Quasi-polynomial functions on bounded chains

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Abstract— *Two emergent properties in aggregation theory are investigated, namely horizontal maxitivity and comonotonic maxitivity (as well as their dual counterparts) which are commonly defined by means of certain functional equations. We present complete descriptions of the function classes axiomatized by each of these properties, up to weak versions of monotonicity, in the cases of horizontal maxitivity and minitivity. While studying the classes axiomatized by combinations of these properties, we introduce the concept of quasi-polynomial function which appears as a natural extension of the well-established notion of polynomial function. We present further axiomatizations for this class both in terms of functional equations and natural relaxations of homogeneity and median decomposability. As noteworthy particular cases, we investigate those subclasses of quasi-term functions and quasi-weighted maximum and minimum functions, and present characterizations accordingly.*

Keywords— Discrete Sugeno integral, quasi-polynomial function, horizontal maxitivity and minitivity, comonotonic maxitivity and minitivity, functional equation.

1   Introduction

Aggregation functions arise wherever aggregating information is important: applied and pure mathematics (probability, statistics, decision theory, functional equations), operations research, computer science, and many applied fields (economics and finance, pattern recognition and image processing, data fusion, etc.). For recent references, see Beliakov et al. [1] and Grabisch et al. [10].

A noteworthy aggregation function is the so-called discrete Sugeno integral, which was introduced by Sugeno [15, 16] and which has been widely investigated in aggregation theory, due to its many applications for instance in fuzzy set theory, decision making, and image analysis. For general background, see also the edited book [11].

A convenient way to introduce the discrete Sugeno integral is via the concept of (lattice) polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. As shown by Marichal [13], the discrete Sugeno integrals are exactly those polynomial functions $f : L^n \to L$ which are idempotent, that is, satisfying $f(x, \ldots, x) = x$. Several axiomatizations of the class of discrete Sugeno integrals (as idempotent polynomial functions) have been recently given; see [4].

Of particular interest in aggregation theory, are the so-called horizontal maxitivity and comonotonic maxitivity (as well as their dual counterparts), usually expressed in terms of certain functional equations, and which we now informally describe.

Let L be a bounded chain. For every $\mathbf{x} \in L^n$ and every $c \in L$, consider the horizontal maxitive decomposition of $\mathbf{x}$ obtained by "cutting" it with c, namely $\mathbf{x} = (\mathbf{x} \wedge c) \vee [\mathbf{x}]_c$,

where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise. A function $f \colon L^n \to L$ is said to be *horizontally maxitive* if

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c)$$

for every $\mathbf{x} \in L^n$ and every $c \in L$.

A function $f \colon L^n \to L$ is said to be *comonotonic maxitive* if, for any two vectors $\mathbf{x}$ and $\mathbf{x}'$ in the same standard simplex of L^n, we have

$$f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}').$$

As we are going to see (Lemma 6 below), these (as well as their duals) are closely related and constitute properties shared by discrete Sugeno integrals. Still, and as it will become evident, no combination of these with their dual forms suffices to fully describe the class of Sugeno integrals. Thus, and given their emergence in aggregation theory, it is natural to ask which classes of functions are axiomatized by combinations of these properties or, in fact, by each of these properties.

In this paper, we answer this question for both the maxitive and minitive comonotonic properties, and for horizontal maxitivity and minitivity properties, up to certain weak variants of monotonicity. While looking at combinations of the latter properties, we reach a natural generalization of polynomial functions, which we call *quasi-polynomial functions* and which are best described by the following equation

$$f(x_1, \ldots, x_n) = p(\varphi(x_1), \ldots, \varphi(x_n)),$$

where p is a polynomial function and φ a nondecreasing function (see Theorem 10 below). Searching for alternative descriptions, we introduce weaker versions of well-established properties, such as homogeneity and median decomposability, to provide further axiomatizations of the class of quasi-polynomial functions, accordingly.

This paper is organized as follows. We start by recalling basic notions and terminology in lattice function theory, as well as present some known results, needed throughout this paper (Section 2). In Section 3, we study the properties of horizontal maxitivity and comonotonic maxitivity, as well as their dual forms, and determine those function classes axiomatized by each of this properties. Combinations of the latter are then considered in Section 4.1, where the notion of quasi-polynomial function is introduced. In Section 4.2, we propose weaker versions of homogeneity and median decomposability, and provide further characterizations of quasi-polynomial functions, accordingly. In Section 5, we introduce and axiomatize few noteworthy subclasses of quasi-polynomial functions, namely, those of quasi-term functions and those quasi-weighted maximum and minimum functions.

