Polynomial functions on bounded chains

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Abstract— *We are interested in representations and characterizations of lattice polynomial functions $f\colon L^n \to L$, where L is a given bounded distributive lattice. In an earlier paper [4, 5], we investigated certain representations and provided various characterizations of these functions both as solutions of certain functional equations and in terms of necessary and sufficient conditions. In the present paper, we investigate these representations and characterizations in the special case when L is a chain, i.e., a totally ordered lattice. More precisely, we discuss representations of lattice polynomial functions given in terms of standard simplices and we present new axiomatizations of these functions by relaxing some of the conditions given in [4, 5] and by considering further conditions, namely comonotonic minitivity and maxitivity.*

Keywords— Lattice polynomial function, discrete Sugeno integral, normal form, homogeneity, strong idempotency, median decomposability, comonotonicity.

1 Introduction

In [5], the class of (*lattice*) *polynomial functions*, i.e., functions representable by combinations of variables and constants using the lattice operations $\wedge$ and $\vee$, was considered and characterized both as solutions of certain functional equations and in terms of necessary and sufficient conditions rooted in aggregation theory.

Formally, let L be a bounded distributive lattice with operations $\wedge$ and $\vee$, and with least and greatest elements 0 and 1, respectively. An *n-ary polynomial function* on L is any function $f : L^n \to L$ which can be obtained by finitely many applications of the following rules:

(i) For each $i \in [n] = \{1, \ldots, n\}$ and each $c \in L$, the projection $\mathbf{x} \mapsto x_i$ and the constant function $\mathbf{x} \mapsto c$ are polynomial functions from L^n to L.

(ii) If f and g are polynomial functions from L^n to L, then $f \vee g$ and $f \wedge g$ are polynomial functions from L^n to L.

Polynomial functions are also called lattice functions (Goodstein [9]), algebraic functions (Burris and Sankappanavar [3]) or weighted lattice polynomial functions (Marichal [15]). Polynomial functions obtained from projections by finitely many applications of (ii) are referred to as *lattice term functions*.

The recent interest by aggregation theorists in this class of polynomial functions is partially motivated by its connection to noteworthy aggregation functions such as the (*discrete*) *Sugeno integral*, which was introduced by Sugeno [18, 19] and widely investigated in aggregation theory, due to the many applications in fuzzy set theory, data fusion, decision making, image analysis, etc. As shown in [15], the discrete Sugeno integrals are nothing other than those polynomial functions $f : L^n \to L$ which are idempotent, that is, satisfying $f(x, \ldots, x) = x$. For general background on aggregation theory, see [1, 11] and for a recent reference, see [10].

In this paper, we refine our previous results in the particular case when L is a chain, by relaxing our conditions and proposing weak analogues of those properties used in [4, 5], and then providing characterizations of polynomial functions, accordingly. Moreover, and motivated by the axiomatizations of the discrete Sugeno integrals established by de Campos and Bolaños [6] (in the case when $L = [0, 1]$ is the unit real interval), we present further and alternative characterizations of polynomial functions given in terms of comonotonic minitivity and maxitivity. As particular cases, we consider the subclass of discrete Sugeno integrals.

The current paper is organized as follows. We start in §2 by introducing the basic notions needed in this paper and presenting the characterizations of lattice polynomial functions on arbitrary (possibly infinite) bounded distributive lattices, established in [4, 5]. Those characterizations are reassembled in Theorem 1. We discuss representations of polynomial functions in normal form (such as the classical disjunctive and conjunctive normal forms) and introduce variant representations in the case of chains and given in terms of standard simplices in §3. In §4, we present characterizations of polynomial functions on chains given in terms of weak analogues of the properties used in Theorem 1 as well as in terms of comonotonic minitivity and maxitivity. Axiomatizations for the subclass of discrete Sugeno integrals are then presented in §5.

2 Basic notions and terminology

Throughout this paper, let L be a bounded chain with operations $\wedge$ and $\vee$, and with least and greatest elements 0 and 1, respectively. A subset S of a chain L is said to be *convex* if for every $a, b \in S$ and every $c \in L$ such that $a \leqslant c \leqslant b$, we have $c \in S$. For any subset $S \subseteq L$, we denote by $\overline{S}$ the convex hull of S, that is, the smallest convex subset of L containing S. For every $a, b \in S$ such that $a \leqslant b$, the *interval* $[a, b]$ is the set $[a, b] = \{c \in L : a \leqslant c \leqslant b\}$. For any integer $n \geqslant 1$, let $[n] = \{1, \ldots, n\}$.

