On three properties of the discrete Choquet integral

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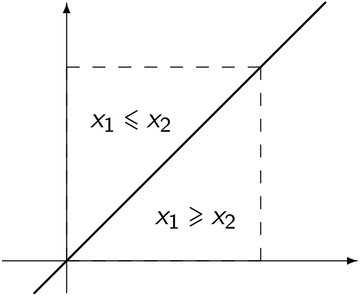
Standard triangulation of $[0,1]^n$

Let σ be a permutation on $[n] = \{1, \ldots, n\}$ $(\sigma \in S_n)$

$$\mathbb{R}^n_\sigma = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)} \right\}$$

$$[0,1]^n_\sigma = \mathbb{R}^n_\sigma \cap [0,1]^n$$

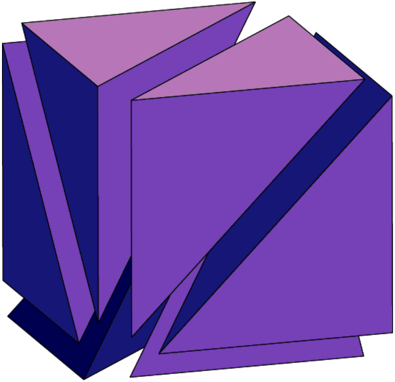
Example : $n = 2$



$2! = 2$ *permutations* (2 *triangles*)

Standard triangulation of $[0,1]^n$

Example : $n = 3$



$3! = 6$ *permutations* (6 *simplexes*)

Lovász extension

Note: Each simplex $[0,1]^n_\sigma$ has exactly $n+1$ vertices

Consider a function $\phi\colon \{0,1\}^n \to \mathbb{R}$ such that $\phi(\mathbf{0}) = 0$

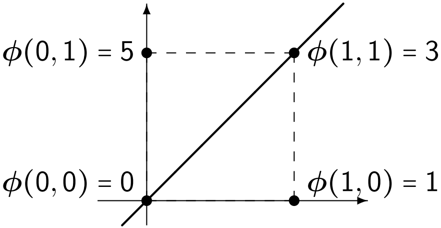
Definition (Lovász, 1983)

The *Lovász extension* of a function $\phi\colon \{0,1\}^n \to \mathbb{R}$ is the function $f_\phi\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each subdomain $\mathbb{R}^n_\sigma$ is the unique linear function which coincides with ϕ at the $n+1$ vertices of the simplex $[0,1]^n_\sigma$

By definition, $f_\phi|_{\{0,1\}^n} = \phi$

Lovász extension

Example :



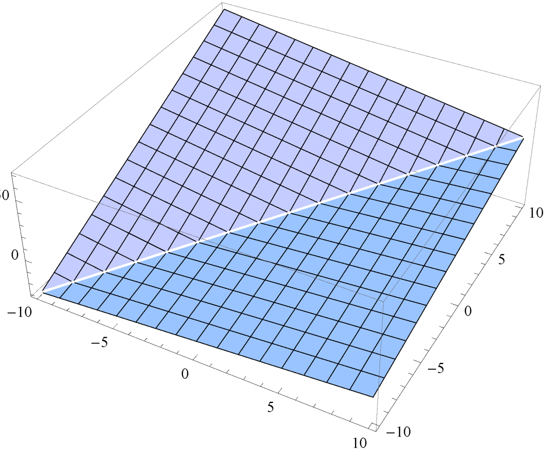
$$x_1 \geqslant x_2 \quad \Rightarrow \quad f_\phi(x_1, x_2) = x_1 + 2x_2$$
$$x_1 \leqslant x_2 \quad \Rightarrow \quad f_\phi(x_1, x_2) = -2x_1 + 5x_2$$

On $\mathbb{R}^2$:

$$\boxed{f_\phi(x_1, x_2) = x_1 + 5x_2 - 3\min(x_1, x_2)}$$

$\Rightarrow$ *f_ϕ is piecewise linear and continuous*

Lovász extension



$$f_\phi(x_1, x_2) = x_1 + 5x_2 - 3\min(x_1, x_2)$$

Lovász extension

In general :

f_ϕ can always be written in the form

$$f_\phi(\mathbf{x}) = \sum_{S \subseteq [n]} a_\phi(S) \min_{i \in S} x_i \qquad (\mathbf{x} \in \mathbb{R}^n)$$

where the coefficients $a_\phi(S)$ are real numbers

$\Rightarrow$ *f_ϕ is always piecewise linear and continuous*

Lovász extension

... and on the "extended" simplex $\mathbb{R}^n_\sigma$?

