FROM DISCRETE SUGENO INTEGRALS TO GENERALIZED LATTICE POLYNOMIAL FUNCTIONS: AXIOMATIZATIONS AND REPRESENTATIONS

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Summary

Two emergent properties in aggregation theory are investigated, namely horizontal maxitivity and comonotonic maxitivity (as well as their dual counterparts) which are commonly defined by means of certain functional equations. We present complete descriptions of the function classes axiomatized by each of these properties, up to weak versions of monotonicity, in the cases of horizontal maxitivity and minitivity. While studying the classes axiomatized by combinations of these properties, we introduce the concept of quasi-polynomial function which appears as a natural extension of the well-established notion of polynomial function. We present further axiomatizations for this class both in terms of functional equations and natural relaxations of homogeneity and median decomposability.

Keywords: Aggregation function, discrete Sugeno integral, polynomial function, quasi-polynomial function, horizontal maxitivity and minitivity, comonotonic maxitivity and minitivity, functional equation.

1 INTRODUCTION

Aggregation functions arise wherever aggregating information is important: applied and pure mathematics (probability, statistics, decision theory, functional equations), operations research, computer science, and many applied fields (economics and finance, pattern recognition and image processing, data fusion, etc.). For recent references, see Beliakov et al. [1] and Grabisch et al. [8].

A noteworthy aggregation function is the so-called discrete Sugeno integral, which was introduced by Sugeno [12, 13] and which has been widely investigated in aggregation theory, due to its many applications for instance in fuzzy set theory, decision making, and image analysis. For general background, see also the edited book [9].

A convenient way to introduce the discrete Sugeno integral is via the concept of (lattice) polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. As shown by Marichal [10], the discrete Sugeno integrals are exactly those polynomial functions $f : L^n \to L$ which are idempotent, that is, satisfying $f(x, \ldots, x) = x$. Several axiomatizations of the class of discrete Sugeno integrals (as idempotent polynomial functions) have been recently given; see [4].

Of particular interest in aggregation theory, are the so-called horizontal maxitivity and comonotonic maxitivity (as well as their dual counterparts), usually expressed in terms of certain functional equations, and which we now informally describe.

Let L be a bounded chain. For every $\mathbf{x} \in L^n$ and every $c \in L$, consider the horizontal maxitive decomposition of $\mathbf{x}$ obtained by "cutting" it with c, namely $\mathbf{x} = (\mathbf{x} \wedge c) \vee [\mathbf{x}]_c$, where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise. A function $f\colon L^n \to L$ is said to be *horizontally maxitive* if

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c)$$

for every $\mathbf{x} \in L^n$ and every $c \in L$.

A function $f\colon L^n \to L$ is said to be *comonotonic maxitive* if, for any two vectors $\mathbf{x}$ and $\mathbf{x}'$ in the same standard simplex of L^n, we have

$$f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}').$$

As we are going to see (Lemma 6 below), these (as well as their duals) are closely related and constitute properties shared by discrete Sugeno integrals. Still, and as it will become evident, no combination of these with their dual forms suffices to fully describe the class of Sugeno integrals. Thus, and given their emergence in aggregation theory, it is natural to ask which classes of functions are axiomatized by combinations of these properties or, in fact, by each of these properties.

In this paper, we answer this question for both the maxitive and minitive comonotonic properties, and for horizontal maxitivity and minitivity properties, up to certain weak variants of monotonicity. While looking at combinations of the latter properties, we reach a natural generalization of polynomial functions, which we call *quasi-polynomial functions* and which are best described by the following equation

$$f(x_1,\ldots,x_n) = p(\varphi(x_1),\ldots,\varphi(x_n)),$$

where p is a polynomial function and φ a nondecreasing function (see Theorem 10 below). Searching for alternative descriptions, we introduce weaker versions of well-established properties, such as homogeneity and median decomposability, to provide further axiomatizations of the class of quasi-polynomial functions, accordingly.