For any bounded chain L, we regard the Cartesian product L^n, $n \geqslant 1$, as a distributive lattice endowed with the operations $\wedge$ and $\vee$ given by

$$\begin{aligned}(a_1, \ldots, a_n) \wedge (b_1, \ldots, b_n) &= (a_1 \wedge b_1, \ldots, a_n \wedge b_n),\\ (a_1, \ldots, a_n) \vee (b_1, \ldots, b_n) &= (a_1 \vee b_1, \ldots, a_n \vee b_n).\end{aligned}$$

The elements of L are denoted by lower case letters $a, b, c, \ldots$, and the elements of L^n, $n > 1$, by bold face letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

We also use $\mathbf{0}$ and $\mathbf{1}$ to denote the least element and greatest element, respectively, of L^n. For $c \in L$ and $\mathbf{x} = (x_1, \ldots, x_n) \in L^n$, set

$$\mathbf{x}\wedge c = (x_1\wedge c, \ldots, x_n\wedge c) \quad \text{and} \quad \mathbf{x}\vee c = (x_1\vee c, \ldots, x_n\vee c).$$

The *range* of a function $f\colon L^n \to L$ is defined by $\mathcal{R}_f = \{f(\mathbf{x}) : \mathbf{x} \in L^n\}$. A function $f\colon L^n \to L$ is said to be *nondecreasing* (*in each variable*) if, for every $\mathbf{a}, \mathbf{b} \in L^n$ such that $\mathbf{a} \leqslant \mathbf{b}$, we have $f(\mathbf{a}) \leqslant f(\mathbf{b})$. Note that if f is nondecreasing, then $\overline{\mathcal{R}}_f = [f(\mathbf{0}), f(\mathbf{1})]$.

Let S be a nonempty subset of L. A function $f\colon L^n \to L$ is said to be

- *S-idempotent* if for every $c \in S$, we have $f(c, ..., c) = c$.
- *S-min homogeneous* if for every $\mathbf{x} \in L^n$ and $c \in S$,
$$f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c. \tag{1}$$
- *S-max homogeneous* if for every $\mathbf{x} \in L^n$ and $c \in S$,
$$f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c. \tag{2}$$
- *horizontally S-minitive* if for every $\mathbf{x} \in L^n$ and $c \in S$,
$$f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c), \tag{3}$$
where $[\mathbf{x}]^c$ is the n-tuple whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise.
- *horizontally S-maxitive* if for every $\mathbf{x} \in L^n$ and $c \in S$,
$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c), \tag{4}$$
where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise.
- *median decomposable* if, for every $\mathbf{x} \in L^n$ and $k \in [n]$,
$$f(\mathbf{x}) = \operatorname{median}\big(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\big), \tag{5}$$
where $\operatorname{median}(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $\mathbf{x}_k^c = (x_1, \ldots, x_{k-1}, c, x_{k+1}, \ldots, x_n)$ for $c \in L$.
- *strongly idempotent* if, for every $\mathbf{x} \in L^n$ and $k \in [n]$,
$$f(x_1, \ldots, x_{k-1}, f(\mathbf{x}), x_{k+1}, \ldots, x_n) = f(\mathbf{x}).$$

Remark 1. In the case when $S = L$ is the real interval $[0, 1]$, the concepts of S-min and S-max homogeneity were introduced by Fodor and Roubens [8] to specify certain classes of aggregation functions, and the concept of horizontal S-maxitivity was introduced by Benvenuti et al. [2] as a general property of the Sugeno integral. The concept of median decomposability was introduced in [15] and that of strong idempotency in [4, 5] as properties of polynomial functions.

We say that a function $f\colon L^n \to L$ has a *componentwise convex range* if either $n = 1$ and f has a convex range, or $n > 1$ and for every $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$ and every $k \in [n]$, the unary function $f_{\mathbf{a}}^k\colon L \to L$, given by $f_{\mathbf{a}}^k(x) = f(\mathbf{a}_k^x)$ has a convex range.

The following theorem reassembles the various characterizations of polynomial functions, established in [4], in the particular case when L is a bounded chain.

Theorem 1. *Let $f\colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) f is a polynomial function.

(ii) f is median decomposable.

(iii) f is nondecreasing, strongly idempotent, has a convex range and a componentwise convex range.

(iv) f is nondecreasing, $\overline{\mathcal{R}}_f$-min homogeneous, and $\overline{\mathcal{R}}_f$-max homogeneous.

(v) f is nondecreasing, $\overline{\mathcal{R}}_f$-min homogeneous, and horizontally $\overline{\mathcal{R}}_f$-maxitive.