2 Basic notions and preliminary results

In this section we recall basic terminology as well as some results needed in the current paper. For general background we refer the reader to, e.g., Burris and Sankappanavar [3] and Rudeanu [14].

2.1 General background

Throughout this paper, let L be a bounded chain with operations $\wedge$ and $\vee$, and with least and greatest elements 0 and 1, respectively. A subset S of a chain L is said to be *convex* if for every $a, b \in S$ and every $c \in L$ such that $a \leqslant c \leqslant b$, we have $c \in S$. For any subset $S \subseteq L$, we denote by $\overline{S}$ the convex hull of S, that is, the smallest convex subset of L containing S. For every $a, b \in S$ such that $a \leqslant b$, the *interval* $[a, b]$ is the set $[a, b] = \{c \in L : a \leqslant c \leqslant b\}$. For any integer $n \geqslant 1$, let $[n] = \{1, \ldots, n\}$.

For any bounded chain L, we regard the Cartesian product L^n, $n \geqslant 1$, as a distributive lattice endowed with the operations $\wedge$ and $\vee$ given by

$$(a_1, \ldots, a_n) \wedge (b_1, \ldots, b_n) = (a_1 \wedge b_1, \ldots, a_n \wedge b_n),$$
$$(a_1, \ldots, a_n) \vee (b_1, \ldots, b_n) = (a_1 \vee b_1, \ldots, a_n \vee b_n).$$

The elements of L are denoted by lower case letters $a, b, c, \ldots$, and the elements of L^n, $n > 1$, by bold face letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$. We also use $\mathbf{0}$ and $\mathbf{1}$ to denote the least element and greatest element, respectively, of L^n. For $c \in L$ and $\mathbf{x} = (x_1, \ldots, x_n) \in L^n$, set

$$\mathbf{x} \wedge c = (x_1 \wedge c, \ldots, x_n \wedge c) \quad \text{and} \quad \mathbf{x} \vee c = (x_1 \vee c, \ldots, x_n \vee c).$$

The *range* of a function $f \colon L^n \to L$ is defined by $\mathcal{R}_f = \{f(\mathbf{x}) : \mathbf{x} \in L^n\}$. A function $f \colon L^n \to L$ is said to be *nondecreasing* (*in each variable*) if, for every $\mathbf{a}, \mathbf{b} \in L^n$ such that $\mathbf{a} \leqslant \mathbf{b}$, we have $f(\mathbf{a}) \leqslant f(\mathbf{b})$. The *diagonal section* of f, denoted δ_f, is defined as the unary function given by $\delta_f(x) = f(x, \ldots, x)$. Note that if f is nondecreasing, then δ_f is nondecreasing and $\overline{\mathcal{R}}_{\delta_f} = \overline{\mathcal{R}}_f = [f(\mathbf{0}), f(\mathbf{1})]$.

2.2 Polynomial functions and their representations

In this paper the so-called polynomial functions will play a fundamental role. Formally, an *n-ary polynomial function* on L is any function $f : L^n \to L$ which can be obtained by finitely many applications of the following rules:

(i) For each $i \in [n]$ and each $c \in L$, the projection $\mathbf{x} \mapsto x_i$ and the constant function $\mathbf{x} \mapsto c$ are polynomial functions from L^n to L.

(ii) If f and g are polynomial functions from L^n to L, then $f \vee g$ and $f \wedge g$ are polynomial functions from L^n to L.

Polynomial functions are also called lattice functions (Goodstein [9]), algebraic functions (Burris and Sankappanavar [3]) or weighted lattice polynomial functions (Marichal [13]). Idempotent polynomial functions (i.e., satisfying $f(c, \ldots, c) = c$ for every $c \in L$) are referred to by aggregation theorists as (*discrete*) *Sugeno integrals*, and those obtained from projections by finitely many applications of (ii) are usually referred to as (*lattice*) *term functions*. A noteworthy term function that we shall make use in the sequel is the median function, defined by

$$\text{median}(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

As observed by Goodstein [9] (see also Rudeanu [14]), polynomial functions are exactly those functions which can be represented by formulas in disjunctive and conjunctive normal forms. In fact, each polynomial function $f \colon L^n \to L$ is uniquely determined by its restriction to $\{0, 1\}^n$. Due to their relevance in the sequel, we recall some known results concerning normal form representations of polynomial functions in the special case where L is a chain. The following result is due to Goodstein [9].