Back to the example :

$$\begin{aligned} x_1 \geqslant x_2 \quad &\Rightarrow \quad f_\phi(x_1, x_2) = x_1 + 2x_2 \\ x_1 \leqslant x_2 \quad &\Rightarrow \quad f_\phi(x_1, x_2) = -2x_1 + 5x_2 \end{aligned}$$

In general :

$$f_\phi(\mathbf{x}) = x_{\sigma(1)}\,\phi(\mathbf{1}) + \sum_{i=2}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)})\,\phi(\mathbf{1}_{\{\sigma(i),\ldots,\sigma(n)\}}) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma)$$

Choquet integral

We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a *Lovász extension* if there exists a function $\phi\colon \{0,1\}^n \to \mathbb{R}$ such that $f = f_\phi$.

Definition

An n-variable *Choquet integral* is a nondecreasing (in each variable) Lovász extension vanishing at the origin

Three properties of the discrete Choquet integral

Two n-tuples $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ are said to be *comonotonic* if there exists $\sigma \in S_n$ such that $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_\sigma$

Comonotonic additivity (Dellacherie, 1971)

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *comonotonically additive* if, for every comonotonic n-tuples $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, we have

$$f(\mathbf{x} + \mathbf{x}') = f(\mathbf{x}) + f(\mathbf{x}')$$

Proposition. (Schmeidler, 1986)
Every n-variable Choquet integral is comonotonically additive

Three properties of the discrete Choquet integral

Axiomatization (Schmeidler, 1986; De Campos and Bolaños, 1992)

A function $f : \mathbb{R}^n \to \mathbb{R}$ *is a* **Choquet integral** *if and only if it satisfies the following properties :*

(*i*) f *is* **comonotonically additive**

(*ii*) f *is nondecreasing*

(*iii*) f *is positively homogeneous,*
i.e., $f(c\mathbf{x}) = c\, f(\mathbf{x})$ *for every* $\mathbf{x} \in \mathbb{R}^n$ *and every* $c > 0$

(*iv*) $f(\mathbf{0}) = 0$

Three properties of the discrete Choquet integral

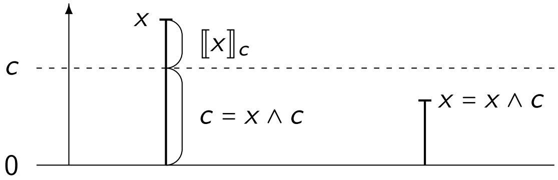
For every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, let

$$\mathbf{x} \wedge c = \big(\min(x_1, c), \ldots, \min(x_n, c)\big)$$

$$[\![\mathbf{x}]\!]_c = \mathbf{x} - \mathbf{x} \wedge c$$

Horizontal min-additivity of $\mathbf{x}$:

$$\mathbf{x} = \mathbf{x} \wedge c + [\![\mathbf{x}]\!]_c$$



Three properties of the discrete Choquet integral

Horizontal min-additivity (Šipoš, 1979)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *horizontally min-additive* if, for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) + f([\![\mathbf{x}]\!]_c)$$

Three properties of the discrete Choquet integral

Axiomatization (Benvenuti et al., 2002)

A function $f : \mathbb{R}^n \to \mathbb{R}$ *is a* **Choquet integral** *if and only if it satisfies the following properties :*

(*i*) f *is* **horizontally min-additive**

(*ii*) f *is nondecreasing*

(*iii*) $f(c\mathbf{1}_S) = c\,f(\mathbf{1}_S)$ *for every* $c \geqslant 0$ *and every* $S \subseteq [n]$

Three properties of the discrete Choquet integral

Horizontal max-additivity

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *horizontally max-additive* if, for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \vee c) + f([\![\mathbf{x}]\!]^c)$$

where $[\![\mathbf{x}]\!]^c = \mathbf{x} - \mathbf{x} \vee c$

Which is the class of functions that are

1. comonotonically additive ?
2. horizontally min-additive ?
3. horizontally max-additive ?