(vi) f is nondecreasing, horizontally $\overline{\mathcal{R}}_f$-minitive, and $\overline{\mathcal{R}}_f$-max homogeneous.

(vii) f is nondecreasing, $\overline{\mathcal{R}}_f$-idempotent, horizontally $\overline{\mathcal{R}}_f$-minitive, and horizontally $\overline{\mathcal{R}}_f$-maxitive.

3 Representations of polynomial functions

Polynomial functions are known to be exactly those functions which can be represented by formulas in disjunctive and conjunctive normal forms. This fact was first observed by Goodstein [9] who, in fact, showed that each polynomial function $f\colon L^n \to L$ is uniquely determined by its restriction to $\{0, 1\}^n$. For a recent reference, see Rudeanu [17].

In this section we recall and refine some known results concerning normal forms of polynomial functions and, in the special case when L is a chain, we present variant representations given in terms of standard simplices of L^n.

The following three results are due to Goodstein [9].

Corollary 2. *Every polynomial function is completely determined by its restriction to $\{0, 1\}^n$.*

Corollary 3. *A function $g\colon \{0, 1\}^n \to L$ can be extended to a polynomial function $f\colon L^n \to L$ if and only if it is nondecreasing. In this case, the extension is unique.*

Proposition 4. *Let $f\colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) f is a polynomial function.

(ii) There exists $\alpha\colon 2^{[n]} \to L$ such that
$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \big(\alpha(I) \wedge \bigwedge_{i \in I} x_i\big).$$

(iii) There exists $\beta\colon 2^{[n]} \to L$ such that
$$f(\mathbf{x}) = \bigwedge_{I \subseteq [n]} \big(\beta(I) \vee \bigvee_{i \in I} x_i\big).$$

We shall refer to the expressions given in (ii) and (iii) of Proposition 4 as the *disjunctive normal form* (DNF) representation and the *conjunctive normal form* (CNF) representation, respectively, of the polynomial function f.

Remark 2. By requiring α and β to be nonconstant functions from $2^{[n]}$ to $\{0, 1\}$ and satisfying $\alpha(\varnothing) = 0$ and $\beta(\varnothing) = 1$, respectively, we obtain the analogue of Proposition 4 for term functions.

For each polynomial function $f\colon L^n \to L$, set

$$\mathrm{DNF}(f) = \Big\{\alpha \in L^{2^{[n]}} : f(\mathbf{x}) = \bigvee_{I\subseteq[n]} \big(\alpha(I) \wedge \bigwedge_{i\in I} x_i\big)\Big\},$$

$$\mathrm{CNF}(f) = \Big\{\beta \in L^{2^{[n]}} : f(\mathbf{x}) = \bigwedge_{I\subseteq[n]} \big(\beta(I) \vee \bigvee_{i\in I} x_i\big)\Big\}.$$

A complete description of the sets DNF(f) and CNF(f) can be found in [5, §3.1]. As we are concerned by the case when L is a chain, we recall the description only in this special case; see [15, §3].

For each $I \subseteq [n]$, let $\mathbf{e}_I$ be the element of L^n whose ith component is 1, if $i \in I$, and 0, otherwise. Let $\alpha_f\colon 2^{[n]} \to L$ be the function given by $\alpha_f(I) = f(\mathbf{e}_I)$ and consider the function $\alpha_f^*\colon 2^{[n]} \to L$ defined by

$$\alpha_f^*(I) = \begin{cases} \alpha_f(I), & \text{if } \bigvee_{J\subsetneq I} \alpha_f(J) < \alpha_f(I),\\ 0, & \text{otherwise.}\end{cases}$$

Dually, Let $\beta_f\colon 2^{[n]} \to L$ be the function given by $\beta_f(I) = f(\mathbf{e}_{[n]\setminus I})$ and consider the function $\beta_f^*\colon 2^{[n]} \to L$ defined by

$$\beta_f^*(I) = \begin{cases} \beta_f(I), & \text{if } \bigwedge_{J\subsetneq I} \beta_f(J) > \beta_f(I),\\ 1, & \text{otherwise.}\end{cases}$$

Proposition 5. *Let $f\colon L^n \to L$ be a polynomial function. Then*

(i) DNF$(f) = [\alpha_f^*, \alpha_f]$ and f has a unique DNF representation if and only if $\bigvee_{J\subsetneq I} \alpha_f(J) < \alpha_f(I)$ for every $I \subseteq [n]$,

(ii) CNF$(f) = [\beta_f, \beta_f^*]$ and f has a unique CNF representation if and only if $\bigwedge_{J\subsetneq I} \beta_f(J) > \beta_f(I)$ for every $I \subseteq [n]$.