Proposition 1. *(a) Every polynomial function is completely determined by its restriction to $\{0, 1\}^n$.*

(b) A function $g \colon \{0, 1\}^n \to L$ can be extended to a polynomial function $f \colon L^n \to L$ if and only if it is nondecreasing. In this case, the extension is unique.

(c) For any $f \colon L^n \to L$, the following are equivalent:

  (i) f is a polynomial function.

  (ii) There exists $\alpha \colon 2^{[n]} \to L$ such that

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(\alpha(I) \wedge \bigwedge_{i \in I} x_i \Big). \qquad (1)$$

  (iii) There exists $\beta \colon 2^{[n]} \to L$ such that

$$f(\mathbf{x}) = \bigwedge_{I \subseteq [n]} \Big(\beta(I) \vee \bigvee_{i \in I} x_i \Big). \qquad (2)$$

The expressions given in (1) and (2) are usually referred to as the *disjunctive normal form* (DNF) representation and the *conjunctive normal form* (CNF) representation, respectively, of the polynomial function f.

Remark 1. By requiring α and β to be nonconstant functions from $2^{[n]}$ to $\{0, 1\}$ and satisfying $\alpha(\varnothing) = 0$ and $\beta(\varnothing) = 1$, respectively, we obtain the analogue of (c) of Proposition 1 for term functions.

As observed by Marichal [13], the DNF and CNF representations of polynomial functions $f \colon L^n \to L$ are not necessarily unique. For instance, we have

$$x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2).$$

However, from among all the possible set functions α (resp. β) defining the DNF (resp. CNF) representation of f, only one is isotone (resp. antitone), namely the function $\alpha_f \colon 2^{[n]} \to L$ (resp. $\beta_f \colon 2^{[n]} \to L$) defined by

$$\alpha_f(I) = f(\mathbf{e}_I) \qquad (\text{resp. } \beta_f(I) = f(\mathbf{e}_{[n] \setminus I})), \qquad (3)$$

where $\mathbf{e}_I$ denotes the element of $\{0, 1\}^n$ whose ith component is 1 if and only if $i \in I$.

In the case when L is a chain, it was shown in [4] that the DNF and CNF representations of polynomial functions $f \colon L^n \to L$ can be refined and given in terms of standard simplices of L^n. Let σ be a permutation on $[n]$. The *standard simplex* of L^n associated with σ is the subset $L^n_\sigma \subset L^n$ defined by

$$L^n_\sigma = \{(x_1, \ldots, x_n) \in L^n \colon x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \cdots \leqslant x_{\sigma(n)}\}.$$

For each $i \in [n]$, define $S^\uparrow_\sigma(i) = \{\sigma(i), \ldots, \sigma(n)\}$ and $S^\downarrow_\sigma(i) = \{\sigma(1), \ldots, \sigma(i)\}$. As a matter of convenience, set $S^\uparrow_\sigma(n+1) = S^\downarrow_\sigma(0) = \varnothing$.

Proposition 2. *For any function $f\colon L^n \to L$, the following conditions are equivalent:*

(i) *f is a polynomial function.*

(ii) *For any permutation σ on $[n]$ and every $\mathbf{x} \in L_\sigma^n$, we have*

$$f(\mathbf{x}) = \bigvee_{i=1}^{n+1} \big(\alpha_f(S_\sigma^\uparrow(i)) \wedge x_{\sigma(i)}\big),$$

where $x_{\sigma(n+1)} = 1$.

(iii) *For any permutation σ on $[n]$ and every $\mathbf{x} \in L_\sigma^n$, we have*

$$f(\mathbf{x}) = \bigwedge_{i=0}^{n} \big(\beta_f(S_\sigma^\downarrow(i)) \wedge x_{\sigma(i)}\big),$$

where $x_{\sigma(0)} = 0$.

3 Motivating characterizations

Even though horizontal maxitivity and comonotonic maxitivity, as well as their dual counterparts, play an important role in aggregation theory (as properties shared by noteworthy classes of aggregation functions), they have not yet been described independently. In this section we investigate each of these properties and determine their corresponding function classes (up to weak versions of monotonicity, in the cases of horizontal maxitivity and minitivity).

3.1 Horizontal maxitivity and minitivity

Recall that a function $f\colon L^n \to L$ is said to be

- *horizontally maxitive* if, for every $\mathbf{x} \in L^n$ and every $c \in L$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c),$$

 where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise.

- *horizontally minitive* if, for every $\mathbf{x} \in L^n$ and every $c \in L$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c),$$

 where $[\mathbf{x}]^c$ is the n-tuple whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise.

