable

$$\frac{\partial A}{\partial x_k}(\mathbf{x}) = \text{rate of change w.r.t. } x_k \text{ of A at } \mathbf{x}$$

$\Rightarrow$ Average rate of change w.r.t. x_k of A

$$\phi_k(A) = \int_{[0,1]^n} \frac{\partial A}{\partial x_k}(\mathbf{x})\, d\mathbf{x}$$

Examples:

$$\begin{aligned} \phi_k(\mathrm{WAM}_{\mathbf{w}}) &= w_k \\ \phi_k(\mathrm{WGM}_{\mathbf{w}}) &= \begin{cases} \prod_{i \neq k} \frac{1}{w_i + 1} & \text{if } w_k > 0 \\ 0 & \text{if } w_k = 0 \end{cases} \end{aligned}$$

Importance indices

Note that

$$\int_0^1 \frac{\partial \mathrm{A}}{\partial x_k}(\mathbf{x})\, dx_k = \mathrm{A}(1_k, \mathbf{x}) - \mathrm{A}(0_k, \mathbf{x}) = \Delta_k \mathrm{A}(\mathbf{x})$$

where

$$\begin{aligned} (1_k, \mathbf{x}) &= (x_1, \ldots, \overset{(k)}{1}, \ldots, x_n) \\ (0_k, \mathbf{x}) &= (x_1, \ldots, \overset{(k)}{0}, \ldots, x_n) \end{aligned}$$

Therefore

$$\phi_k(\mathrm{A}) = \int_{[0,1]^n} \frac{\partial \mathrm{A}}{\partial x_k}(\mathbf{x})\, d\mathbf{x} = \int_{[0,1]^n} \Delta_k \mathrm{A}(\mathbf{x})\, d\mathbf{x}$$

The assumption of differentiability is no longer necessary!

Importance indices

Definition.

- *Importance index of coordinate k on* A

$$\phi_k(\mathrm{A}) = \int_{[0,1]^n} \Delta_k \mathrm{A}(\mathbf{x})\, d\mathbf{x}$$

- *Normalized importance index of coordinate k on* A

$$\overline{\phi}_k(\mathrm{A}) = \frac{\phi_k(\mathrm{A})}{\sum_{i=1}^n \phi_i(\mathrm{A})}$$

Examples:

$$\begin{aligned} \overline{\phi}_k(\mathrm{WAM}_{\mathbf{w}}) &= w_k \\ \overline{\phi}_k(\mathrm{OWA}_{\mathbf{w}}) &= 1/n \end{aligned}$$

where $\mathrm{OWA}_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^n w_i x_{(i)}$, with $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$

Importance indices

A more general case: Choquet integral $A = \mathcal{C}_\mu$
Let $[n] = \{1, \ldots, n\}$

$$\mathcal{C}_\mu(\mathbf{x}) = a_\varnothing + \sum_{i \in [n]} a_{\{i\}} x_i + \sum_{\{i,j\} \subseteq [n]} a_{\{i,j\}} (x_i \wedge x_j) + \cdots + a_{[n]} (x_1 \wedge \cdots \wedge x_n)$$

where the set function $a : 2^{[n]} \to \mathbb{R}$ is such that $\mathcal{C}_\mu$ is an aggregation function

$$\overline{\phi}_k(\mathcal{C}_\mu) = \phi_k^\mu := \sum_{S \ni k} \frac{1}{|S|}\, a_S$$

(Shapley value for k associated to μ)

Importance indices

Analyzing the distribution of the influences of the input variables...

Case of $\text{WAM}_{\mathbf{w}}$:

- If $\mathbf{w} = (w_1, \ldots, w_n) = (\frac{1}{n}, \ldots, \frac{1}{n})$

$\longrightarrow$ uniform contribution (evenness, regularity, dispersion)

$$H_S(\mathbf{w}) = -\frac{1}{\ln n} \sum_{i=1}^{n} w_i \ln w_i = 1$$

- If $\mathbf{w} = (w_1, \ldots, w_n) = (0, 0, \ldots, 1, \ldots, 0)$

$\longrightarrow$ Dirac measure (irregularity, concentration)

$$H_S(\mathbf{w}) = -\frac{1}{\ln n} \sum_{i=1}^{n} w_i \ln w_i = 0$$

Importance indices

Definition. *Index of uniformity of arguments contribution*, associated with A:

$$AC(\mathsf{A}) = H_S(\overline{\phi}_1(\mathsf{A}), \ldots, \overline{\phi}_n(\mathsf{A}))$$