In particular, α_f and β_f are the unique isotone and antitone, respectively, maps in DNF(f) and CNF(f), respectively.

In the case of chains, the DNF and CNF representations of polynomial functions $f\colon L^n \to L$ can be refined and given in terms of standard simplices of L^n (see Proposition 6 below). Recall that

$$\mathrm{median}(x_1, \ldots, x_{2n+1}) = \bigvee_{\substack{I\subseteq[2n+1]\\ |I|=n+1}} \bigwedge_{i\in I} x_i.$$

Let σ be a permutation on $[n]$. The *standard simplex* of L^n associated with σ is the subset $L^n_\sigma \subset L^n$ defined by

$$L^n_\sigma = \{(x_1, \ldots, x_n) \in L^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}.$$

For each $i \in [n]$, define $S^{\uparrow}_\sigma(i) = \{\sigma(i), \ldots, \sigma(n)\}$ and $S^{\downarrow}_\sigma(i) = \{\sigma(1), \ldots, \sigma(i)\}$. As a matter of convenience, set $S^{\uparrow}_\sigma(n+1) = S^{\downarrow}_\sigma(0) = \varnothing$.

Proposition 6. *Let $f\colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) f is a polynomial function.

(ii) For any permutation σ on $[n]$ and every $\mathbf{x} \in L^n_\sigma$, we have

$$\begin{aligned} f(\mathbf{x}) &= \bigvee_{i\in[n+1]} \big(\alpha_f(S^{\uparrow}_\sigma(i)) \wedge x_{\sigma(i)}\big)\\ &= \bigwedge_{i\in[n+1]} \big(\alpha_f(S^{\uparrow}_\sigma(i)) \vee x_{\sigma(i-1)}\big)\\ &= \mathrm{median}\big(x_1, ..., x_n, \alpha_f(S^{\uparrow}_\sigma(1)), ..., \alpha_f(S^{\uparrow}_\sigma(n+1))\big),\end{aligned}$$

where $x_{\sigma(0)} = 0$ and $x_{\sigma(n+1)} = 1$.

(iii) For any permutation σ on $[n]$ and every $\mathbf{x} \in L^n_\sigma$, we have

$$\begin{aligned} f(\mathbf{x}) &= \bigvee_{i\in[n+1]} \big(\beta_f(S^{\downarrow}_\sigma(i-1)) \wedge x_{\sigma(i)}\big)\\ &= \bigwedge_{i\in[n+1]} \big(\beta_f(S^{\downarrow}_\sigma(i-1)) \wedge x_{\sigma(i-1)}\big)\\ &= \mathrm{median}\big(x_1, \ldots, x_n, \beta_f(S^{\downarrow}_\sigma(0)), \ldots, \beta_f(S^{\downarrow}_\sigma(n))\big),\end{aligned}$$

where $x_{\sigma(0)} = 0$ and $x_{\sigma(n+1)} = 1$.

Remark 3. The equivalence between (i) and (ii) of Proposition 6 was already observed in [15, §5]. Prior to this, Propositions 5 and 6 were already established in [14] for idempotent polynomial functions (discrete Sugeno integrals) in the case when L is the unit real interval $[0, 1]$; see also [13, §4.3].

4 Characterizations of polynomial functions

In this section, we propose weak analogues of the properties used in Theorem 1 and provide characterizations of polynomial functions on chains, accordingly. Moreover, we introduce further properties, namely, comonotonic minitivity and maxitivity, which we then use to provide further characterizations of polynomial functions.

For integers $0 \leqslant p \leqslant q \leqslant n$, define $L_n^{(p,q)} = \{\mathbf{x} \in L^n : |\{x_1, \ldots, x_n\} \cap \{0, 1\}| \geqslant p$ and $|\{x_1, \ldots, x_n\}| \leqslant q\}$. For instance, $L_n^{(0,2)}$ is the set of Boolean vectors of L^n that are two-sided trimmed by constant vectors, that is

$$L_n^{(0,2)} = \bigcup_{\substack{\mathbf{e}\in\{0,1\}^n\\ c,d\in L}} \{\mathrm{median}(c, \mathbf{e}, d)\},$$

where the median is taken componentwise.

4.1 Weak homogeneity

Let S be a nonempty subset of L. We say that a function $f\colon L^n \to L$ is *weakly S-min homogeneous* (resp. *weakly S-max homogeneous*) if (1) (resp. (2)) holds for every $\mathbf{x} \in L_n^{(0,2)}$ and every $c \in S$.

