# Using Choquet integral in Machine Learning: what can MCDA bring?

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# Plan

## The context: MultiCriteria Decision Aid (MCDA)

Aim: to help a decision-maker (DM) to select one or more alternatives among several alternatives evaluated on $|N|$ criteria often contradictory.

$\Longrightarrow$ We need to construct a preference relation over the set of all alternatives $X$

## In practice, using MultiAttribute Utility Theory (MAUT):

- People suppose $\succsim_X$ representable by an overall utility function $U$:

$$x \succsim_X y \Leftrightarrow U(x) \geq U(y) \quad (1)$$

- People ask to the DM some preferential information $\succsim_{X'}$ (learning examples) on $X' \subseteq X$
- If $\succsim_{X'}$ is representable by $U$, then the model obtained in $X'$ will be then automatically extended to $X$.

Notations:

- $N = \{1, \dots, n\}$ a finite set of criteria.
- $X = X_1 \times \cdots \times X_n$ is the set of actions (alternatives or options),
  - $X_i$ is the set of possible levels on criterion $i$.
- $u_i : X_i \to \mathbb{R}$ is an utility function.
- $\succsim$ is a preferential information of DM (binary relation)
  - $\succ$ is the asymmetric part of $\succsim$.

### Example (Grabisch and Labreuche)

Evaluation of students on three subjects Mathematics (M), Statistics (S) and Language skills (L).

$\Longrightarrow$ with $N = \{1, 2, 3\}$

| Students | 1 : Mathematics (M) | 2 : Statistics (S) | 3 : Language (L) |
|---|---|---|---|
| $E$ | 14 | 16 | 7 |
| $F$ | 14 | 15 | 8 |
| $G$ | 9 | 16 | 7 |
| $H$ | 9 | 15 | 8 |

- All marks are taken from the same scale [0, 20].

## Example

| Students | 1 : Math (M) | 2 : Stat (S) | 3 : Lang (L) |
|---|---|---|---|
| $E$ | 14 | 16 | 7 |
| $F$ | 14 | 15 | 8 |
| $G$ | 9 | 16 | 7 |
| $H$ | 9 | 15 | 8 |

To select the best students, the dean of the faculty expresses his preferences:

- for a student good in Mathematics, Language is more important than Statistics

$$\Longrightarrow E \prec F,$$

- for a student bad in Mathematics, Statistics is more important than Language

$$\Longrightarrow H \prec G.$$

## Example (The Limitation of the Weighted Sum)

| Students | 1 : Math (M) | 2 : Stat (S) | 3 : Lang (L) |
|---|---|---|---|
| $E$ | 14 | 16 | 7 |
| $F$ | 14 | 15 | 8 |
| $G$ | 9 | 16 | 7 |
| $H$ | 9 | 15 | 8 |

- For a student good in M, L is more important than S:

$$E \prec F$$
$$\Downarrow$$
$$u_M(14)\ w_M + u_S(16)\ w_S + u_L(7)\ w_L < u_M(14)\ w_M + u_S(15)\ w_S + u_L(8)\ w_L$$

- For a student bad in M, S is more important than L:

$$H \prec G$$
$$\Downarrow$$
$$u_M(9)\ w_M + u_S(15)\ w_S + u_L(8)\ w_L < u_M(9)\ w_M + u_S(16)\ w_S + u_L(7)\ w_L$$

### Example (The Limitation of the Weighted Sum)

| Students | 1 : Math (M) | 2 : Stat (S) | 3 : Lang (L) |
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**Conclusion:** For all utility functions $u_M$, $u_S$ and $u_L$, the two preferences $E \prec F$ and $H \prec G$ lead to a contradiction with the arithmetic mean model.

$\Longrightarrow$ **We need to try another model**

### Definition (A capacity)

A set function $\mu : 2^N \to [0,1]$ is a *capacity* if:

1. $\mu(\emptyset) = 0$
2. $\mu(N) = 1$
3. $\forall A, B \in 2^N,\ [A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).

