Aggregation on bipolar scales

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1 Introduction

Most of the time aggregation functions are defined on $[0, 1]$, where 0 and 1 represent the lowest and highest scores along each dimension. We may desire to consider a third particular point e of the interval, which will play a particular role, for example a neutral value or an annihilator value (this is the case with uninorms). For convenience, we may always consider that we work on $[-1, 1]$, and 0 corresponds to our particular point e.

The motivation for such a study has its roots in psychology. In many cases, scores or utilities manipulated by humans lie on a *bipolar scale*, that is, a scale with a neutral value making the frontier between good or satisfactory scores, and bad or unsatisfactory scores. With our convention, good scores are positive ones, while negative scores reflect bad scores. Very often our behaviour with positive scores is not the same as with negative ones, hence it becomes important to define aggregation functions that are able to reflect the variety of aggregation behaviours on bipolar scales.

In the sequel, we will consider several ways to define bipolar aggregation functions, starting from some aggregation funnction defined on $[0, 1]$. We first consider associative aggregation functions, and treat separetely the case of minimum, and maximum, then we will turn to nonassociative aggregation functions.

This work is closely related and brings new insights to the following mathemetical and applied fields:

(i) algebraic structures, such as rings, groups and monoids, ordered Abelian groups. In particular, Section 3 offers a incursion into nonassociative algebra, a domain which has been scarcely investigated. Many-valued logics dealing with bipolar notions is also concerned.

(ii) integration, measure theory by providing a new type of integral (Choquet integral w.r.t. a bi-capacity). In the finite case, the notion bi-capacity is related to bi-set functions, which are known in some domains of discrete mathematics and combinatorial optimization (bisubmodular base polyhedron, see, e.g., Fujishige [1]).

(iii) decision making and mathematical economics, since the motivation of this work is rooted there. This work offers a generalization of the well-known Cumulative Prospect Theory (see Section 5).

The material presented here is drawn from Chapter 9 of [7], a forthcoming monograph on aggregation functions written by the authors.

We introduce first the fundamental concept of pseudo-difference.

Definition 1. *Let S be a t-conorm (see [8] for details on t-norms and t-conorms).*

(i) The S-difference is defined for any (a,b) in $[0,1]^2$ by $a \overset{\mathsf{S}}{-} b := \inf\{c \in [0,1] \mid \mathsf{S}(b,c) \geqslant a\}$.

(ii) The pseudo-difference associated to S is defined for any (a,b) in $[0,1]^2$ by

$$a \ominus_{\mathsf{S}} b := \begin{cases} a \overset{\mathsf{S}}{-} b & \text{if } a \geqslant b \\ -(b \overset{\mathsf{S}}{-} a) & \text{if } a < b \\ 0, & \text{if } a = b, \end{cases}$$

Proposition 1. *If S is a continuous Archimedean t-conorm with additive generator s, then $a \overset{\mathsf{S}}{-} b = s^{-1}(0 \vee (s(a) - s(b)))$, and $a \ominus_{\mathsf{S}} b = g^{-1}(g(a) - g(b))$, with $g(x) := s(x)$ for $x \geqslant 0$, and $g(x) := -s(-x)$ for $x \leqslant 0$.*

2 Associative bipolar operators

In this section, we want to define associative and commutative operators where 0 is either a neutral element or an annihilator element, which we call respectively *(symmetric) pseudo-addition* and *(symmetric) pseudo-multiplication*. This section is mainly based on [3].

We denote respectively by $\oplus, \otimes : [-1,1]^2 \to [-1,1]$ these operators.

2.1 Pseudo-additions

Our basic requirements are the following, for any $x, y, z \in [-1,1]$:

A1 Commutativity: $x \oplus y = y \oplus x$
A2 Associativity: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
A3 Neutral element $x \oplus 0 = 0 \oplus x = x$.
A4 Nondecreasing monotonicity: $x \oplus y \leqslant x' \oplus y'$, for any $x \leqslant x'$, $y \leqslant y'$.