For every integer $m \geqslant 1$, every $\mathbf{x} \in L^m$, and every $f\colon L^n \to L$, we define $\langle\mathbf{x}\rangle_f \in L^m$ as the m-tuple

$$\langle\mathbf{x}\rangle_f = \mathrm{median}(f(\mathbf{0}), \mathbf{x}, f(\mathbf{1})),$$

where the right-hand side median is taken componentwise. As observed in [5], for every nonempty subset $S \subseteq L$, we have that f is S-min homogeneous and S-max homogeneous if and only if it satisfies

$$f(\mathrm{median}(r, \mathbf{x}, s)) = \mathrm{median}(r, f(\mathbf{x}), s)$$

for every $\mathbf{x} \in L^n$ and every $r, s \in S$. In particular, if $f(\mathbf{0}), f(\mathbf{1}) \in S$ then, for any $\mathbf{x} \in L^n$ such that $f(\mathbf{0}) \leqslant f(\mathbf{x}) \leqslant f(\mathbf{1})$, we have $f(\mathbf{x}) = f(\langle \mathbf{x} \rangle_f)$.

It was also shown in [5] that, for every nonempty subset $S \subseteq L$, if f is S-min homogeneous and S-max homogeneous, then it is S-idempotent. The following lemma shows that the weak analogue also holds.

Lemma 7. *Let S be a nonempty subset of L. If $f\colon L^n \to L$ is weakly S-min homogeneous and weakly S-max homogeneous, then it is S-idempotent. Moreover, if $f(\mathbf{0}), f(\mathbf{1}) \in S$ then, for any $\mathbf{x} \in L_n^{(0,2)}$ such that $f(\mathbf{0}) \leqslant f(\mathbf{x}) \leqslant f(\mathbf{1})$, we have $f(\mathbf{x}) = f(\langle \mathbf{x} \rangle_f)$.*

As we are going to see, of particular interest is when $S = \overline{\mathcal{R}}_f$. The following result characterizing the class of polynomial functions shows that, in the case of chains, the conditions in (iv) of Theorem 1 can be replaced with their weak analogues.

Theorem 8. *A function $f\colon L^n \to L$ is a polynomial function if and only if it is nondecreasing, weakly $\overline{\mathcal{R}}_f$-min homogeneous, and weakly $\overline{\mathcal{R}}_f$-max homogeneous.*

Remark 4. Note that Theorem 8 does not generally hold in the case of bounded distributive lattices. To see this, let $L = \{0, a, b, 1\}$ where $a \wedge b = 0$ and $a \vee b = 1$, and consider the binary function $f\colon L^2 \to L$ defined by

$$f(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 = 1 \text{ or } x_2 = 1, \\ 1, & \text{if } x_1 = x_2 = b, \\ a, & \text{if } (x_1 = a \text{ and } x_2 \neq 1) \\ & \text{or } (x_2 = a \text{ and } x_1 \neq 1), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that f is nondecreasing and both weakly $\overline{\mathcal{R}}_f$-min homogeneous and weakly $\overline{\mathcal{R}}_f$-max homogeneous, but it is not $\overline{\mathcal{R}}_f$-max homogeneous. For instance, $f(a \vee a, b \vee a) = f(a, 1) = 1 \neq a = f(a, b) \vee a$. Thus, f is not a polynomial function by Theorem 1.

4.2 Weak horizontal minitivity and maxitivity

Let S be a nonempty subset of L. We say that a function $f\colon L^n \to L$ is *weakly horizontally S-minitive* (resp. *weakly horizontally S-maxitive*) if (3) (resp. (4)) holds for every $\mathbf{x} \in L_n^{(0,2)}$ and every $c \in S$.

The following result provides characterizations of the n-ary polynomial functions on a chain L, given in terms of weak homogeneity and weak horizontal minitivity and maxitivity.

Theorem 9. *Let $f\colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) f is a polynomial function.

(ii) f is nondecreasing, weakly $\overline{\mathcal{R}}_f$-min homogeneous, and weakly $\overline{\mathcal{R}}_f$-max homogeneous.

(iii) f is nondecreasing, weakly $\overline{\mathcal{R}}_f$-min homogeneous, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.

(iv) f is nondecreasing, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and weakly $\overline{\mathcal{R}}_f$-max homogeneous.

(v) f is nondecreasing, $\overline{\mathcal{R}}_f$-idempotent, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.

4.3 Weak median decomposability

As we saw in Theorem 1, in the case of distributive lattices L, the n-ary polynomial functions on L are exactly those which satisfy the median decomposition formula (5). As we are going to see, in the case of chains, this condition can be relaxed by restricting the satisfaction of (5) by a function $f : L^n \to L$ to the vectors of $L_n^{(0,2)} \cup L_n^{(1,3)}$. In the latter case, we say that $f : L^n \to L$ is *weakly median decomposable*.

