Characterizations of discrete Sugeno integrals as polynomial functions on chains

30th Linz Seminar on Fuzzy Set Theory

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Preliminaries

Let L be a chain with operations $\wedge$ and $\vee$, and with least and greatest elements 0 and 1, respectively. E.g.: $L = [0, 1] \subseteq \mathbb{R}$.

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A polynomial function (on L) is any map $f : L^n \to L$, for $n \geq 1$, obtainable by finitely many applications of the following rules:

1. The projections $\mathbf{x} \mapsto x_i$, $i \in [n]$, and the constant functions $\mathbf{x} \mapsto c$, $c \in L$, are polynomial functions.

2. If $f : L^n \to L$ and $g : L^n \to L$ are polynomial functions, then $f \vee g$ and $f \wedge g$ are polynomial functions.

We say that $f\colon L^n \to L$ is S-idempotent, $S \subseteq L$, if

$$f(c, \ldots, c) = c \text{ for every } c \in S.$$

If $S = L$, then f is said to be idempotent.

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Theorem (Marichal)

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Fact

Each Sugeno integral $f : L^n \to L$ is nondecreasing, i.e., satisfies

$$\text{for every} \quad \mathbf{a}, \mathbf{b} \in L^n, \quad \mathbf{a} \leqslant \mathbf{b} \quad \Rightarrow \quad f(\mathbf{a}) \leqslant f(\mathbf{b}).$$

Examples of nondecreasing functions on $[0,1] \subseteq \mathbb{R}$

1. $f(x_1) = (0.2 \vee x_1) \wedge 0.7$.
2. $\mathrm{median}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1)$.
3. $g(x_1, x_2) = x_1 \cdot x_2$.

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Problem

Characterize (discrete) Sugeno integrals on chains.

Homogeneity (Fodor and Roubens)

A function $f\colon L^n \to L$ is said to be

1. min homogenous if, for every $\mathbf{x} \in L^n$ and $c \in L$,

$$f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c$$

2. max homogenous if, for every $\mathbf{x} \in L^n$ and $c \in L$,

$$f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c$$

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Fact

If f is min and max homogeneous, then it is idempotent.

However, note that...

Every idempotent Boolean function is min and max homog. but not all are nondecreasing, e.g.,

$$f(x_1, x_2, x_3) = (x_1 \wedge \overline{x_2} \wedge \overline{x_3}) \vee (x_1 \wedge x_2 \wedge x_3)$$

Let $L_2^n = \{\mathbf{x} \in L^n \colon |\{x_1, \ldots, x_n\}| \leqslant 2\}$.

For example...

1. $(c, c, c), (0, 1, 1), (c, d, c) \in L_2^3$ but,
2. but $(0, 1, c), (0, d, c), (c, d, e) \notin L_2^3$.

Weak variants of homogeneity

We say that a function $f\colon L^n \to L$ is

(w) weakly min homogenous if it is min homog. on L_2^n

(w) weakly max homogenous if it is max homog. on L_2^n

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(b) Boolean min homogeneous if it is min homog. on $\{0,1\}^n$

(b) Boolean max homogeneous if it is max homog. on $\{0,1\}^n$

First characterization of Sugeno integrals

Theorem (C. and Marichal)

Let $f\colon L^n \to L$ be a nondecreasing function. The following are equivalent:

(i) f is a Sugeno integral.

(ii) f is min and max homogenous.

(iii) f is weakly min and weakly max homogenous.

(iv) f is Boolean min and Boolean max homogenous.

Minitive and maxitive vector decompositions

For $c \in L$, define the upper c-cut and lower c-cut of $\mathbf{x} \in L^n$ by

1. $[\mathbf{x}]^c$ whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise
2. $[\mathbf{x}]_c$ whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise

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Example:

Let $\mathbf{x} = (0.1, 0.4, 0.7) \in [0,1]^3$ and $c = 0.5$. Then

(i) $[\mathbf{x}]^c = (0.1, 0.4, 1)$

(ii) $[\mathbf{x}]_c = (0, 0, 0.7)$

Minitive and maxitive vector decompositions

Fact

For every $\mathbf{x} \in L^n$ and $c \in L$, we have

(i) Minitive decomposition: $\mathbf{x} = (\mathbf{x} \vee c) \wedge [\mathbf{x}]^c$

(ii) Maxitive decomposition: $\mathbf{x} = (\mathbf{x} \wedge c) \vee [\mathbf{x}]_c$

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Example:

Let $\mathbf{x} = (0.1, 0.4, 0.7) \in [0,1]^3$ and $c = 0.5$. Then

(i) $(\mathbf{x} \vee c) \wedge [\mathbf{x}]^c = (0.5, 0.5, 0.7) \wedge (0.1, 0.4, 1) = \mathbf{x}$

(ii) $(\mathbf{x} \wedge c) \vee [\mathbf{x}]_c = (0.1, 0.4, 0.5) \vee (0, 0, 0.7) = \mathbf{x}$

Horizontal minitivity and maxitivity (Benvenuti, Mesiar and Vivona)

A function $f\colon L^n \to L$ is said to be

1. horizontally minitive if for every $\mathbf{x} \in L^n$ and $c \in L$,

$$f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c)$$

2. horizontally maxitive if for every $\mathbf{x} \in L^n$ and $c \in L$,

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Note that...