The above requirements mean that we recognize $\oplus$ as a t-conorm when restricted to $[0,1]$, which we denote by S. Since $[-1,1]$ is a symmetric interval, and if 0 plays the role of a neutral element, then we should have

A5 Symmetry: $x \oplus (-x) = 0$, for all $x \in]-1,1[$.

From **A1**, **A2**, and **A5** we easily deduce $(-x) \oplus (-y) = -(x \oplus y)$, $\quad \forall (x,y) \in]-1,0]^2 \cup [0,1[^2$. Then, **A3** and **A4** permit to define $\oplus$ on $[-1,1]$:

$$x \oplus y = \begin{cases} \mathsf{S}(x,y) & \text{if } x, y \in [0,1] \\ -\mathsf{S}(-x,-y) & \text{if } x, y \in [-1,0] \\ x \ominus_{\mathsf{S}} (-y) & \text{if } x \in [0,1[, y \in]-1,0] \\ 1 \text{ or } -1 & \text{if } x = 1, y = -1, \end{cases} \tag{1}$$

with the remaining cases being determined by commutativity. We distinguish several cases for S. We write for convenience $x \oplus (-y) = x \ominus y$ for any $x, y \in [-1,1]^2$.

S *is a strict t-conorm* with additive generator s. Let us rescale $\oplus$ on $[0,1]^2$, calling U the result:

$$\mathsf{U}(z,t) := \frac{((2z-1) \oplus (2t-1)) + 1}{2}. \tag{2}$$

We introduce $g : [-1,1] \to [-\infty, \infty]$ by $g(x) := s(x)$ for positive x, $\quad g(x) := -s(-x)$ for negative x, i.e., g is a symmetrization of s. Then $x \oplus y = g^{-1}(g(x) + g(y))$ for any $x, y \in [-1,1]$, with the convention $\infty - \infty = \infty$ or $-\infty$. Also, U is a generated uninorm that is continuous (except at $(0,1)$ and $(1,0)$), strictly increasing on $]0,1[^2$, has neutral element $\frac{1}{2}$, and is conjunctive (respectively, disjunctive) when the convention $\infty - \infty = -\infty$ (respectively, $\infty - \infty = \infty$). Finally, we have:

Theorem 1. *Let S be a strict t-conorm with additive generator s and $\oplus$ the corresponding pseudo-addition. Then $(]-1,1[, \oplus)$ is an Abelian group.*

S *is a nilpotent t-conorm* with additive generator s. It is easy to see that the construction does not lead to an associative operator.

S *is the maximum operator.* This case will be treated in Section 3.

S *is an ordinal sum of continuous Archimedean t-conorms.* In this case too, associativity cannot hold everywhere.

2.2 Pseudo-multiplications

Our first requirements are, for any $x, y, z \in [-1,1]$

M0 0 is an annihilator element: $x \otimes 0 = 0 \otimes x = 0$
M1 Commutativity: $x \otimes y = y \otimes x$
M2 Associativity: $x \otimes (y \otimes z) = (x \otimes y) \otimes z$.

Let us adopt for the moment the following.

M3 Nondecreasing monotonicity on $[0,1]^2$: $x \otimes y \leqslant x' \otimes y'$, for any $0 \leqslant x \leqslant x' \leqslant 1$, $0 \leqslant y \leqslant y' \leqslant 1$.
M4 Neutral element for positive elements: $x \otimes 1 = 1 \otimes x = x$, for all $x \in [0,1]$,

then axioms **M1** to **M4** make $\otimes$ a t-norm on $[0,1]^2$, and **M0** is deduced from them. If pseudo-addition and pseudo-multiplication are used conjointly, a natural requirement is then distributivity.

M5 Distributivity of $\otimes$ with respect to $\oplus$: $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ and $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ for all $x, y, z \in [-1,1]$.