Lemma 10. *Let $f\colon L^n \to L$ be a nondecreasing function. If f is weakly median decomposable, then it is $\overline{\mathcal{R}}_f$-idempotent.*

Proposition 11. *Let $f\colon L^n \to L$ be a nondecreasing function. The following conditions are equivalent:*

(i) f is weakly median decomposable.

(ii) f weakly $\overline{\mathcal{R}}_f$-min homogeneous and weakly $\overline{\mathcal{R}}_f$-max homogeneous.

Remark 5. Using the binary function f given in Remark 4, we can see that Proposition 11 does not hold in the general case of bounded distributive lattices. Indeed, as observed, f is nondecreasing and both weakly $\overline{\mathcal{R}}_f$-min homogeneous and weakly $\overline{\mathcal{R}}_f$-max homogeneous, but $f(b, b) = 1 \neq b = \mathrm{median}(f(0, b), b, f(1, b))$ which shows that f is not weakly median decomposable.

From Proposition 11 and Theorem 8, we obtain the following description of polynomial functions given in terms of weak median decomposability.

Theorem 12. *A nondecreasing function $f\colon L^n \to L$ is a polynomial function if and only if it is weakly median decomposable.*

Remark 6. Note that Theorem 12 does not hold if weak median decomposability would have been defined in terms of vectors in $L_n^{(0,2)}$ only. To see this, let $L = \{0, c, 1\}$ and consider the following nondecreasing function $f : L^3 \to L$, defined by

$$f(x_1, x_2, x_3) = \begin{cases} 1, & \text{if } \mathrm{median}(x_1, x_2, x_3) = 1, \\ c, & \text{if } \mathrm{median}(x_1, x_2, x_3) \\ & \quad = x_1 \wedge x_2 \wedge x_3 = c, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to this that f is median decomposable for vectors in $L_n^{(0,2)}$, but it is not a polynomial function, e.g., we have $f(0, c, c) = 0$ but $f(0, 1, 1) \wedge c = c$.

4.4 Strong idempotency and componentwise range convexity

By Theorem 1, a nondecreasing function $f\colon L^n \to L$ is a polynomial function if and only if it is strongly idempotent, has a convex range, and a componentwise convex range. In the case of chains, the condition requiring a convex range becomes redundant, since it becomes a consequence of componentwise range convexity. Thus, we obtain the following characterization, which weakens condition (iii) of Theorem 1 when L is a chain.

Theorem 13. *A function $f: L^n \to L$ is a polynomial function if and only if it is nondecreasing, strongly idempotent, and has a componentwise convex range.*

Remark 7. None of the conditions provided in Theorem 13 can be dropped off. For instance, let L be the real interval $[0,1]$. Clearly, the unary function $f(x) = x^2$ is nondecreasing and has a componentwise convex range, but it is not strongly idempotent. On the other hand, the function $f: L^2 \to L$ defined by

$$f(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 = x_2 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is nondecreasing and strongly idempotent but it does not have a componentwise convex range, e.g., both $f_{\mathbf{1}}^1$ and $f_{\mathbf{1}}^2$ do not have convex ranges.

In the special case of real interval lattices, i.e., where $L = [a,b]$ for reals $a \leqslant b$, the property of having a convex range, as well as the property of having a componentwise convex range, are consequences of continuity. More precisely, for nondecreasing functions $f: [a,b]^n \to \mathbb{R}$, being continuous reduces to being continuous in each variable, and this latter property is equivalent to having a componentwise convex range. In fact, since polynomial functions are continuous, the condition of having a componentwise convex range can be replaced in Theorem 13 by continuity in each variable. Also, we can add continuity and replace $\overline{\mathcal{R}}_f$ by $\mathcal{R}_f$ in Theorems 8 and 9.

Corollary 14. *Assume that L is a bounded real interval $[a,b]$. A function $f: L^n \to L$ is a polynomial function if and only if it is nondecreasing, strongly idempotent, and continuous (in each variable).*

4.5 Comonotonic maxitivity and minitivity

Two vectors $\mathbf{x}$ and $\mathbf{x}'$ in L^n are said to be *comonotonic* if $\mathbf{x}, \mathbf{x}' \in L_\sigma^n$ for some permutation σ on $[n]$. A function $f: L^n \to L$ is said to be

- *comonotonic minitive* if, for any two comonotonic vectors $\mathbf{x}, \mathbf{x}' \in L^n$, we have
$$f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}').$$
- *comonotonic maxitive* if, for any two comonotonic vectors $\mathbf{x}, \mathbf{x}' \in L^n$, we have
$$f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}').$$

Note that for any $\mathbf{x} \in L^n$ and any $c \in L$, we have that $\mathbf{x}$ and $(c, \ldots, c)$ are comonotonic and that $\mathbf{x} \vee c$ and $[\mathbf{x}]^c$ are comonotonic. These facts lead to the following result.