This paper is organized as follows. We start by recalling basic notions and terminology in lattice function theory, as well as present some known results, needed throughout this paper (Section 2). In Section 3, we study the properties of horizontal maxitivity and comonotonic maxitivity, as well as their dual forms, and determine those function classes axiomatized by each of this properties. Combinations of the latter are then considered in Section 4.1, where the notion of quasi-polynomial function is introduced. In Section 4.2, we propose weaker versions of homogeneity and median decomposability, and provide further characterizations of quasi-polynomial functions, accordingly.

2 BASIC NOTIONS AND PRELIMINARY RESULTS

In this section we recall basic terminology as well as some results needed in the current paper. For general background we refer the reader to, e.g., Burris and Sankappanavar [3] and Rudeanu [11].

2.1 GENERAL BACKGROUND

Throughout this paper, let L be a bounded chain with operations $\wedge$ and $\vee$, and with least and greatest elements 0 and 1, respectively. A subset S of a chain L is said to be *convex* if for every $a,b\in S$ and every $c\in L$ such that $a\leqslant c\leqslant b$, we have $c\in S$. For any subset $S\subseteq L$, we denote by $\overline{S}$ the convex hull of S, that is, the smallest convex subset of L containing S. For every $a,b\in S$ such that $a\leqslant b$, the *interval* $[a,b]$ is the set $[a,b]=\{c\in L : a\leqslant c\leqslant b\}$. For any integer $n\geqslant 1$, let $[n]=\{1,\ldots,n\}$.

For any bounded chain L, we regard the Cartesian product L^n, $n\geqslant 1$, as a distributive lattice endowed with the operations $\wedge$ and $\vee$ given by

$$\begin{aligned}(a_1,\ldots,a_n)\wedge(b_1,\ldots,b_n) &= (a_1\wedge b_1,\ldots,a_n\wedge b_n),\\ (a_1,\ldots,a_n)\vee(b_1,\ldots,b_n) &= (a_1\vee b_1,\ldots,a_n\vee b_n).\end{aligned}$$

The elements of L are denoted by lower case letters $a,b,c,\ldots$, and the elements of L^n, $n>1$, by bold face letters $\mathbf{a},\mathbf{b},\mathbf{c},\ldots$. We also use $\mathbf{0}$ and $\mathbf{1}$ to denote the least element and greatest element, respectively, of L^n. For $c\in L$ and $\mathbf{x}=(x_1,\ldots,x_n)\in L^n$, set

$$\begin{aligned}\mathbf{x}\wedge c &= (x_1\wedge c,\ldots,x_n\wedge c),\\ \mathbf{x}\vee c &= (x_1\vee c,\ldots,x_n\vee c).\end{aligned}$$

The *range* of a function $f\colon L^n\to L$ is defined by $\mathcal{R}_f=\{f(\mathbf{x}) : \mathbf{x}\in L^n\}$. A function $f\colon L^n\to L$ is said to be *nondecreasing* (*in each variable*) if, for every $\mathbf{a},\mathbf{b}\in L^n$ such that $\mathbf{a}\leqslant\mathbf{b}$, we have $f(\mathbf{a})\leqslant f(\mathbf{b})$. The *diagonal section* of f, denoted δ_f, is defined as the unary function given by $\delta_f(x)=f(x,\ldots,x)$. Note that if f is nondecreasing, then δ_f is nondecreasing and $\overline{\mathcal{R}}_{\delta_f}=\overline{\mathcal{R}}_f=[f(\mathbf{0}),f(\mathbf{1})]$.

2.2 POLYNOMIAL FUNCTIONS AND THEIR REPRESENTATIONS

In this paper the so-called polynomial functions will play a fundamental role. Formally, an *n-ary polynomial function* on L is any function $f : L^n\to L$ which can be obtained by finitely many applications of the following rules:

(i) For each $i\in[n]$ and each $c\in L$, the projection $\mathbf{x}\mapsto x_i$ and the constant function $\mathbf{x}\mapsto c$ are polynomial functions from L^n to L.

(ii) If f and g are polynomial functions from L^n to L, then $f\vee g$ and $f\wedge g$ are polynomial functions from L^n to L.