### Definition (The Choquet integral)

For an alternative $x := (x_1, ..., x_n) \in X$, the expression of the Choquet integral w.r.t. a capacity $\mu$ is given by:

$$C_\mu((u_1(x_1), \ldots, u_n(x_n))) := \sum_{i=1}^{n} (u_{\tau(i)}(x_{\tau(i)}) - u_{\tau(i-1)}(x_{\tau(i-1)}))\ \mu(\{\tau(i), \ldots, \tau(n)\})$$

where $\tau$ is a permutation on $N$ such that

- $u_{\tau(1)}(x_{\tau(1)}) \leq u_{\tau(2)}(x_{\tau(2)}) \leq \cdots \leq u_{\tau(n-1)}(x_{\tau(n-1)}) \leq u_{\tau(n)}(x_{\tau(n)})$
- $u_{\tau(0)}(x_{\tau(0)}) := 0$

# Plan

Example (The Limitation of the Choquet integral)

| Students | 1 : Math (M) | 2 : Stat (S) | 3 : Lang (L) |
|---|---|---|---|
| $E$ | 14 | 16 | 7 |
| $F$ | 14 | 15 | 8 |
| $G$ | 9 | 16 | 7 |
| $H$ | 9 | 15 | 8 |

**Hypothesis:** utility functions are fixed here and are equal to the marks obtained.

- For a student good in M, L is more important than S

$$[E \prec F] \Rightarrow 7 + 7\mu(\{M, S\}) + 2\mu(\{S\}) < 8 + 6\mu(\{M, S\}) + \mu(\{S\})$$

- For a student bad in M, S is more important than L

$$[H \prec G] \Rightarrow 8 + \mu(\{M, S\}) + 6\mu(\{S\}) < 7 + 2\mu(\{M, S\}) + 7\mu(\{S\})$$

## Example (The Limitation of the Choquet integral)

| Students | 1 : Math (M) | 2 : Stat (S) | 3 : Lang (L) |
|---|---|---|---|
| $E$ | 14 | 16 | 7 |
| $F$ | 14 | 15 | 8 |
| $G$ | 9 | 16 | 7 |
| $H$ | 9 | 15 | 8 |

- For a student good in M, L is more important than S

$$[E \prec F] \Rightarrow \mu(\{M,S\}) + \mu(\{S\}) < 1$$

- For a student bad in M, S is more important than L

$$[H \prec G] \Rightarrow \mu(\{M,S\}) + \mu(\{S\}) > 1$$

**Conclusion:** The two preferences, $E \prec F$ and $H \prec G$, are not representable by a Choquet integral $C_\mu$

### Remark

- *We tried to find a capacity by assuming that the utility functions $u_i$ are are equal to the marks obtained.*
- *If $u_M(a) = u_S(a) = u_L(a)$, for all $a \in [0, 20]$, then $E \prec F$ and $H \prec G$ remain not representable by $C_\mu$ because $E, F, G, H$ are comonotone and thus the preferences $E \prec F$ and $H \prec G$ violate comonotone additivity.*

## Example

Let us assume the following utility functions:

| | 1 : Mathematics (M) | 2 : Statistics (S) | 3 : Language (L) |
|---|---|---|---|
| $E$ | $u_M(14) = 16$ | $u_S(16) = 16$ | $u_L(7) = 7$ |
| $F$ | $u_M(14) = 16$ | $u_S(15) = 15$ | $u_L(8) = 8$ |
| $G$ | $u_M(9) = 9$ | $u_S(16) = 16$ | $u_L(7) = 7$ |
| $H$ | $u_M(9) = 9$ | $u_S(15) = 15$ | $u_L(8) = 8$ |

- Here for the DM, the interpretation of "a good mark" in mathematics and "a good mark" in statistics is different.
- Such an interpretation is not in contradiction with the definition of commensurate scales: for $x_i \in X_i$ and $x_j \in X_j$,

$$u_i(x_i) \geq u_j(x_j) \text{ iff the DM considers } x_i \text{ at least as good as } x_j$$

.

## Example

| | 1 : Mathematics (M) | 2 : Statistics (S) | 3 : Language (L) |
|---|---|---|---|
| $E$ | $u_M(14) = 16$ | $u_S(16) = 16$ | $u_L(7) = 7$ |
| $F$ | $u_M(14) = 16$ | $u_S(15) = 15$ | $u_L(8) = 8$ |
| $G$ | $u_M(9) = 9$ | $u_S(16) = 16$ | $u_L(7) = 7$ |
| $H$ | $u_M(9) = 9$ | $u_S(15) = 15$ | $u_L(8) = 8$ |

Now the two preferences, $E \prec F$ and $H \prec G$, can be modeled by $C_\mu$.