Let us consider the following weak forms of nondecreasing monotonicity:

$(\mathbf{P_1})$ $f(\mathbf{e} \wedge c) \leqslant f(\mathbf{e}' \wedge c)$ for every $\mathbf{e}, \mathbf{e}' \in \{0,1\}^n$ such that $\mathbf{e} \leqslant \mathbf{e}'$ and every $c \in L$.

$(\mathbf{D_1})$ $f(\mathbf{e} \vee c) \leqslant f(\mathbf{e}' \vee c)$ for every $\mathbf{e}, \mathbf{e}' \in \{0,1\}^n$ such that $\mathbf{e} \leqslant \mathbf{e}'$ and every $c \in L$.

$(\mathbf{P_2})$ $f(\mathbf{e} \wedge c) \leqslant f(\mathbf{e} \wedge c')$ for every $\mathbf{e} \in \{0,1\}^n$ and every $c, c' \in L$ such that $c \leqslant c'$.

$(\mathbf{D_2})$ $f(\mathbf{e} \vee c) \leqslant f(\mathbf{e} \vee c')$ for every $\mathbf{e} \in \{0,1\}^n$ and every $c, c' \in L$ such that $c \leqslant c'$.

Theorem 3. *A function $f\colon L^n \to L$ is horizontally maxitive and satisfies $\mathbf{P_1}$ if and only if there exists $g\colon L^n \to L$ satisfying $\mathbf{P_2}$ such that*

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} g\Big(\mathbf{e}_I \wedge \bigwedge_{i \in I} x_i\Big).$$

In this case, we can choose $g = f$.

Similarly, we obtain the following dual characterization:

Theorem 4. *A function $f\colon L^n \to L$ is horizontally minitive and satisfies $\mathbf{D_1}$ if and only if there exists $g\colon L^n \to L$ satisfying $\mathbf{D_2}$ such that*

$$f(\mathbf{x}) = \bigwedge_{I \subseteq [n]} g\Big(\mathbf{e}_{[n]\setminus I} \vee \bigvee_{i \in I} x_i\Big).$$

In this case, we can choose $g = f$.

From Theorems 3 and 4 we have the following corollary.

Corollary 5. *A function $f\colon L^n \to L$ is horizontally maxitive (resp. horizontally minitive) and satisfies $\mathbf{P_1}$ (resp. $\mathbf{D_1}$) if and only if there are unary nondecreasing functions $\varphi_I\colon L \to L$, for $I \subseteq [n]$, such that*

$$
\begin{aligned}
f(\mathbf{x}) &= \bigvee_{I \subseteq [n]} \big(\alpha_f(I) \wedge \bigwedge_{i \in I} \varphi_I(x_i)\big) \\
(\text{resp. } f(\mathbf{x}) &= \bigwedge_{I \subseteq [n]} \big(\beta_f(I) \vee \bigvee_{i \in I} \varphi_I(x_i)\big)),
\end{aligned}
$$

where the set function α_f (resp. β_f) is defined in (3). In this case, we can choose $\varphi_I(x) = f(\mathbf{e}_I \wedge x)$ (resp. $\varphi_I(x) = f(\mathbf{e}_{[n]\setminus I} \vee x)$) for every $I \subseteq [n]$.

Remark 2. (i) Theorem 3 (resp. Theorem 4) provides the description of those horizontally maxitive (resp. horizontally minitive) functions which are nondecreasing.

(ii) Every Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ satisfying $f(\mathbf{0}) \leqslant f(\mathbf{x})$ (resp. $f(\mathbf{x}) \leqslant f(\mathbf{1})$) is horizontally maxitive (resp. horizontally minitive). Moreover, not all such functions are nondecreasing, thus showing that condition $\mathbf{P_1}$ (resp. $\mathbf{D_1}$) is necessary in Theorem 3 (resp. Theorem 4).

(iii) As shown in [4], polynomial functions $f\colon L^n \to L$ are exactly those $\overline{\mathcal{R}}_f$-idempotent (i.e., satisfying $f(c, \ldots, c) = c$ for every $c \in \overline{\mathcal{R}}_f$) which are nondecreasing, horizontally maxitive, and horizontally minitive.

(iv) The concept of horizontal maxitivity was introduced, in the case when L is the real interval $[0,1]$, by Benvenuti et al. [2] as a general property of the Sugeno integral.

