Characterizations of discrete Sugeno integrals as lattice polynomial functions

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Abstract. We survey recent characterizations of the class of lattice polynomial functions and of the subclass of discrete Sugeno integrals defined on bounded chains.

1 Introduction

We are interested in the so-called discrete Sugeno integral, which was introduced by Sugeno [9, 10] and widely investigated in aggregation theory, due to the many applications in fuzzy set theory, data fusion, decision making, pattern recognition, image analysis, etc. For general background, see [1, 6] and for a recent reference, see [5].

A convenient way to introduce the discrete Sugeno integral is via the concept of (lattice) polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. More precisely, given a bounded chain L, by an n-ary polynomial function we simply mean a function $f : L^n \to L$ defined recursively as follows:

(i) For each $i \in [n] = \{1, \ldots, n\}$ and each $c \in L$, the projection $\mathbf{x} \mapsto x_i$ and the constant function $\mathbf{x} \mapsto c$ are polynomial functions from L^n to L.
(ii) If f and g are polynomial functions from L^n to L, then $f \vee g$ and $f \wedge g$ are polynomial functions from L^n to L.
(iii) Any polynomial function from L^n to L is obtained by finitely many applications of the rules (i) and (ii).

As shown by Marichal [7], the discrete Sugeno integrals are exactly those polynomial functions $f : L^n \to L$ which are idempotent, that is, satisfying $f(x, \ldots, x) = x$.

In this paper, we are interested in defining this particular class of lattice polynomial functions by means of properties which appear naturally in aggregation theory. We start in §2 by introducing the basic notions needed in this paper and presenting general characterizations of lattice polynomial functions as obtained in Couceiro and Marichal [2, 3]. In §3, we particularize these characterizations to axiomatize the subclass of discrete Sugeno integrals.

2 Characterizations of polynomial functions

Let L be a bounded chain and let S be a nonempty subset of L. A function $f\colon L^n \to L$ is said to be

- *S-idempotent* if for every $c \in S$, $f(c, \ldots, c) = c$.
- *S-min homogenous* if $f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c$ for all $\mathbf{x} \in L^n$ and $c \in S$.
- *S-max homogenous* if $f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c$ for all $\mathbf{x} \in L^n$ and $c \in S$.
- *horizontally S-minitive* if $f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c)$ for all $\mathbf{x} \in L^n$ and $c \in S$, where $[\mathbf{x}]^c$ is the n-tuple whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise.

- *horizontally S-maxitive* if $f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c)$ for all $\mathbf{x} \in L^n$ and $c \in S$, where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise.
- *median decomposable* if $f(\mathbf{x}) = \text{median}\big(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\big)$ for all $\mathbf{x} \in L^n$ and $k \in [n]$, where $\mathbf{x}_k^c = (x_1, \ldots, x_{k-1}, c, x_{k+1}, \ldots, x_n)$ for all $c \in L$.
- *strongly idempotent* if $f(x_1, \ldots, x_{k-1}, f(\mathbf{x}), x_{k+1}, \ldots, x_n) = f(\mathbf{x})$ for all $\mathbf{x} \in L^n$ and $k \in [n]$.

Two vectors $\mathbf{x}$ and $\mathbf{x}'$ in L^n are said to be *comonotonic*, denoted $\mathbf{x} \sim \mathbf{x}'$, if $(x_i - x_j)(x'_i - x'_j) \geqslant 0$ for every $i, j \in [n]$. A function $f\colon L^n \to L$ is said to be

- *comonotonic minitive* if $f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}')$ whenever $\mathbf{x} \sim \mathbf{x}'$.
- *comonotonic maxitive* if $f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}')$ whenever $\mathbf{x} \sim \mathbf{x}'$.

For integers $0 \leqslant p \leqslant q \leqslant n$, define

$$L_n^{(p,q)} = \{\mathbf{x} \in L^n \colon |\{x_1, \ldots, x_n\} \cap \{0,1\}| \geqslant p \text{ and } |\{x_1, \ldots, x_n\}| \leqslant q\}.$$

For instance, $(c,d,c) \in L_n^{(0,2)}$, $(0,c,d), (1,c,d) \in L_n^{(1,3)}$, but $(0,1,c,d) \notin L_n^{(0,2)} \cup L_n^{(1,3)}$.