Indeed these utility functions lead to the system

$$\begin{cases} E \prec F \Rightarrow 2\mu(\{M,S\}) - \mu(\{M\}) < 1 \\ H \prec G \Rightarrow \mu(\{M,S\}) + \mu(\{S\}) > 1 \end{cases}$$

for which a solution is $\mu(\{M,S\}) = \mu(\{M\}) = \mu(\{S\}) = 0.6$.

## Our objective

- To show the limitation of the Choquet integral in MCDA, it will be interesting to look for an example where the utility functions are not fixed a priori.

- Concretely we looked for a relation $\succsim$ (learning examples) which is not representable by $C_\mu$ for any capacity $\mu$ and any utility functions $u_i$.

    - It was not an easy task!!!

## Example (A counter-example with 2 criteria)

Let $X_1 = \{a_1, b_1, c_1, d_1, e_1, f_1\}$ and $X_2 = \{a_2, b_2, c_2, d_2, e_2, f_2\}$ such that

$$\begin{aligned} u_1(a_1) \le u_1(b_1) \le u_1(c_1) \le u_1(d_1) \le u_1(e_1) \le u_1(f_1) \\ u_2(a_2) \le u_2(b_2) \le u_2(c_2) \le u_2(d_2) \le u_2(e_2) \le u_2(f_2) \end{aligned} \quad (2)$$

Suppose that the relation $\succsim$ is such that:

$$\begin{aligned} (a_1, e_2) \sim (b_1, d_2) \\ (c_1, d_2) \sim (a_1, f_2) \\ (c_1, e_2) \not\sim (b_1, f_2) \end{aligned} \quad (3)$$

and

$$\begin{aligned} (d_1, b_2) \sim (e_1, a_2) \\ (f_1, a_2) \sim (d_1, c_2) \\ (f_1, b_2) \not\sim (e_1, c_2) \end{aligned} \quad (4)$$

Then $\succsim$ cannot be represented using the Choquet integral model.

## Example (A counter-example with 2 criteria)

The proof is based on two remarks:

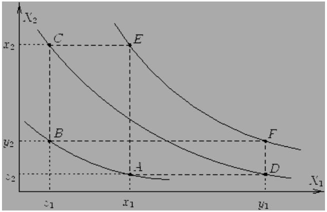
- $\succsim$ representable by $C_\mu$ if there is $u_1$, $u_2$ and positive real numbers $\lambda_1, \omega_1$, such that:

$$x \succsim y \iff V(x) \geq V(y),$$

  where $V$ is a real valued function on $X$ such that:

$$V(x) = \begin{cases} \lambda_1 u_1(x_1) + (1-\lambda_1) u_2(x_2) & \text{if } u_1(x_1) \geq u_2(x_2), \\ \omega_1 u_1(x_1) + (1-\omega_1) u_2(x_2) & \text{otherwise.} \end{cases}$$

- The relations (3) and (4) violates the Thomsen condition



## Example (A counter-example with 3 criteria)

Let be $x_1^1, x_1^2, \ldots, x_1^{11} \in X_1$ with $x_1^1 < x_1^2 < \cdots < x_1^{11}$ (we suppose an order on $X_1$).

Let be: $y_2, z_2 \in X_2$ and $y_3, z_3 \in X_3$.

We assume that the DM provides $\succsim$ as follows:

$$\begin{array}{ll}
(x_1^1, y_2, y_3) \succ (x_1^1, z_2, z_3) & (x_1^2, y_2, y_3) \prec (x_1^2, z_2, z_3) \\
(x_1^3, y_2, y_3) \succ (x_1^3, z_2, z_3) & (x_1^4, y_2, y_3) \prec (x_1^4, z_2, z_3) \\
(x_1^5, y_2, y_3) \succ (x_1^5, z_2, z_3) & (x_1^6, y_2, y_3) \prec (x_1^6, z_2, z_3) \\
(x_1^7, y_2, y_3) \succ (x_1^7, z_2, z_3) & (x_1^8, y_2, y_3) \prec (x_1^8, z_2, z_3) \\
(x_1^9, y_2, y_3) \succ (x_1^9, z_2, z_3) & (x_1^{10}, y_2, y_3) \prec (x_1^{10}, z_2, z_3) \\
(x_1^{11}, y_2, y_3) \succ (x_1^{11}, z_2, z_3) &
\end{array}$$

The idea of this example is to introduce sufficiently many comparisons such that there necessarily exist three comparisons of comonotonic alternatives leading to a contradiction.