There are horizontally min. and max. functions which are not

1. idempotent: constant functions
2. nondecreasing: $(x_1 \wedge \overline{x_2} \wedge \overline{x_3}) \vee (x_1 \wedge x_2 \wedge x_3)$

Weak horizontal minitivity and maxitivity

A function $f\colon L^n \to L$ is said to be

1. weakly horizontally minitive if it is horizontally min. on L_2^n
2. weakly horizontally maxitive if it is horizontally max. on L_2^n

Comonotonic vectors (Hardy)

Two vectors $\mathbf{x}, \mathbf{x}' \in L^n$ are said to be comonotonic, $\mathbf{x} \sim \mathbf{x}'$, if there is a permutation σ on $[n]$ such that

$$x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \cdots \leqslant x_{\sigma(n)} \quad \text{and} \quad x'_{\sigma(1)} \leqslant x'_{\sigma(2)} \leqslant \cdots \leqslant x'_{\sigma(n)}.$$

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Lemma (C. and Marichal)

For any $\mathbf{x} \leqslant \mathbf{x}'$ in L^n, there are at most $2n-1$ vectors $\mathbf{y}^i$ s. t.

$$\mathbf{y}^i \sim \mathbf{y}^{i+1} \quad \text{and} \quad \mathbf{x} \leqslant \mathbf{y}^1 \leqslant \cdots \leqslant \mathbf{y}^{2n-1} \leqslant \mathbf{x}'.$$

Comonotonic minitivity and maxitivity (Bolaños and Campos)

A function $f\colon L^n \to L$ is said to be

1. comonotonic minitive if, for every $\mathbf{x} \sim \mathbf{x}'$, we have

$$f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}')$$

2. comonotonic maxitive if, for every $\mathbf{x} \sim \mathbf{x}'$, we have

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Proposition (C. and Marichal)

If f is comon. minitive or maxitive, then it is nondecreasing.

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Proposition (C. and Marichal)

If f is comon. minitive or maxitive, then it is nondecreasing.

However...

Constant functions are comonotonic minitive and maxitive, but not idempotent.

Second characterization of Sugeno integrals

Theorem (C. and Marichal)

Let $f\colon L^n \to L$ be a function. The following are equivalent:

(i) f is a Sugeno integral.

(ii) f is nondecreasing, idempotent, and **A** and **B**.

where: **A** is (weakly) min homogeneous, (weakly) horizontally min. or comonotonic min., and **B** is one of the dual properties.

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To avoid redundancy...

1. If **A** is comonotonic min. or **B** is comonotonic max., then remove "nondecreasing".

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To avoid redundancy...

1. If **A** is comonotonic min. or **B** is comonotonic max., then remove "nondecreasing".
2. If **A** is (weakly) min homog. or **B** is (weakly) max homog., then replace idempotency by $\{1\}$- or $\{0\}$-idempot., resp.

Median decomposibility (Marichal)

For $c \in L$ and $k \in [n]$, set $\mathbf{x}_k^c = (x_1, \ldots, x_{k-1}, c, x_{k+1}, \ldots, x_n)$.

A function $f\colon L^n \to L$ is median decomposable if

$$f(\mathbf{x}) = \mathrm{median}\left(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\right), \text{ for every } \mathbf{x} \in L^n, k \in [n].$$

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Theorem (Marichal)

Sugeno integrals are exactly those functions which are idempotent and median decomposable.

Weak median decomposibility

Let $\overline{L_2^n} = \{\mathbf{x} \in L^n \colon |\{x_1, \ldots, x_n\} \setminus \{0,1\}| \leqslant 2\}$.

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There are functions which are...

1. median decomposable on $\overline{L_2^n}$ but not nondecreasing.

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There are functions which are...

1. median decomposable on $\overline{L_2^n}$ but not nondecreasing.
2. nondecreasing and median decomposable on L_2^n and which are not Sugeno integrals.

Third characterization of Sugeno integrals

Theorem (C. and Marichal)

Let $f\colon L^n \to L$ be a function. The following are equivalent:

(i) f is a Sugeno integral.

(ii) f is $\{0,1\}$-idempotent and median decomposable.

(iii) f is nondecreasing, $\{0,1\}$-idempotent and weakly median decomposable.

Strong idempotency and componentwise range convexity

A function $f\colon L^n \to L$ is strongly idempotent if

$$f(x_1, \ldots, x_{k-1}, f(\mathbf{x}), x_{k+1}, \ldots, x_n) = f(\mathbf{x}), \text{ for every } \mathbf{x} \in L^n, k \in [n].$$

Strong idempotency and componentwise range convexity

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A function $f\colon L^n \to L$ has a componentwise convex range if, for every $\mathbf{a} = (a_1, \ldots, a_n) \in L^n$ and $k \in [n]$, the map

$$x \mapsto f(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n)$$

has a convex range.

Fourth characterization of Sugeno integrals

Theorem (C. and Marichal)

Let $f\colon L^n \to L$ be a function. The following are equivalent:

(i) f is a Sugeno integral.

(ii) f is nondecreasing, $\{0,1\}$-idempotent, strongly idempotent and has a componentwise convex range.

Fourth characterization of Sugeno integrals

Theorem (C. and Marichal)

Let $f\colon L^n \to L$ be a function. The following are equivalent:

(i) f is a Sugeno integral.

(ii) f is nondecreasing, $\{0,1\}$-idempotent, strongly idempotent and has a componentwise convex range.

Observe that...

In the case $L = [a,b] \subseteq \mathbb{R}$, componentwise range convexity can be replaced by continuity.

Thank you for your attention!

Results appearing in http://arxiv.org/abs/0811.0309