Then under **A1** to **A4**, and **M1** to **M4**, axiom **M5** can be satisfied on $[0,1]^2$ if and only if $\oplus = \vee$. Finally we can show:

Proposition 2. *Under M1 to M5 and A3, A5, $\otimes$ has the form $x \otimes y = \text{sign}(x \cdot y)\mathsf{T}(|x|,|y|)$, for some t-norm T.*

If distributivity is not needed, we can impose monotonicity of $\otimes$ on the whole domain $[-1,1]^2$:

M3' Nondecreasing monotonicity for $\otimes$: $x \otimes y \leqslant x' \otimes y'$, $-1 \leqslant x \leqslant x' \leqslant 1$, $-1 \leqslant y \leqslant y' \leqslant 1$.

Then, if we impose in addition

M4' Neutral element for negative numbers: $(-1) \otimes x = x$ for all $x \leqslant 0$,

up to a rescaling in $[0,1]^2$, $\otimes$ is a nullnorm with $a = 1/2$. In summary, we have shown the following.

Proposition 3. *Under M1, M2, M3', M4 and M4', $\otimes$ has the following form:*

$$x \otimes y = \begin{cases} \mathsf{T}(x,y) & \text{if } x,y \geqslant 0 \\ \mathsf{S}(x+1,y+1) - 1 & \text{if } x,y \leqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

for some t-norm T and t-conorm S.

3 Symmetric minimum and maximum

The previous section has shown that except for strict t-conorms, there was no way to build a pseudo-addition fulfilling requirements **A1** to **A5**. Hence extending the maximum on $[-1,1]^2$ in this way is not possible. However, we will show that this is in fact almost possible (see [2] for details).

3.1 The symmetric maximum

Our basic requirements are:

SM1 $\varotimes$ coincide with $\vee$ on $(L^+)^2$.
SM2 Commutativity
SM3 Associativity
SM4 0 is a neutral element
SM5 $-x$ is the symmetric image of x, i.e. $x \varotimes (-x) = 0$.

These requirements are already contradictory. In fact, **SM1** and **SM5** imply that associativity (**SM3**) cannot hold. The following can be shown.

Proposition 4. *Under conditions (SM1), (SM5) and (SM6), no operation is associative on a larger domain than $\varotimes$ defined by:*

$$x \varotimes y = \begin{cases} -(|x| \vee |y|) & \text{if } y \neq -x \text{ and } |x| \vee |y| = -x \text{ or } = -y \\ 0 & \text{if } y = -x \\ |x| \vee |y| & \text{otherwise.} \end{cases} \tag{3}$$

Except for the case $y = -x$, $x \varotimes y$ equals the absolutely larger one of the two elements x and y.

3.2 Symmetric minimum

The case of the symmetric minimum is less problematic. The following requirements determine it uniquely.

Sm1 $\varoslash$ coincides with $\wedge$ on $(L^+)^2$
Sm2 Rule of signs: $-(x \varoslash y) = (-x) \varoslash y = x \varoslash (-y)$, for all $x, y \in L$.

Under **Sm1** and **Sm2**, we get

$$x \varoslash y := \begin{cases} -(|x| \wedge |y|) & \text{if } \operatorname{sign}(x) \neq \operatorname{sign}(y) \\ |x| \wedge |y| & \text{otherwise.} \end{cases} \tag{4}$$

As for pseudo-multiplications, we could as well impose a different rule of signs, namely $-(x \varoslash y) = (-x) \varoslash (-y)$, and impose monotonicity on the whole domain. This would give, up to a rescaling, a nullnorm, namely $\mathsf{Med}_{0.5}(x,y) := \mathsf{Med}(x,y,0.5)$.

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4 Separable aggregation functions

We consider here not necessarily associative functions A. A simple way to build bipolar aggregation functions is the following. Let A^+, A^- be given aggregation functions on $[0,1]^n$. We define A on $[-1,1]^n$ by $A(\mathbf{x}) := \psi(A^+(\mathbf{x}^+), A^-(\mathbf{x}^-))$, $\quad \forall \mathbf{x} \in [-1,1]^n$, where $\mathbf{x}^+ := \mathbf{x} \vee \mathbf{0}$, $\mathbf{x}^- := (-\mathbf{x})^+$, and ψ is a pseudo-difference (Definition 1). A bipolar aggregation function defined as above is called *separable*.
   We give as illustration three cases of interest.