Example (A counter-example with 3 criteria)

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We assume that the DM provides $\succsim$ as follows:

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(x_1^5, y_2, y_3) \succ (x_1^5, z_2, z_3) & (x_1^6, y_2, y_3) \prec (x_1^6, z_2, z_3) \\
(x_1^7, y_2, y_3) \succ (x_1^7, z_2, z_3) & (x_1^8, y_2, y_3) \prec (x_1^8, z_2, z_3) \\
(x_1^9, y_2, y_3) \succ (x_1^9, z_2, z_3) & (x_1^{10}, y_2, y_3) \prec (x_1^{10}, z_2, z_3) \\
(x_1^{11}, y_2, y_3) \succ (x_1^{11}, z_2, z_3) &
\end{array}$$

Then $\succsim$ cannot be represented using the Choquet integral model.

### From the two counter-examples

- We can deduce some necessary conditions to represent a preference by the Choquet integral model when utility functions and capacity are unknown a priori.

- We hope to entirely characterized this model in the future works.

### Proposition

*If the learning examples use only alternatives that belong to an antichain,*

*then any weak order over these alternatives, that does not violates Pareto condition and transitivity,*

*is representable by a Choquet integral and partial utility functions.*

# Plan

1. Settings
   - Context and notations
   - The Choquet integral model
2. Learning the utility functions and the capacity simultaneously
   - The limitation of Choquet integral: a classical example
   - Examples non representable by a Choquet integral
   - Sufficient conditions for two criteria
3. Interpretation of the Choquet integral model
   - Interpretation of a capacity
   - Interpretation of the Choquet integral model
4. Conclusion

### Definition (The Shapley value)

The Shapley value for $i \in N$:

$$\phi_i(\mu) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} [\mu(S \cup \{i\}) - \mu(S)]$$

- The Shapley value allocates to agent $i$ her expected marginal cost over all possible orderings of agents;
- Coefficient $\frac{|S|!(n-|S|-1)!}{n!}$ is the probability that coalition $S$ corresponds precisely to the set of players preceding player $i$ in a giving ordering.
- In the context of MCDA, the Shapley value can be seen as the mean importance of criteria.

### Definition (Interaction index)

The *interaction index* between criteria $i$ and $j$ is defined by

$$I_{ij}(\mu) := \sum_{A \subset N \setminus \{i,j\}} \frac{|A|!(n-|A|-2)!}{(n-1)!} \Delta_{i,j}\mu(A)$$

where $\Delta_{i,j}\mu(A) := \mu(A \cup \{i,j\}) - \mu(A \cup \{i\}) - \mu(A \cup \{j\}) + \mu(A)$.

A positive (resp. negative) interaction depicts a positive (resp. negative) synergy between criteria – both criteria need to be satisfied (resp. it is sufficient that only one criterion is met).

The interpretation of the Shapley interaction indices for the Choquet integral is basically due to J.L. Marichal:

**Lemma**

$$I_S(\mu) = \int_{[0,1]^n} \frac{\partial^{|S|} C_\mu(z)}{\partial z_S} \, dz$$

*where the partial derivative is piecewise continuous.*

- The interaction index among criteria $S$ is the integral over $[0,1]^n$ of the partial derivative of the Choquet integral w.r.t. criteria in $S$.
- This can be easily extended to the cases when the set of feasible alternatives is not $[0,1]^n$ but a subset.
- The Shapley value appears as the mean of relative amplitude of the range of $C_\mu$ w.r.t. criterion $i$, when the remaining variables take random values.

# Plan

## Using Choquet integral in Machine Learning: what can MCDA bring?

- The conviction that it important to learn not only the capacity but also partial utility functions (in the same time).
  - Research Challenge 1: We hope to provide necessary and sufficient conditions to represent a preference by the Choquet integral model when utility functions and capacity are unknown a priori.
  - Research Challenge 2: This problem is a non linear problem: how to solve it?
- Shapley value and interaction index can be used to interpret the capacity, but also the Choquet integral model.