3.2 Comonotonic maxitivity and minitivity

Two vectors $\mathbf{x}, \mathbf{x}' \in L^n$ are said to be *comonotonic* if there exists a permutation σ on $[n]$ such that $\mathbf{x}, \mathbf{x}' \in L_\sigma^n$. A function $f\colon L^n \to L$ is said to be

- *comonotonic maxitive* if, for any two comonotonic vectors $\mathbf{x}, \mathbf{x}' \in L^n$, we have

$$f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}').$$

- *comonotonic minitive* if, for any two comonotonic vectors $\mathbf{x}, \mathbf{x}' \in L^n$, we have

$$f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}').$$

Note that for any $\mathbf{x} \in L^n$ and any $c \in L$, the vectors $\mathbf{x} \vee c$ and $[\mathbf{x}]^c$ are comonotonic. As a consequence, if a function $f \colon L^n \to L$ is comonotonic maxitive (resp. comonotonic minitive), then it is horizontally maxitive (resp. horizontally minitive). It was also observed in [4] that if f is comonotonic maxitive or comonotonic minitive, then it is nondecreasing. Moreover, we obtain the following result.

Lemma 6. *A function $f \colon L^n \to L$ is comonotonic maxitive (resp. comonotonic minitive) if and only if it is horizontally maxitive (resp. horizontally minitive) and satisfies* $\mathbf{P_1}$ *(resp.* $\mathbf{D_1}$*).*

Combining Theorems 3 and 4 with Lemma 6, we immediately obtain the descriptions of the classes of comonotonic maxitive and comonotonic minitive functions.

Theorem 7. *A function $f \colon L^n \to L$ is comonotonic maxitive if and only if there exists $g \colon L^n \to L$ satisfying* $\mathbf{P_2}$ *such that*

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} g\Big(\mathbf{e}_I \wedge \bigwedge_{i \in I} x_i\Big).$$

In this case, we can choose $g = f$.

Theorem 8. *A function $f \colon L^n \to L$ is comonotonic minitive if and only if there exists $g \colon L^n \to L$ satisfying* $\mathbf{D_2}$ *such that*

$$f(\mathbf{x}) = \bigwedge_{I \subseteq [n]} g\Big(\mathbf{e}_{[n] \setminus I} \vee \bigvee_{i \in I} x_i\Big).$$

In this case, we can choose $g = f$.

As before, we have the following corollary.

Corollary 9. *A function $f \colon L^n \to L$ is comonotonic maxitive (resp. comonotonic minitive) if and only if there are unary nondecreasing functions $\varphi_I \colon L \to L$, for $I \subseteq [n]$, such that*

$$
\begin{aligned}
f(\mathbf{x}) &= \bigvee_{I \subseteq [n]} \Big(\alpha_f(I) \wedge \bigwedge_{i \in I} \varphi_I(x_i)\Big) \\
(\text{resp. } f(\mathbf{x}) &= \bigwedge_{I \subseteq [n]} \big(\beta_f(I) \vee \bigvee_{i \in I} \varphi_I(x_i)\big)),
\end{aligned}
$$

where the set function α_f (resp. β_f) is defined in (3). In this case, we can choose $\varphi_I(x) = f(\mathbf{e}_I \wedge x)$ (resp. $\varphi_I(x) = f(\mathbf{e}_{[n] \setminus I} \vee x)$) for every $I \subseteq [n]$.

Remark 3. (i) An alternative description of comonotonic maxitive (resp. comonotonic minitive) functions was obtained in Grabisch et al. [10, §2.5] in the case when L is a real interval.

(ii) It was shown in [4] that polynomial functions $f \colon L^n \to L$ are exactly those $\overline{\mathcal{R}}_f$-idempotent functions which are comonotonic maxitive and comonotonic minitive.

(ii) Comonotonic minitivity and maxitivity were introduced in the context of Sugeno integrals in de Campos et al. [5].

4 Quasi-polynomial functions

Motivated by the results of Section 3 concerning horizontal maxitivity and comonotonic maxitivity, as well as their dual counterparts, we now study combinations of these properties. This will lead to a relaxation of the notion of polynomial function, which we will refer to as *quasi-polynomial function*. Accordingly, we introduce weaker variants of well-established properties, such as homogeneity and median decomposability, which are then used to provide further axiomatizations of the class of quasi-polynomial functions.

4.1 Motivation and definition

We start by looking at combinations of those properties studied in Section 3. These are considered in the following result.

Theorem 10. *Let $f \colon L^n \to L$ be a function. The following assertions are equivalent:*

(i) f is horizontally maxitive, horizontally minitive, and satisfies $\mathbf{P_1}$ *or* $\mathbf{D_1}$*.*

(ii) f is comonotonic maxitive and comonotonic minitive.