Polynomial functions are also called lattice functions (Goodstein [7]), algebraic functions (Burris and Sankappanavar [3]) or weighted lattice polynomial functions (Marichal [10]). Idempotent polynomial functions (i.e., satisfying $f(c,\ldots,c)=c$ for every $c\in L$) are referred to by aggregation theorists as (*discrete*) *Sugeno integrals*.

As observed by Goodstein [7] (see also Rudeanu [11]), polynomial functions are exactly those functions which can be represented by formulas in disjunctive and conjunctive normal forms. In fact, each polynomial function $f\colon L^n \to L$ is uniquely determined by its restriction to $\{0,1\}^n$. Due to their relevance in the sequel, we recall some known results concerning normal form representations of polynomial functions in the special case where L is a chain. The following result is due to Goodstein [7].

Proposition 1. *(a) Every polynomial function is completely determined by its restriction to* $\{0,1\}^n$.

(b) A function $g\colon \{0,1\}^n \to L$ *can be extended to a polynomial function* $f\colon L^n \to L$ *if and only if it is nondecreasing. In this case, the extension is unique.*

(c) For any $f\colon L^n \to L$*, the following are equivalent:*

(i) f *is a polynomial function.*

(ii) There exists $\alpha\colon 2^{[n]} \to L$ *such that*

$$f(\mathbf{x}) = \bigvee_{I\subseteq[n]} \big(\alpha(I) \wedge \bigwedge_{i\in I} x_i\big). \qquad (1)$$

(iii) There exists $\beta\colon 2^{[n]} \to L$ *such that*

$$f(\mathbf{x}) = \bigwedge_{I\subseteq[n]} \big(\beta(I) \vee \bigvee_{i\in I} x_i\big). \qquad (2)$$

The expressions given in (1) and (2) are usually referred to as the *disjunctive normal form* (DNF) representation and the *conjunctive normal form* (CNF) representation, respectively, of the polynomial function f.

As observed by Marichal [10], the DNF and CNF representations of polynomial functions $f\colon L^n \to L$ are not necessarily unique. However, from among all the possible set functions α (resp. β) defining the DNF (resp. CNF) representation of f, only one is isotone (resp. antitone), namely the function $\alpha_f\colon 2^{[n]} \to L$ (resp. $\beta_f\colon 2^{[n]} \to L$) defined by

$$\alpha_f(I) = f(\mathbf{e}_I) \qquad (\text{resp. } \beta_f(I) = f(\mathbf{e}_{[n]\setminus I})), \qquad (3)$$

where $\mathbf{e}_I$ denotes the element of $\{0,1\}^n$ whose ith component is 1 if and only if $i \in I$.

In the case when L is a chain, it was shown in [4] that the DNF and CNF representations of polynomial functions $f\colon L^n \to L$ can be refined and given in terms of standard simplices of L^n. Let σ be a permutation on $[n]$. The *standard simplex* of L^n associated with σ is the subset $L^n_\sigma \subset L^n$ defined by

$$L^n_\sigma = \{\mathbf{x} = (x_1,\ldots,x_n) \in L^n \colon x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}.$$

For each $i \in [n]$, define $S^{\uparrow}_\sigma(i) = \{\sigma(i),\ldots,\sigma(n)\}$ and $S^{\downarrow}_\sigma(i) = \{\sigma(1),\ldots,\sigma(i)\}$. As a matter of convenience, set $S^{\uparrow}_\sigma(n+1) = S^{\downarrow}_\sigma(0) = \varnothing$.

Proposition 2. *For any function* $f\colon L^n \to L$*, the following conditions are equivalent:*

(i) f *is a polynomial function.*

(ii) For any permutation σ *on* $[n]$ *and every* $\mathbf{x} \in L^n_\sigma$,

$$f(\mathbf{x}) = \bigvee_{i=1}^{n+1} \big(\alpha_f(S^{\uparrow}_\sigma(i)) \wedge x_{\sigma(i)}\big),$$

where $x_{\sigma(n+1)} = 1$.

