Quasi-Lovász extensions and their symmetric counterparts

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Order simplexes

Let σ be a permutation on $[n] = \{1, \dots, n\}$ $(\sigma \in S_n)$

$$\mathbb{R}^n_\sigma = \left\{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)} \right\}$$

$$[0,1]^n_\sigma = \mathbb{R}^n_\sigma \cap [0,1]^n$$

Example : $n = 2$ ($2! = 2$ permutations $\Rightarrow$ 2 simplexes!)

In general: The hypercube $[0,1]^n$ has exactly $n!$ simplexes, and each simplex $[0,1]^n_\sigma$ has exactly $n+1$ vertices.

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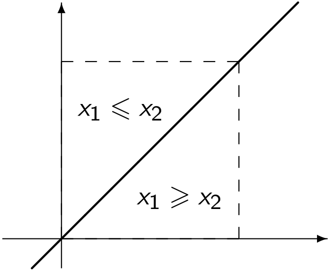

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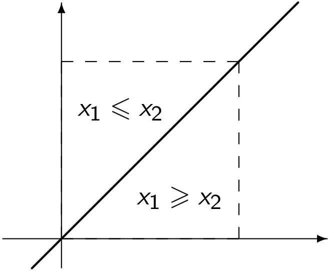

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Lovász extension

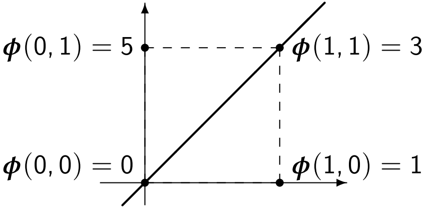
Let $\boldsymbol{\phi}\colon \{0,1\}^n \to \mathbb{R}$ be a (*pseudo-Boolean*) function s. t. $\boldsymbol{\phi}(\mathbf{0}) = 0$.

Definition (Lovász, 1983)

The *Lovász extension* of $\boldsymbol{\phi}\colon \{0,1\}^n \to \mathbb{R}$ is the function $f_{\boldsymbol{\phi}}\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each $\mathbb{R}^n_\sigma$ is the unique linear function which coincides with $\boldsymbol{\phi}$ at the $n+1$ vertices of the simplex $[0,1]^n_\sigma$

In particular: $f_{\boldsymbol{\phi}}|_{\{0,1\}^n} = \boldsymbol{\phi}$

Example :



$$x_1 \geqslant x_2 \quad \Rightarrow \quad f_\phi(x_1, x_2) = x_1 + 2x_2$$
$$x_1 \leqslant x_2 \quad \Rightarrow \quad f_\phi(x_1, x_2) = -2x_1 + 5x_2$$

On $\mathbb{R}^2$:

$$f_\phi(x_1, x_2) = x_1 + 5x_2 - 3\min(x_1, x_2)$$

Representations of Lovász extensions

In general: f_{ϕ} can always be written in the form

$$f_{\phi}(\mathbf{x}) = \sum_{S \subseteq [n]} a_{\phi}(S) \min_{i \in S} x_i \qquad (\mathbf{x} \in \mathbb{R}^n)$$

where the coefficients $a_{\phi}(S)$ are given by the Möbius transform of ϕ

Consequence: f_{ϕ} is always piecewise linear and continuous!

Representations of Lovász extensions

... and on each order simplex $\mathbb{R}^n_\sigma$?

In general :

$$f_{\boldsymbol{\phi}}(\mathbf{x}) = x_{\sigma(1)}\,\boldsymbol{\phi}(\mathbf{1}) + \sum_{i=2}^{n}(x_{\sigma(i)} - x_{\sigma(i-1)})\,\boldsymbol{\phi}(\mathbf{1}_{\{\sigma(i),\ldots,\sigma(n)\}}) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma)$$

Choquet integral

$f \colon \mathbb{R}^n \to \mathbb{R}$ is a Lovász extension if there is $\boldsymbol{\phi} \colon \{0,1\}^n \to \mathbb{R}$ s.t. $f = f_{\boldsymbol{\phi}}$.

Definition

A **Choquet integral** is a nondecreasing Lovász extension (vanishing at 0).

Generalization: Quasi-Lovász extensions

Let I be a real interval containing 0.