(iii) f is horizontally maxitive and comonotonic minitive.

(iv) f is comonotonic maxitive and horizontally minitive.

(v) There exist a polynomial function $p \colon L^n \to L$ and a nondecreasing function $\varphi \colon L \to L$ such that

$$f(x_1, \ldots, x_n) = p(\varphi(x_1), \ldots, \varphi(x_n)).$$

If these conditions hold then we can choose for p the unique polynomial function p_f extending $f|_{\{0,1\}^n}$ and for φ the diagonal section δ_f of f.

Theorem 10 motivates the following definition.

Definition 11. We say that a function $f \colon L^n \to L$ is a *quasi-polynomial function* (resp. a *discrete quasi-Sugeno integral*, a *quasi-term function*) if there exist a polynomial function (resp. a discrete Sugeno integral, a term function) $p \colon L^n \to L$ and a nondecreasing function $\varphi \colon L \to L$ such that $f = p \circ \varphi$, that is,

$$f(x_1, \ldots, x_n) = p(\varphi(x_1), \ldots, \varphi(x_n)). \qquad (4)$$

Remark 4. (i) Note that each quasi-polynomial function $f \colon L^n \to L$ can be represented as a combination of constants and a nondecreasing unary function φ (applied to the projections $\mathbf{x} \mapsto x_i$) using the lattice operations $\vee$ and $\wedge$.

(ii) In the setting of decision-making under uncertainty, the nondecreasing function φ in (4) can be thought of as a *utility function* and the corresponding quasi-polynomial function as a (qualitative) *global preference functional*; see for instance Dubois et al. [6].

Note that the functions p and φ in (4) are not necessarily unique. For instance, if f is a constant $c \in L$, then we could choose $p \equiv c$ and φ arbitrarily, or p idempotent and $\varphi \equiv c$. We now describe all possible choices for p and φ. For any integers $m, n \geqslant 1$, any vector $\mathbf{x} \in L^m$, and any function $f \colon L^n \to L$, we define $\langle \mathbf{x} \rangle_f \in L^m$ as the m-tuple

$$\langle \mathbf{x} \rangle_f = \text{median}(f(\mathbf{0}), \mathbf{x}, f(\mathbf{1})),$$

where the right-hand side median is taken componentwise.

Proposition 12. *Let* $f\colon L^n \to L$ *be a quasi-polynomial function and let* $p_f\colon L^n \to L$ *be the unique polynomial function extending* $f|_{\{0,1\}^n}$. *We have*

$$\{(p,\varphi)\colon f = p \circ \varphi\} = \{(p,\varphi)\colon p_f = \langle p\rangle_f \text{ and } \delta_f = \langle\varphi\rangle_p\},$$

where p and φ stand for polynomial and unary nondecreasing functions, respectively. In particular, we have $f = p_f \circ \delta_f$.

It was shown in Marichal [13] that every polynomial function $p\colon L^n \to L$ can be represented as $\langle q\rangle_p$ for some discrete Sugeno integral $q\colon L^n \to L$. Combining this with Proposition 12, we obtain the next result.

Corollary 13. *The class of quasi-polynomial functions is exactly the class of discrete quasi-Sugeno integrals.*

4.2 Further axiomatizations

We now recall some properties of polynomial functions, namely homogeneity and median decomposability, and we propose weaker variants of these to provide alternative axiomatizations of the class of quasi-polynomial functions.

4.2.1 Quasi-homogeneity

Let S be a subset of L. A function $f\colon L^n \to L$ is said to be *S-max homogeneous* (resp. *S-min homogeneous*) if for every $\mathbf{x} \in L^n$ and every $c \in S$, we have

$$f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c \qquad (\text{resp. } f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c).$$

Although polynomial functions $p\colon L^n \to L$ share both of these properties for any $S \subseteq \overline{\mathcal{R}}_p$, this is not the case for quasi-polynomial functions. For instance, let $f_1, f_2\colon [0,1] \to [0,1]$ be respectively given by $f_1(x) = x^2$ and $f_2(x) = \sqrt{x}$. Clearly, f_1 and f_2 are quasi-polynomial functions but, e.g., for $x = c$, we have

$$f_1(x \vee c) < f_1(x) \vee c \quad \text{and} \quad f_2(x \wedge c) > f_2(x) \wedge c.$$

This example motivates the following relaxations. We say that a function $f\colon L^n \to L$ is *quasi-max homogeneous* (resp. *quasi-min homogeneous*) if for every $\mathbf{x} \in L^n$ and $c \in L$, we have

$$f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee \delta_f(c) \qquad (\text{resp. } f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge \delta_f(c)).$$

Observe that if f is $\overline{\mathcal{R}}_f$-idempotent (i.e., satisfying $f(c,\dots,c) = c$ for every $c \in \overline{\mathcal{R}}_f$), then $\overline{\mathcal{R}}_f$-min homogeneity (resp. $\overline{\mathcal{R}}_f$-max homogeneity) is equivalent to quasi-min homogeneity (resp. quasi-max homogeneity).