(iii) For any permutation σ *on* $[n]$ *and every* $\mathbf{x} \in L^n_\sigma$,

$$f(\mathbf{x}) = \bigwedge_{i=0}^{n} \big(\beta_f(S^{\downarrow}_\sigma(i)) \wedge x_{\sigma(i)}\big),$$

where $x_{\sigma(0)} = 0$.

3 MOTIVATING CHARACTERIZATIONS

Even though horizontal maxitivity and comonotonic maxitivity, as well as their dual counterparts, play an important role in aggregation theory (as properties shared by noteworthy classes of aggregation functions), they have not yet been described independently. In this section we investigate each of these properties and determine their corresponding function classes (up to weak versions of monotonicity, in the cases of horizontal maxitivity and minitivity).

3.1 HORIZONTAL MAXITIVITY AND MINITIVITY

Recall that a function $f\colon L^n \to L$ is said to be

- *horizontally maxitive* if, for every $\mathbf{x} \in L^n$ and every $c \in L$, we have

  $$f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c),$$

  where $[\mathbf{x}]_c$ is the n-tuple whose ith component is 0, if $x_i \leqslant c$, and x_i, otherwise.

- *horizontally minitive* if, for every $\mathbf{x} \in L^n$ and every $c \in L$, we have

  $$f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c),$$

  where $[\mathbf{x}]^c$ is the n-tuple whose ith component is 1, if $x_i \geqslant c$, and x_i, otherwise.

Let us consider the following weak forms of nondecreasing monotonicity:

($\mathbf{P_1}$) $f(\mathbf{e} \wedge c) \leqslant f(\mathbf{e}' \wedge c)$ for every $\mathbf{e}, \mathbf{e}' \in \{0,1\}^n$ such that $\mathbf{e} \leqslant \mathbf{e}'$ and every $c \in L$.

($\mathbf{D_1}$) $f(\mathbf{e} \vee c) \leqslant f(\mathbf{e}' \vee c)$ for every $\mathbf{e}, \mathbf{e}' \in \{0,1\}^n$ such that $\mathbf{e} \leqslant \mathbf{e}'$ and every $c \in L$.

($\mathbf{P_2}$) $f(\mathbf{e} \wedge c) \leqslant f(\mathbf{e} \wedge c')$ for every $\mathbf{e} \in \{0,1\}^n$ and every $c, c' \in L$ such that $c \leqslant c'$.

($\mathbf{D_2}$) $f(\mathbf{e} \vee c) \leqslant f(\mathbf{e} \vee c')$ for every $\mathbf{e} \in \{0,1\}^n$ and every $c, c' \in L$ such that $c \leqslant c'$.

Theorem 3. *A function $f\colon L^n \to L$ is horizontally maxitive and satisfies $\mathbf{P_1}$ if and only if there exists $g\colon L^n \to L$ satisfying $\mathbf{P_2}$ such that*

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} g\Big(\mathbf{e}_I \wedge \bigwedge_{i \in I} x_i\Big). \qquad (4)$$

In this case, we can choose $g = f$.

Similarly, we obtain the dual characterization:

Theorem 4. *A function $f\colon L^n \to L$ is horizontally minitive and satisfies $\mathbf{D_1}$ if and only if there exists $g\colon L^n \to L$ satisfying $\mathbf{D_2}$ such that*

$$f(\mathbf{x}) = \bigwedge_{I \subseteq [n]} g\Big(\mathbf{e}_{[n]\setminus I} \vee \bigvee_{i \in I} x_i\Big).$$

In this case, we can choose $g = f$.