Measure of:
- Regularity of the importances of the variables in the comput. of A
- Uniformity of the increasing monotonicity of A

Case of the Choquet integral $\mathsf{A} = \mathcal{C}_\mu$

$$AC(\mathcal{C}_\mu) = H_S(\phi_1^\mu, \ldots, \phi_n^\mu)$$

(Yager, 1999)

$$AC(\mathrm{WAM}_{\mathbf{w}}) = H_S(\mathbf{w}) \qquad AC(\mathrm{OWA}_{\mathbf{w}}) = 1$$

Importance indices

Generalization: What is the influence of variables $\{x_j, x_k\}$ over A?
Is it $\overline{\phi}_j(\mathsf{A}) + \overline{\phi}_k(\mathsf{A})$?

Definition.
Importance index of coordinates j and k on A

$$\phi_{\{j,k\}}(\mathsf{A}) = \int_{[0,1]^n} \big(\mathsf{A}(1_{\{j,k\}}, \mathbf{x}) - \mathsf{A}(0_{\{j,k\}}, \mathbf{x})\big)\, d\mathbf{x}$$

Examples:

$$\begin{aligned}
\phi_{\{j,k\}}(\mathsf{WAM}_{\mathbf{w}}) &= w_j + w_k \\
\phi_{\{j,k\}}(\mathsf{OWA}_{\mathbf{w}}) &= \frac{1}{n-1}\sum_{i=1}^{n} w_i \min(i, 2, n-i+1, n-1)
\end{aligned}$$

Typical value

Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ is a random vector uniformly distributed over $[0,1]^n$

Definition. *Average value of* A *over* $[0,1]^n$

$$E(\mathrm{A}(\mathbf{X})) = \overline{\mathrm{A}} = \int_{[0,1]^n} \mathrm{A}(\mathbf{x})\, d\mathbf{x}$$

(we assume A integrable)

Examples:

$$\begin{aligned}
\overline{\mathrm{GM}} &= \int_{[0,1]^n} \mathrm{GM}(\mathbf{x})\, d\mathbf{x} = \left(\frac{n}{n+1}\right)^n \\
\overline{\mathrm{Min}} &= \int_{[0,1]^n} \mathrm{Min}(\mathbf{x})\, d\mathbf{x} = \frac{1}{n+1} \\
\overline{\mathrm{Max}} &= \int_{[0,1]^n} \mathrm{Max}(\mathbf{x})\, d\mathbf{x} = \frac{n}{n+1}
\end{aligned}$$

Typical value

Suppose that A satisfies

$$\mathsf{Min} \leqslant \mathsf{A} \leqslant \mathsf{Max}$$

Then

$$\overline{\mathsf{Min}} \leqslant \overline{\mathsf{A}} \leqslant \overline{\mathsf{Max}}$$

Refinement: Relative position of $\overline{\mathsf{A}}$ within the interval $[\overline{\mathsf{Min}}, \overline{\mathsf{Max}}]$
$\rightarrow$ *Global orness value* (Dujmović, 1974)

$$\mathrm{orness}(\mathsf{A}) = \frac{\overline{\mathsf{A}} - \overline{\mathsf{Min}}}{\overline{\mathsf{Max}} - \overline{\mathsf{Min}}} \in [0,1]$$

Example: Geometric mean

$$\mathrm{orness}(\mathsf{GM}) = -\frac{1}{n-1} + \frac{n+1}{n-1}\left(\frac{n}{n+1}\right)^n$$

Typical value

Suppose that A satisfies

$$0 \leqslant \mathsf{A} \leqslant \mathsf{Min}$$

Then

$$0 \leqslant \overline{\mathsf{A}} \leqslant \overline{\mathsf{Min}}$$

Refinement: Relative position of $\overline{\mathsf{A}}$ within the interval $[0, \overline{\mathsf{Min}}]$
$\longrightarrow$ *Global idempotency value* (Kolesárova, 2006)

$$\mathrm{idemp}(\mathsf{A}) = \frac{\overline{\mathsf{A}} - 0}{\overline{\mathsf{Min}} - 0} = \frac{\overline{\mathsf{A}}}{\overline{\mathsf{Min}}} \in [0, 1]$$

Example: Product

$$\mathrm{idemp}(\Pi) = \frac{n+1}{2^n}$$

All these behavioral parameters can be found in...

M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, Cambridge University Press, Cambridge, UK, 2009

Thank you for your attention!