Definition

A quasi-Lovász extension is a function $f\colon I^n \to \mathbb{R}$ defined by

$$f \; = \; L \circ \varphi \; = \; L \circ (\varphi, \ldots, \varphi),$$

where

1. $L\colon \mathbb{R}^n \to \mathbb{R}$ is a Lovász extension
2. $\varphi\colon I \to \mathbb{R}$ is a nondecreasing function satisfying $\varphi(0) = 0$.

In DMU: φ is a utility function and f an overall preference functional.

Symmetric Lovász extension

For $\mathbf{x} \in \mathbb{R}^n$, set $\mathbf{x}^+ = \mathbf{x} \vee 0$ and $\mathbf{x}^- = (-\mathbf{x})^+$

Definition

The symmetric Lovász extension of $\phi : \{0,1\}^n \to \mathbb{R}$ is defined by

$$\check{f}_\phi(\mathbf{x}) = f_\phi(\mathbf{x}^+) - f_\phi(\mathbf{x}^-)$$

For a (nonsymmetric) Lovász extension:

$$f_\phi(\mathbf{x}) = f_\phi(\mathbf{x}^+) - f_{\phi^d}(\mathbf{x}^-)$$

where $\phi^d(\mathbf{1}_A) = \phi(\mathbf{1}) - \phi(\mathbf{1} - \mathbf{1}_A) = \phi(\mathbf{1}) - \phi(\mathbf{1}_{[n]\setminus A})$.

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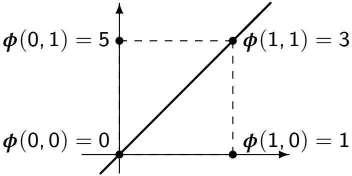
$$\check{f}_{\boldsymbol{\phi}}(\mathbf{x}) = f_{\boldsymbol{\phi}}(\mathbf{x}^+) - f_{\boldsymbol{\phi}}(\mathbf{x}^-)$$

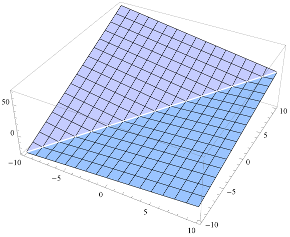
Immediate consequences:

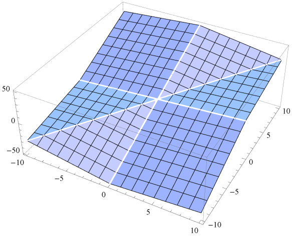
- $\check{f}_{\boldsymbol{\phi}}$ is piecewise linear and continuous
- $\check{f}_{\boldsymbol{\phi}}(c\mathbf{x}) = c\,\check{f}_{\boldsymbol{\phi}}(\mathbf{x})$ for every $c \in \mathbb{R}$
- **Šipoš (1979)**: A symmetric Choquet integral is a nondecreasing symmetric Lovász extension

Symmetric Lovász extension

Example :







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$$\check{f}_{\phi}(\mathbf{x}) = f_{\phi}(\mathbf{x}^+) - f_{\phi}(\mathbf{x}^-)$$

Symmetrizing quasi-Lovász extensions

Let I be a real interval centered at 0: $-x \in I$ whenever $x \in I$.

Definition

A symmetric quasi-Lovász extension is a function $f\colon I^n \to \mathbb{R}$ defined by

$$f = \check{L} \circ \varphi = \check{L} \circ (\varphi, \ldots, \varphi)$$

1. $\check{L}\colon \mathbb{R}^n \to \mathbb{R}$ is a symmetric Lovász extension
2. $\varphi\colon I \to \mathbb{R}$ is a nondecreasing *odd* function.

Comonotonic modularity

$\mathbf{x}, \mathbf{x}' \in I^n$ are comonotonic if $\mathbf{x}, \mathbf{x}' \in I^n_\sigma = I^n \cap \mathbb{R}^n_\sigma$ for some $\sigma \in S_n$.