From Theorems 3 and 4 we get:

Corollary 5. *A function $f\colon L^n \to L$ is horizontally maxitive (resp. horizontally minitive) and satisfies $\mathbf{P_1}$ (resp. $\mathbf{D_1}$) if and only if there are unary nondecreasing functions $\varphi_I\colon L \to L$, for $I \subseteq [n]$, such that*

$$\begin{aligned} f(\mathbf{x}) &= \bigvee_{I \subseteq [n]} \big(\alpha_f(I) \wedge \bigwedge_{i \in I} \varphi_I(x_i)\big) \\ \textit{(resp. } f(\mathbf{x}) &= \bigwedge_{I \subseteq [n]} \big(\beta_f(I) \vee \bigvee_{i \in I} \varphi_I(x_i)\big)\textit{)}, \end{aligned}$$

where the set function α_f (resp. β_f) is defined in (3). In this case, we can choose $\varphi_I(x) = f(\mathbf{e}_I \wedge x)$ (resp. $\varphi_I(x) = f(\mathbf{e}_{[n]\setminus I} \vee x)$) for every $I \subseteq [n]$.

Remark 1. (i) Theorem 3 (resp. Theorem 4) provides the description of those horizontally maxitive (resp. horizontally minitive) functions which are nondecreasing.

(ii) Every Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ satisfying $f(\mathbf{0}) \leqslant f(\mathbf{x})$ (resp. $f(\mathbf{x}) \leqslant f(\mathbf{1})$) is horizontally maxitive (resp. horizontally minitive). Moreover, not all such functions are nondecreasing, thus showing that condition $\mathbf{P_1}$ (resp. $\mathbf{D_1}$) is necessary in Theorem 3 (resp. Theorem 4).

(iii) As shown in [**?**], polynomial functions $f\colon L^n \to L$ are exactly those $\overline{\mathcal{R}}_f$-idempotent (i.e., satisfying $f(c, \ldots, c) = c$ for every $c \in \overline{\mathcal{R}}_f$) which are nondecreasing, horizontally maxitive, and horizontally minitive.

(iv) The concept of horizontal maxitivity was introduced, in the case when L is the real interval $[0,1]$, by Benvenuti et al. [2] as a general property of the Sugeno integral.

3.2 COMONOTONIC MAXITIVITY AND MINITIVITY

Two vectors $\mathbf{x}, \mathbf{x}' \in L^n$ are said to be *comonotonic* if there exists a permutation σ on $[n]$ such that $\mathbf{x}, \mathbf{x}' \in L^n_\sigma$. A function $f\colon L^n \to L$ is said to be

- *comonotonic maxitive* if, for any two comonotonic vectors $\mathbf{x}, \mathbf{x}' \in L^n$, we have
$$f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}').$$
- *comonotonic minitive* if, for any two comonotonic vectors $\mathbf{x}, \mathbf{x}' \in L^n$, we have
$$f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}').$$

Note that for any $\mathbf{x} \in L^n$ and any $c \in L$, the vectors $\mathbf{x} \vee c$ and $[\mathbf{x}]^c$ are comonotonic. As a consequence, if a function $f\colon L^n \to L$ is comonotonic maxitive (resp. comonotonic minitive), then it is horizontally maxitive (resp. horizontally minitive). It was also observed in [4] that if f is comonotonic maxitive or comonotonic minitive, then it is nondecreasing. Moreover, we obtain the following result.

Lemma 6. *A function $f\colon L^n \to L$ is comonotonic maxitive (resp. comonotonic minitive) if and only if it is horizontally maxitive (resp. horizontally minitive) and satisfies $\mathbf{P_1}$ (resp. $\mathbf{D_1}$).*

Combining Theorems 3 and 4 with Lemma 6, we immediately obtain the descriptions of the classes of comonotonic maxitive and minitive functions.

Theorem 7. *A function $f\colon L^n \to L$ is comonotonic maxitive if and only if there exists $g\colon L^n \to L$ satisfying $\mathbf{P_2}$ such that*

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} g\Big(\mathbf{e}_I \wedge \bigwedge_{i \in I} x_i\Big).$$

In this case, we can choose $g = f$.

Theorem 8. *A function $f\colon L^n \to L$ is comonotonic minitive if and only if there exists $g\colon L^n \to L$ satisfying $\mathbf{D_2}$ such that*

$$f(\mathbf{x}) = \bigwedge_{I \subseteq [n]} g\Big(\mathbf{e}_{[n]\setminus I} \vee \bigvee_{i \in I} x_i\Big).$$

In this case, we can choose $g = f$.