Let S be a nonempty subset of L. We say that a function $f\colon L^n \to L$ is

- *weakly S-min homogenous* if $f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$.
- *weakly S-max homogenous* if $f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$.
- *weakly horizontally S-minitive* if $f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c)$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$, where $[\mathbf{x}]^c$ is the n-tuple whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise.
- *weakly horizontally S-maxitive* if $f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c)$ for all $\mathbf{x} \in L_n^{(0,2)}$ and $c \in S$, where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise.
- *weakly median decomposable* if $f(\mathbf{x}) = \text{median}\big(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\big)$ for all $\mathbf{x} \in L_n^{(0,2)} \cup L_n^{(1,3)}$ and $k \in [n]$.

A subset S of a lattice L is said to be *convex* if for every $a, b \in S$ and every $c \in L$ such that $a \leqslant c \leqslant b$, we have $c \in S$. For any subset $S \subseteq L$, we denote by $\overline{S}$ the convex hull of S, that is, the smallest convex subset of L containing S. The *range* of a function $f\colon L^n \to L$ is defined by $\mathcal{R}_f = \{f(\mathbf{x}) : \mathbf{x} \in L^n\}$.

A function $f\colon L^n \to L$ is said to be *nondecreasing* (*in each variable*) if, for every $\mathbf{a}, \mathbf{b} \in L^n$ such that $\mathbf{a} \leqslant \mathbf{b}$, we have $f(\mathbf{a}) \leqslant f(\mathbf{b})$. Note that if f is nondecreasing, then $\overline{\mathcal{R}}_f = [f(\mathbf{0}), f(\mathbf{1})]$. We say that a function $f\colon L^n \to L$ has a *componentwise convex range* if, for every $\mathbf{a} \in L^n$ and $k \in [n]$, the function $x \mapsto f_{\mathbf{a}}^k(x) = f(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n)$ has a convex range.

Theorem 1. *Let $f\colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) *f is a polynomial function.*
(ii) *f is median decomposable.*
(ii-w) *f is nondecreasing and weakly median decomposable.*
(iii) *f is nondecreasing, strongly idempotent, has a componentwise convex range.*
(iv) *f is nondecreasing, $\overline{\mathcal{R}}_f$-min homogeneous, and $\overline{\mathcal{R}}_f$-max homogeneous.*
(iv-w) *f is nondecreasing, weakly $\overline{\mathcal{R}}_f$-min homogeneous, and weakly $\overline{\mathcal{R}}_f$-max homogeneous.*
(v) *f is nondecreasing, $\overline{\mathcal{R}}_f$-min homogeneous, and horizontally $\overline{\mathcal{R}}_f$-maxitive.*
(v-w) *f is nondecreasing, weakly $\overline{\mathcal{R}}_f$-min homogeneous, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.*
(vi) *f is nondecreasing, horizontally $\overline{\mathcal{R}}_f$-minitive, and $\overline{\mathcal{R}}_f$-max homogeneous.*
(vi-w) *f is nondecreasing, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and weakly $\overline{\mathcal{R}}_f$-max homogeneous.*
(vii) *f is nondecreasing, $\overline{\mathcal{R}}_f$-idempotent, horizontally $\overline{\mathcal{R}}_f$-minitive, and horizontally $\overline{\mathcal{R}}_f$-maxitive.*

(vii-w) *f is nondecreasing, $\overline{\mathcal{R}}_f$-idempotent, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.*
(viii) *f is $\overline{\mathcal{R}}_f$-min homogeneous and comonotonic maxitive.*
(viii-w) *f is weakly $\overline{\mathcal{R}}_f$-min homogeneous and comonotonic maxitive.*
(ix) *f is comonotonic minitive and $\overline{\mathcal{R}}_f$-max homogeneous.*
(ix-w) *f is comonotonic minitive and weakly $\overline{\mathcal{R}}_f$-max homogeneous.*
(x) *f is $\overline{\mathcal{R}}_f$-idempotent, horizontally $\overline{\mathcal{R}}_f$-minitive, and comonotonic maxitive.*
(x-w) *f is $\overline{\mathcal{R}}_f$-idempotent, weakly horizontally $\overline{\mathcal{R}}_f$-minitive, and comonotonic maxitive.*
(xi) *f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and horizontally $\overline{\mathcal{R}}_f$-maxitive.*
(xi-w) *f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and weakly horizontally $\overline{\mathcal{R}}_f$-maxitive.*
(xii) *f is $\overline{\mathcal{R}}_f$-idempotent, comonotonic minitive, and comonotonic maxitive.*