Definition

$f\colon I^n \to \mathbb{R}$ is comonotonically modular if for all comonotonic $\mathbf{x}, \mathbf{x}' \in I^n$

$$f(\mathbf{x}) + f(\mathbf{x}') = f(\mathbf{x} \vee \mathbf{x}') + f(\mathbf{x} \wedge \mathbf{x}')$$

1. **If** $f(\mathbf{0}) = 0$, **then**

$$f(\mathbf{x}) = f(\mathbf{x}^+) + f(-\mathbf{x}^-) \quad (\text{take } \mathbf{x}' = \mathbf{0})$$

2. **If** f is odd, **then**

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Complete description of comonotonically modular functions

For $0 \in I \subseteq \mathbb{R}$, let $I_- = I \cap \mathbb{R}_-$ and $I_+ = I \cap \mathbb{R}_+$

Theorem: For any $f\colon I^n \to \mathbb{R}$ s.t. $f(\mathbf{0}) = 0$, T.F.A.E.:

1. f is comonotonically modular.
2. There are comonotonically modular $g\colon I_+^n \to \mathbb{R}$ and $h\colon I_-^n \to \mathbb{R}$ s.t.

$$f(\mathbf{x}) = g(\mathbf{x}^+) + h(-\mathbf{x}^-).$$

Furthermore: For every $\mathbf{x} \in I_\sigma^n$ s.t. $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$

$$g(\mathbf{x}^+) = \sum_{p+1 \leqslant i \leqslant n} \left(g(x_{\sigma(i)}\mathbf{1}_{\{\sigma(i),\ldots,\sigma(n)\}}) - g(x_{\sigma(i)}\mathbf{1}_{\{\sigma(i+1),\ldots,\sigma(n)\}})\right)$$

$$h(-\mathbf{x}^-) = \sum_{1 \leqslant i \leqslant p} \left(h(x_{\sigma(i)}\mathbf{1}_{\{\sigma(1),\ldots,\sigma(i)\}}) - h(x_{\sigma(i)}\mathbf{1}_{\{\sigma(1),\ldots,\sigma(i-1)\}})\right)$$

In this case: We can choose $g = f|_{I_+^n}$ and $h = f|_{I_-^n}$.

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Comonotonic modularity $\Leftrightarrow$ comonotonic separability

Corollary:

f is comonotonically modular **iff** it is **comonotonically separable**:

for every $\sigma \in S_n$, there are $f_i^\sigma \colon I \to \mathbb{R}$, $i \in [n]$, s.t.

$$f(\mathbf{x}) = \sum_{i=1}^{n} f_i^\sigma(x_{\sigma(i)}) \qquad \text{for } \mathbf{x} \in I^n \cap \mathbb{R}_\sigma^n.$$

Axiomatization of symmetric quasi-Lovász extensions

Definition

$f\colon I^n \to \mathbb{R}$ is oddly homogeneous if there is a nondecreasing odd function $\varphi\colon I \to \mathbb{R}$ s.t. for every $x \in I$ and $A \subseteq [n]$

$$f(x\mathbf{1}_A) = \varphi(x)f(\mathbf{1}_A)$$

Theorem: Assume that I is centered at 0 with $[-1,1] \subseteq I \subseteq \mathbb{R}$...

If $f\colon I^n \to \mathbb{R}$ is nonconstant, then T.F.A.E:

1. f is symmetric quasi-Lovász with $f(\mathbf{1}_A) \neq 0$ for some $A \subseteq [n]$.
2. f is comonotonically modular and oddly homogeneous.

Moreover: $f = \check{L}_{f|_{\{0,1\}^n}} \circ \varphi_f$ where $\varphi_f(x) = \frac{f(x\mathbf{1}_A)}{f(\mathbf{1}_A)}$.

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Final remarks

1. For *nonsymmetric* quasi-Lovász extensions "odd homogeneity" is replaced by: there is $\varphi\colon I \to \mathbb{R}$ nondecreasing s.t.

$$f(x\mathbf{1}_A) \;=\; \mathrm{sign}(x)\,\varphi(x)\,f(\mathrm{sign}(x)\,\mathbf{1}_A)$$

2. For *symmetric* and *nonsymmetric* Lovász extensions by:

$$f(x\mathbf{1}_A) = x\,f(\mathbf{1}_A) \quad \text{and} \quad f(x\mathbf{1}_A) = \mathrm{sign}(x)\,x\,f(\mathrm{sign}(x)\,\mathbf{1}_A)$$

3. Condition $f(\mathbf{0}) = 0$ can be dropped off: $f_0 \;=\; f - f(\mathbf{0})$

Thank you for your attention!