As before, we have the following corollary.

Corollary 9. *A function $f\colon L^n \to L$ is comonotonic maxitive (resp. comonotonic minitive) if and only if there are unary nondecreasing functions $\varphi_I\colon L \to L$, for $I \subseteq [n]$, such that*

$$f(\mathbf{x}) = \bigvee_{I\subseteq[n]} \big(\alpha_f(I) \wedge \bigwedge_{i\in I} \varphi_I(x_i)\big)$$

$$\text{(resp. } f(\mathbf{x}) = \bigwedge_{I\subseteq[n]} \big(\beta_f(I) \vee \bigvee_{i\in I} \varphi_I(x_i)\big)\text{)},$$

where the set function α_f (resp. β_f) is defined in (3). In this case, we can choose $\varphi_I(x) = f(\mathbf{e}_I \wedge x)$ (resp. $\varphi_I(x) = f(\mathbf{e}_{[n]\setminus I} \vee x)$) for every $I \subseteq [n]$.

Remark 2. (i) An alternative description of comonotonic maxitive (resp. comonotonic minitive) functions was obtained in Grabisch et al. [8, §2.5] in the case when L is a real interval.

(ii) It was shown in [4] that polynomial functions $f\colon L^n \to L$ are exactly those $\overline{\mathcal{R}}_f$-idempotent functions which are comonotonic maxitive and comonotonic minitive.

(ii) Comonotonic minitivity and maxitivity were introduced in the context of Sugeno integrals in de Campos et al. [5].

4 QUASI-POLYNOMIAL FUNCTIONS

Motivated by the results of Section 3 concerning horizontal maxitivity and comonotonic maxitivity, as well as their dual counterparts, we now study combinations of these properties. This will lead to a relaxation of the notion of polynomial function, which we will refer to as *quasi-polynomial function*. Accordingly, we introduce weaker variants of well-established properties, such as homogeneity and median decomposability, which are then used to provide further axiomatizations of the class of quasi-polynomial functions.

4.1 MOTIVATION AND DEFINITION

Combinations of those properties studied in Section 3 are considered in the following result.

Theorem 10. *Let $f\colon L^n \to L$ be a function. The following assertions are equivalent:*

(i) f is horizontally maxitive, horizontally minitive, and satisfies $\mathbf{P_1}$ or $\mathbf{D_1}$.

(ii) f is comonotonic maxitive and minitive.

(iii) f is horizontally maxitive and comon. minitive.

(iv) f is comon. maxitive and horizontally minitive.

(v) There exist a polynomial function $p\colon L^n \to L$ and a nondecreasing function $\varphi\colon L \to L$ such that

$$f(x_1,\ldots,x_n) = p(\varphi(x_1),\ldots,\varphi(x_n)).$$

If these conditions hold then we can choose for p the unique polynomial function p_f extending $f|_{\{0,1\}^n}$ and for φ the diagonal section δ_f of f.

Theorem 10 motivates the following definition.

Definition 11. We say that a function $f\colon L^n \to L$ is a *quasi-polynomial function* (resp. a *discrete quasi-Sugeno integral*, a *quasi-term function*) if there exist a polynomial function (resp. a discrete Sugeno integral, a term function) $p\colon L^n \to L$ and a nondecreasing function $\varphi\colon L \to L$ such that $f = p \circ \varphi$, that is,

$$f(x_1,\ldots,x_n) = p(\varphi(x_1),\ldots,\varphi(x_n)). \tag{5}$$

Remark 3. (i) Note that each quasi-polynomial function $f\colon L^n \to L$ can be represented as a combination of constants and a nondecreasing unary function φ (applied to the projections $\mathbf{x} \mapsto x_i$) using the lattice operations $\vee$ and $\wedge$.

(ii) In the setting of decision-making under uncertainty, the nondecreasing function φ in (5) can be thought of as a *utility function* and the corresponding quasi-polynomial function as a (qualitative) *global preference functional*; see for instance Dubois et al. [6].