Lemma 15. *Let S be a nonempty subset of L. If a function $f: L^n \to L$ is comonotonic minitive (resp. comonotonic maxitive), then it is horizontally S-minitive (resp. horizontally S-maxitive). Moreover, if f is S-idempotent, then it is S-min homogeneous (resp. S-max homogeneous).*

Let σ be a permutation on $[n]$. Clearly, every comonotonic minitive (or comonotonic maxitive) function $f: L^n \to L$ is nondecreasing on the standard simplex L_σ^n. The following lemma shows that this fact can be extended to the whole domain L^n.

Lemma 16. *If $f: L^n \to L$ is comonotonic minitive or comonotonic maxitive, then it is nondecreasing. Furthermore, every nondecreasing unary function is comonotonic minitive and comonotonic maxitive.*

We now have the following characterization of polynomial functions.

Theorem 17. *Let $f: L^n \to L$ be a function. The following conditions are equivalent:*

(i) f is a polynomial function.

(ii) f is weakly $\overline{\mathcal{R}}_f$-min homogeneous and comonotonic maxitive.

(iii) f is comonotonic minitive and weakly $\overline{\mathcal{R}}_f$-max homogeneous.

(iv) f is $\overline{\mathcal{R}}_f$-idempotent, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and comonotonic maxitive.

(v) f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.

(vi) f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and comonotonic maxitive.

Remark 8. (i) As already observed in the remark following Theorem 9, the weak horizontal $\overline{\mathcal{R}}_f$-minitivity (resp. weak horizontal $\overline{\mathcal{R}}_f$-maxitivity) can be replaced with weak horizontal L-minitivity (resp. weak horizontal L-maxitivity) in the assertions (iv)–(v) of Theorem 17.

(ii) The condition requiring $\overline{\mathcal{R}}_f$-idempotency is necessary in conditions (iv)–(vi) of Theorem 17. For instance, let L be the unit interval $[0,1]$. Clearly, the unary function $f(x) = x^2$ is nondecreasing and thus comonotonic minitive and comonotonic maxitive. By Lemma 15, it is also horizontally $\overline{\mathcal{R}}_f$-minitive and horizontally $\overline{\mathcal{R}}_f$-maxitive. However, it is not a polynomial function.

(iii) The concept of comonotonic vectors appeared as early as 1952 in Hardy et al. [12]. Comonotonic minitivity and maxitivity were introduced in the context of Sugeno integrals in de Campos et al. [7]. An interpretation of these properties was given by Ralescu and Ralescu [16] in the framework of aggregation of fuzzy subsets.

5 Discrete Sugeno integrals

In this final section, we consider a noteworthy subclass of polynomial functions, namely, that of discrete Sugeno integrals, and provide its characterizations.

A function $f: L^n \to L$ is said to be *idempotent* if it is L-idempotent.

Fact 18. *A polynomial function is $\{0,1\}$-idempotent if and only if it is idempotent.*

In [15, §4], $\{0,1\}$-idempotent polynomial functions are referred to as *discrete Sugeno integrals*. They coincide exactly with those functions $\mathcal{S}_\mu: L^n \to L$ for which there is a fuzzy measure μ such that

$$\mathcal{S}_\mu(\mathbf{x}) = \bigvee_{I \subseteq [n]} \big(\mu(I) \wedge \bigwedge_{i \in I} x_i\big).$$

(A *fuzzy measure* μ is simply a set function $\mu\colon 2^{[n]} \to L$ satisfying $\mu(I) \leqslant \mu(I')$ whenever $I \subseteq I'$, and $\mu(\varnothing) = 0$ and $\mu([n]) = 1$.)

The following proposition shows how polynomial functions relate to Sugeno integrals; see [15, Proposition 12].

Proposition 19. *For any polynomial function $f\colon L^n \to L$ there is a fuzzy measure $\mu\colon 2^{[n]} \to L$ such that $f(\mathbf{x}) = \langle \mathcal{S}_\mu(\mathbf{x})\rangle_f$.*

We say that a function $f\colon L^n \to L$ is *Boolean min homogeneous* (resp. *Boolean max homogeneous*) if (1) (resp. (2)) holds for every $\mathbf{x} \in \{0,1\}^n$ and every $c \in L$. Note that every weakly L-min homogeneous (resp. weakly L-max homogeneous) function is Boolean min homogeneous (resp. Boolean max homogeneous). The following result provides a variant of Theorem 8.