Lemma 14. *Let* $f\colon L^n \to L$ *be nondecreasing and quasi-min homogeneous (resp. quasi-max homogeneous). Then f is quasi-max homogeneous (resp. quasi-min homogeneous) if and only if it is horizontally maxitive (resp. horizontally minitive).*

Combining Theorem 10 and Lemma 14, we obtain a characterization of quasi-polynomial functions in terms of quasi-min homogeneity and quasi-max homogeneity.

Theorem 15. *A function* $f\colon L^n \to L$ *is a quasi-polynomial function if and only if it is nondecreasing, quasi-max homogeneous, and quasi-min homogeneous.*

4.2.2 Quasi-median decomposability

A function $f\colon L^n \to L$ is said to be *median decomposable* [13] if, for every $\mathbf{x} \in L^n$ and every $k \in [n]$, we have

$$f(\mathbf{x}) = \text{median}\left(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\right),$$

where $\mathbf{x}_k^c = (x_1,\dots,x_{k-1},c,x_{k+1},\dots,x_n)$ for any $c \in L$. As Marichal [13] showed, the class of polynomial functions are exactly those functions which are median decomposable.

In complete analogy with the previous subsection we propose the following weaker variant of median decomposability. We say that a function $f\colon L^n \to L$ is *quasi-median decomposable* if, for every $\mathbf{x} \in L^n$ and every $k \in [n]$, we have

$$f(\mathbf{x}) = \text{median}\left(f(\mathbf{x}_k^0), \delta_f(x_k), f(\mathbf{x}_k^1)\right).$$

Note that every nondecreasing unary function is quasi-median decomposable.

Observe that $\vee$ and $\wedge$, as well as any nondecreasing function $\varphi\colon L \to L$, are quasi-median decomposable. Also, it is easy to see that any combination of constants and a nondecreasing unary function φ using $\vee$ and $\wedge$ is quasi-median decomposable and hence, by Remark 4 (i), every quasi-polynomial function is quasi-median decomposable. Our following result asserts that quasi-median decomposable functions $f\colon L^n \to L$ with a nondecreasing diagonal section δ_f are exactly the quasi-polynomial functions.

Theorem 16. *A function* $f\colon L^n \to L$ *is a quasi-polynomial function if and only if δ_f is nondecreasing and f is quasi-median decomposable.*

5 Some special classes of quasi-polynomial functions

In this final section we consider few noteworthy subclasses of quasi-polynomial functions, namely those of quasi-term functions and quasi-weighted maximum and minimum functions, and provide characterizations accordingly.

5.1 Quasi-term functions

We say that a function $f\colon L^n \to L$ is

- *conservative* if, for every $\mathbf{x} \in L^n$, we have $f(\mathbf{x}) \in \{x_1,\dots,x_n\}$.

- *quasi-conservative* if, for every $\mathbf{x} \in L^n$, we have $f(\mathbf{x}) \in \{\delta_f(x_1),\dots,\delta_f(x_n)\}$.

Note that, if f is idempotent, then it is quasi-conservative if and only if it is conservative.

Theorem 17. *A quasi-polynomial function* $f\colon L^n \to L$ *is a quasi-term function if and only if it is quasi-conservative.*

5.2 Quasi-weighted maximum and minimum functions

A function $f\colon L^n \to L$ is said to be a *weighted maximum function* if there are $v_0, v_1, \dots, v_n \in L$ such that

$$f(\mathbf{x}) = v_0 \vee \bigvee_{i \in [n]} (v_i \wedge x_i). \tag{5}$$

Similarly, $f\colon L^n \to L$ is said to be a *weighted minimum function* if there are $w_0, w_1, \dots, w_n \in L$ such that

$$f(\mathbf{x}) = w_0 \wedge \bigwedge_{i \in [n]} (w_i \vee x_i). \tag{6}$$

We say that a function $f\colon L^n \to L$ is a *quasi-weighted maximum function* (resp. a *quasi-weighted minimum function*) if there exist a weighted maximum function (resp. a weighted minimum function) $p\colon L^n \to L$ and a nondecreasing function $\varphi\colon L \to L$ such that $f = p \circ \varphi$.