Remark 1. In the special case when L is a bounded real interval $[a,b]$, by requiring continuity in each of the conditions of Theorem 1, we can replace $\overline{\mathcal{R}}_f$ with $\mathcal{R}_f$ and remove componentwise convexity in (iii) of Theorem 1.

3 Characterizations of discrete Sugeno integrals

Recall that discrete Sugeno integrals are exactly those lattice polynomial functions which are idempotent. In fact, $\{0,1\}$-idempotency suffices to completely characterize this subclass of polynomial functions.

We say that a function $f\colon L^n \to L$ is

- *Boolean min homogeneous* if $f(\mathbf{x}\wedge c) = f(\mathbf{x})\wedge c$ for all $\mathbf{x}\in\{0,1\}^n$ and $c\in L$.
- *Boolean max homogeneous* if $f(\mathbf{x}\vee c) = f(\mathbf{x})\vee c$ for all $\mathbf{x}\in\{0,1\}^n$ and $c\in L$.

Theorem 2. *Let $f\colon L^n \to L$ be a function. The following conditions are equivalent:*

(i) *f is a discrete Sugeno integral.*
(ii) *f is $\{0,1\}$-idempotent and median decomposable.*
(ii-w) *f is nondecreasing, $\{0,1\}$-idempotent, and weakly median decomposable.*
(iii) *f is nondecreasing, $\{0,1\}$-idempotent, strongly idempotent, has a componentwise convex range.*
(iv) *f is nondecreasing, Boolean min homogeneous, and Boolean max homogeneous.*
(v) *f is nondecreasing, $\{1\}$-idempotent, L-min homogeneous, and horizontally L-maxitive.*
(v-w) *f is nondecreasing, $\{1\}$-idempotent, weakly L-min homogeneous, and weakly horizontally L-maxitive.*
(vi) *f is nondecreasing, $\{0\}$-idempotent, horizontally L-minitive, and L-max homogeneous.*
(vi-w) *f is nondecreasing, $\{0\}$-idempotent, weakly horizontally L-minitive, and weakly L-max homogeneous.*
(vii) *f is nondecreasing, L-idempotent, horizontally L-minitive, and horizontally L-maxitive.*
(vii-w) *f is nondecreasing, L-idempotent, weakly horizontally L-minitive, and weakly horizontally L-maxitive.*
(viii) *f is $\{1\}$-idempotent, L-min homogeneous, and comonotonic maxitive.*
(viii-w) *f is $\{1\}$-idempotent, weakly L-min homogeneous, and comonotonic maxitive.*
(ix) *f is $\{0\}$-idempotent, comonotonic minitive, and L-max homogeneous.*
(ix-w) *f is $\{0\}$-idempotent, comonotonic minitive, and weakly L-max homogeneous.*
(x) *f is L-idempotent, horizontally L-minitive, and comonotonic maxitive.*

(x-w) f *is L-idempotent, weakly horizontally L-minitive, and comonotonic maxitive.*
(xi) f *is L-idempotent, comonotonic minitive, and horizontally L-maxitive.*
(xi-w) f *is L-idempotent, comonotonic minitive, and weakly horizontally L-maxitive.*
(xii) f *is L-idempotent, comonotonic minitive, and comonotonic maxitive.*

Remark 2. (i) As in Remark 1, when L is a bounded real interval $[a,b]$, componentwise convexity can be replaced with continuity in (iii) of Theorem 2.

(ii) The characterizations given in (iv) and (xii) of Theorem 2 were previously established, in the case of real variables, by Marichal [8, §4.3]. The one given in $(viii)$ was established, also in the case of real variables, by de Campos and Bolaños [4] with the redundant assumption of nondecreasing monotonicity.

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