Note that the functions p and φ in (5) are not necessarily unique. For instance, if f is a constant $c \in L$, then we could choose $p \equiv c$ and φ arbitrarily, or p idempotent and $\varphi \equiv c$. We now describe all possible choices for p and φ. For any integers $m, n \geqslant 1$, any vector $\mathbf{x} \in L^m$, and any function $f\colon L^n \to L$, we define $\langle \mathbf{x} \rangle_f \in L^m$ as the m-tuple

$$\langle \mathbf{x} \rangle_f = \text{median}(f(\mathbf{0}), \mathbf{x}, f(\mathbf{1})),$$

where the median is taken componentwise.

Proposition 12. *Let $f\colon L^n \to L$ be a quasi-polynomial function and let $p_f : L^n \to L$ be the unique polynomial function extending $f|_{\{0,1\}^n}$. We have*

$$\{(p,\varphi)\colon f = p\circ\varphi\} = \{(p,\varphi)\colon p_f = \langle p \rangle_f \text{ and } \delta_f = \langle \varphi \rangle_p\},$$

where p and φ stand for polynomial and unary nondecreasing functions, respectively. In particular, we have $f = p_f \circ \delta_f$.

It was shown in Marichal [10] that every polynomial function $p\colon L^n \to L$ can be represented as $\langle q \rangle_p$ for some discrete Sugeno integral $q\colon L^n \to L$. Combining this with Proposition 12, we obtain the next result.

Corollary 13. *The class of quasi-polynomial functions is exactly the class of discrete quasi-Sugeno integrals.*

4.2 FURTHER AXIOMATIZATIONS

We now propose weaker variants of some properties of polynomial functions, namely, homogeneity and median decomposability, to provide alternative axiomatizations of the class of quasi-polynomial functions. For background see [4].

4.2.1 Quasi-homogeneity

We say that a function $f\colon L^n \to L$ is *quasi-max homogeneous* (resp. *quasi-min homogeneous*) if for every $\mathbf{x} \in L^n$ and $c \in L$, we have

$$f(\mathbf{x}\vee c) = f(\mathbf{x})\vee\delta_f(c) \ \text{(resp. } f(\mathbf{x}\wedge c) = f(\mathbf{x})\wedge\delta_f(c)\text{)}.$$

Lemma 14. *Let $f\colon L^n \to L$ be nondecreasing and quasi-min homogeneous (resp. quasi-max homogeneous). Then f is quasi-max homogeneous (resp. quasi-min homogeneous) if and only if it is horizontally maxitive (resp. horizontally minitive).*

Combining Theorem 10 and Lemma 14, we obtain a characterization of quasi-polynomial functions in terms of quasi-min and quasi-max homogeneity.

Theorem 15. *A function $f\colon L^n \to L$ is a quasi-polynomial function if and only if it is nondecreasing, quasi-max homogeneous, and quasi-min homogeneous.*

4.2.2 Quasi-median decomposability

In complete analogy with the previous subsection we propose the following weaker variant of median decomposability (see [4]). We say that a function $f\colon L^n \to L$ is *quasi-median decomposable* if, for every $\mathbf{x} \in L^n$ and every $k \in [n]$, we have

$$f(\mathbf{x}) = \text{median}\big(f(\mathbf{x}_k^0), \delta_f(x_k), f(\mathbf{x}_k^1)\big).$$

Note that every nondecreasing unary function is quasi-median decomposable.

Observe that $\vee$ and $\wedge$, as well as any nondecreasing function $\varphi\colon L \to L$, are quasi-median decomposable. Also, it is easy to see that any combination of constants and a nondecreasing unary function φ using $\vee$ and $\wedge$ is quasi-median decomposable and hence, by Remark 3 (i), every quasi-polynomial function is quasi-median decomposable. In fact, we have following:

Theorem 16. *A function $f\colon L^n \to L$ is a quasi-polynomial function if and only if δ_f is nondecreasing and f is quasi-median decomposable.*

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