Theorem 20. *A function $f\colon L^n \to L$ is a discrete Sugeno integral if and only if it is nondecreasing, Boolean min homogeneous, and Boolean max homogeneous.*

Remark 9. (i) Theorem 20 as well as the characterization of the discrete Sugeno integrals obtained by combining $\{0,1\}$-idempotency with (vi) in Theorem 17 were presented in the case of real variables by [13, §4.3]; see also [14].

(ii) Even though Theorem 20 can be derived from condition (ii) of Theorem 9 by simply modifying the two homogeneity properties, to proceed similarly with conditions (iii) and (iv), it is necessary to add the conditions of $\{1\}$-idempotency and $\{0\}$-idempotency, respectively (and similarly with conditions (ii) and (iii) of Theorem 17). To see this, let L be a bounded chain with at least three elements and consider the unary functions $f(x) = x \wedge d$ and $g(x) = x \vee d$, where $d \in L \setminus \{0,1\}$. Clearly, f is L-min homogeneous and horizontally L-maxitive and g is L-max homogeneous and horizontally L-minitive. However, neither f nor g is a discrete Sugeno integral. To see that these additions are sufficient, just note that L-min homogeneity (resp. L-max homogeneity) implies $\{0\}$-idempotency (resp. $\{1\}$-idempotency).

(iii) Marichal [13, §2.2.3] showed that, when L is a chain, a nondecreasing and idempotent function $f\colon L^n \to L$ is Boolean min homogeneous (resp. Boolean max homogeneous) if and only if we have $f(\mathbf{e} \wedge c) \in \{f(\mathbf{e}), c\}$ (resp. $f(\mathbf{e} \vee c) \in \{f(\mathbf{e}), c\}$) for every $\mathbf{e} \in \{0,1\}^n$ and every $c \in L$.

References

[1] G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*. Studies in Fuziness and Soft Computing. Springer, Berlin, 2007.

[2] P. Benvenuti, R. Mesiar, and D. Vivona. Monotone set functions-based integrals. In *Handbook of measure theory, Vol. I, II*, pages 1329–1379. North-Holland, Amsterdam, 2002.

[3] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.

[4] M. Couceiro and J.-L. Marichal. Polynomial functions over bounded distributive lattices. `http://arxiv.org/abs/0901.4888`

[5] M. Couceiro and J.-L. Marichal. Characterizations of discrete Sugeno integrals as polynomial functions over distributive lattices. `http://arxiv.org/abs/0808.2619`

[6] L. M. de Campos and M. J. Bolaños. Characterization and comparison of Sugeno and Choquet integrals. *Fuzzy Sets and Systems*, 52(1):61–67, 1992.

[7] L. M. de Campos, M. T. Lamata, and S. Moral. A unified approach to define fuzzy integrals. *Fuzzy Sets and Systems*, 39(1):75–90, 1991.

[8] J. C. Fodor and M. Roubens. Characterization of weighted maximum and some related operations. *Inform. Sci.*, 84(3-4):173–180, 1995.

[9] R. L. Goodstein. The solution of equations in a lattice. *Proc. Roy. Soc. Edinburgh Sect. A*, 67:231–242, 1965/1967.

[10] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, UK, 2009.

[11] M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy measures and integrals*, volume 40 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg, 2000.

[12] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, at the University Press, 1952. 2d ed.

[13] J.-L. Marichal. *Aggregation operators for multicriteria decision aid*. PhD thesis, Institute of Mathematics, University of Liège, Liège, Belgium, December 1998.

[14] J.-L. Marichal. On Sugeno integral as an aggregation function. *Fuzzy Sets and Systems*, 114(3):347–365, 2000.

[15] J.-L. Marichal. Weighted lattice polynomials. *Discrete Mathematics*, 309(4):814–820, 2009.

[16] A. L. Ralescu and D. A. Ralescu. Extensions of fuzzy aggregation. *Fuzzy Sets and Systems*, 86(3):321–330, 1997.

[17] S. Rudeanu. *Lattice functions and equations*. Springer Series in Discrete Mathematics and Theoretical Computer Science. Springer-Verlag London Ltd., London, 2001.

[18] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, Tokyo, 1974.

[19] M. Sugeno. Fuzzy measures and fuzzy integrals—a survey. In M. M. Gupta, G. N. Saridis, and B. R. Gaines, editors, *Fuzzy automata and decision processes*, pages 89–102. North-Holland, New York, 1977.