$A^+ = A^-$ *is a strict t-conorm* S. If $A^+ = A^-$ is a strict t-conorm S with generator s, and $\ominus_S$ is taken as pseudo-difference, we recover the construction of Section 2.

$A^+ = A^-$ *is a continuous t-conorm* S. We know by Section 2 that associativity is lost if S is not strict. Restricting to the binary case, it is always possible to apply the definition of $\oplus$ given by (1), taking the associated pseudo-difference operator $\ominus_S$. For example, considering $S = S_{\mathbf{L}}$, we easily obtain $A(x,y) = ((x+y) \wedge 1) \vee (-1)$.

A^+, A^- *are integral-based aggregation functions*. An interesting case is when A^+, A^- are integral-based aggregation functions, such as the Choquet or Sugeno integrals. Then we recover various definitions of integrals for real-valued functions. Specifically, let us take A^+, A^- to be Choquet integrals with respect to capacities μ^+, μ^-, and ψ is the usual difference $\ominus_{\mathbf{L}}$. Then:

- Taking $\mu^+ = \mu^-$ we obtain the *symmetric Choquet integral* or Šipoš integral $\check{\mathcal{C}}_\mu(\mathbf{x}) := \mathcal{C}_\mu(\mathbf{x}^+) - \mathcal{C}_\mu(\mathbf{x}^-)$.
- Taking $\mu^- = \overline{\mu^+}$ we obtain the *asymmetric Choquet integral* $\mathcal{C}_\mu(\mathbf{x}) := \mathcal{C}_\mu(\mathbf{x}^+) - \mathcal{C}_{\bar{\mu}}(\mathbf{x}^-)$.
- For the general case, we obtain what is called in decision making theory the *Cumulative Prospect Theory (CPT)* model [9] $\mathrm{CPT}_{\mu^+,\mu^-}(\mathbf{x}) := \mathcal{C}_{\mu^+}(\mathbf{x}^+) - \mathcal{C}_{\mu^-}(\mathbf{x}^-)$.

5 Integral-based aggregation functions

It is possible to generalize the above definitions based on the Choquet integral to a much wider model called the Choquet integral w.r.t bicapacities (see [4, 5] and [6] for a general construction).
   We introduce $Q(N) := \{(A,B) \mid A,B \subseteq N, A \cap B = \varnothing\}$. A *bicapacity* w on N is a function $w : Q(N) \to \mathbb{R}$ satisfying $w(\varnothing, \varnothing) = 0$, and $w(A,B) \leqslant w(C,D)$ whenever $A \subseteq C$ and $B \supseteq D$ (monotonicity).

Definition 2. *Let w be a bicapacity and $\mathbf{x} \in \mathbb{R}^n$. The Choquet integral of $\mathbf{x}$ with respect to w is given by $\mathcal{C}_w(\mathbf{x}) := \mathcal{C}_{v_{N_{\mathbf{x}}^+}}(|\mathbf{x}|)$, where $v_{N_{\mathbf{x}}^+}$ is a game on N defined by $v_{N_{\mathbf{x}}^+}(C) := w(C \cap N_{\mathbf{x}}^+, C \cap N_{\mathbf{x}}^-)$, and $N_{\mathbf{x}}^+ := \{i \in N \mid x_i \geqslant 0\}$, $N_{\mathbf{x}}^- = N \setminus N_{\mathbf{x}}^+$.*

The CPT model (and hence the asymmetric and symmetric Choquet integrals) are recovered taking a bicapacity of the form $w(A,B) = \mu_+(A) - \mu_-(B)$, for all $(A,B) \in Q(N)$.

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