To present a axiomatization of each of these classes, we need to recall some terminology. We say that a function $f\colon L^n \to L$ is

- *maxitive* if, for every $\mathbf{x}, \mathbf{x}' \in L^n$, we have $f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}')$.

- *minitive* if, for every $\mathbf{x}, \mathbf{x}' \in L^n$, we have $f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}')$.

We first recall the descriptions of maxitive and minitive functions; see Dubois and Prade [8] and Marichal [12].

Proposition 18. *A function $f\colon L^n \to L$ is maxitive (resp. minitive) if and only if there are nondecreasing unary functions $f_i\colon L \to L$ ($i \in [n]$) such that, for every $\mathbf{x} \in L^n$,*

$$f(\mathbf{x}) = \bigvee_{i \in [n]} f_i(x_i) \qquad (resp. \ f(\mathbf{x}) = \bigwedge_{i \in [n]} f_i(x_i)).$$

Theorem 19. *Let $f\colon L^n \to L$ be a quasi-polynomial function. Then f is a quasi-weighted maximum function (resp. quasi-weighted minimum function) if and only if it is maxitive (resp. minitive).*

Remark 5. (i) Idempotent weighted maximum functions $f\colon L^n \to L$ are those functions (5) for which $v_0 = 0$ and $\vee_{i \in [n]} v_i = 1$. Dually, idempotent weighted minimum functions $f\colon L^n \to L$ are those functions (6) for which $w_0 = 1$ and $\wedge_{i \in [n]} w_i = 0$. These functions were introduced on real intervals by Dubois and Prade [7] in fuzzy set theory.

(ii) As observed in Proposition 12, the underlying weighted maximum function (resp. weighted minimum function) defining a given quasi-weighted maximum function (resp. quasi-weighted minimum function) can be chosen to be idempotent.

References

[1] G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*. Studies in Fuziness and Soft Computing. Springer, Berlin, 2007.

[2] P. Benvenuti, R. Mesiar, and D. Vivona. Monotone set functions-based integrals. In *Handbook of measure theory, Vol. I, II*, pages 1329–1379. North-Holland, Amsterdam, 2002.

[3] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.

[4] M. Couceiro and J.-L. Marichal. Representations and characterizations of polynomial functions on chains. http://arxiv.org/abs/0811.0309

[5] L. M. de Campos, M. T. Lamata, and S. Moral. A unified approach to define fuzzy integrals. *Fuzzy Sets and Systems*, 39(1):75–90, 1991.

[6] D. Dubois, J.-L. Marichal, H. Prade, M. Roubens, and R. Sabbadin. The use of the discrete Sugeno integral in decision-making: a survey. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 9(5):539–561, 2001.

[7] D. Dubois and H. Prade. Weighted minimum and maximum operations in fuzzy set theory. *Inform. Sci.*, 39(2):205–210, 1986.

[8] D. Dubois and H. Prade. Aggregation of possibility measures. In *Multiperson decision making models using fuzzy sets and possibility theory*, volume 18 of *Theory Decis. Lib. Ser. B Math. Statist. Methods*, pages 55–63. Kluwer Acad. Publ., Dordrecht, 1990.

[9] R. L. Goodstein. The solution of equations in a lattice. *Proc. Roy. Soc. Edinburgh Sect. A*, 67:231–242, 1965/1967.

[10] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, UK, to appear in 2009.

[11] M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy measures and integrals*, volume 40 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg, 2000. Theory and applications.

[12] J.-L. Marichal. On Sugeno integral as an aggregation function. *Fuzzy Sets and Systems*, 114(3):347–365, 2000.

[13] J.-L. Marichal. Weighted lattice polynomials. *Discrete Mathematics*, 309(4):814–820, 2009.

[14] S. Rudeanu. *Lattice functions and equations*. Springer Series in Discrete Mathematics and Theoretical Computer Science. Springer-Verlag London Ltd., London, 2001.

[15] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, Tokyo, 1974.

[16] M. Sugeno. Fuzzy measures and fuzzy integrals—a survey. In M. M. Gupta, G. N. Saridis, and B. R. Gaines, editors, *Fuzzy automata and decision processes*, pages 89–102. North-Holland, New York, 1977.