New results

Equivalence

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, the following assertions are equivalent

(*i*) *f is comonotonically additive*

(*ii*) *f is horizontally min-additive*

(*iii*) *f is horizontally max-additive*

New results

Comonotonically additive functions

A function $f:\mathbb{R}^n \to \mathbb{R}$ *is* **comonotonically additive** *if and only if there exists a function* $g:\mathbb{R}^n \to \mathbb{R}$ *satisfying*

(*i*) $x \mapsto g(x\mathbf{1})$ *is additive*

(*ii*) $x \mapsto g(x\mathbf{1}_S)$ *is additive on* $\mathbb{R}_+$ *for every* $S \subseteq [n]$

such that

$$f(\mathbf{x}) = g\big(x_{\sigma(1)}\mathbf{1}\big) + \sum_{i=2}^{n} g\big((x_{\sigma(i)} - x_{\sigma(i-1)})\mathbf{1}_{\{\sigma(i),\ldots,\sigma(n)\}}\big) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma)$$

In this case we can choose $g = f$

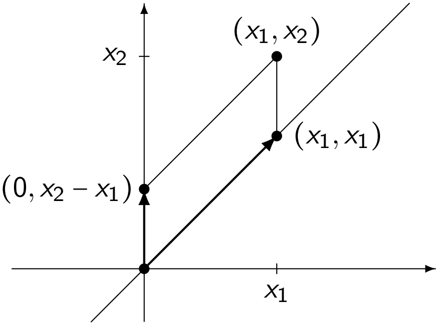
To be compared with the Lovász extension

$$f(\mathbf{x}) = x_{\sigma(1)}\, f(\mathbf{1}) + \sum_{i=2}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)})\, f(\mathbf{1}_{\{\sigma(i),\ldots,\sigma(n)\}}) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma)$$

New results

Example : Comonotonically additive function $f\colon \mathbb{R}^2 \to \mathbb{R}$

$$f|_{x_1 \leqslant x_2}(x_1, x_2) = g(x_1, x_1) + g(0, x_2 - x_1)$$



Standard “parallelogram rule” for vector addition

New results

Lovász extensions

A function $f:\mathbb{R}^n \to \mathbb{R}$ *is a* **Lovász extension** *if and only if the following conditions hold :*

(i) f *is comonotonically additive or horizontally min-additive or horizontally max-additive*

(ii) f *is positively homogeneous,*
i.e., $f(c\mathbf{x}) = c\,f(\mathbf{x})$ *for every* $\mathbf{x} \in \mathbb{R}^n$ *and every* $c > 0$

Remark :

1. Condition (ii) can be relaxed into :
   $x \mapsto f(x\mathbf{1}_S)$ is positively homogeneous for every $S \subseteq [n]$
2. Axiomatizations of the Choquet integral :
   → just add monotonicity (Schmeidler, Benvenuti...)

New results

Condition (ii) can be further relaxed :

Each of the maps

$x \mapsto f(x\mathbf{1})$ on $\mathbb{R}$

$x \mapsto f(x\mathbf{1}_S)$ on $\mathbb{R}_+$ or $\mathbb{R}_-$ $(S \subseteq [n])$

is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure

Symmetric Lovász extension

For every $\mathbf{x} \in \mathbb{R}^n$, set

$$\mathbf{x}^+ = \mathbf{x} \vee 0 \quad \text{and} \quad \mathbf{x}^- = (-\mathbf{x})^+$$

Definition

The *symmetric Lovász extension* of a function $\phi\colon \{0,1\}^n \to \mathbb{R}$ is the function $\check{f}_\phi\colon \mathbb{R}^n \to \mathbb{R}$ defined by

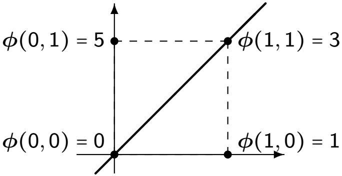
$$\check{f}_\phi(\mathbf{x}) = f_\phi(\mathbf{x}^+) - f_\phi(\mathbf{x}^-)$$

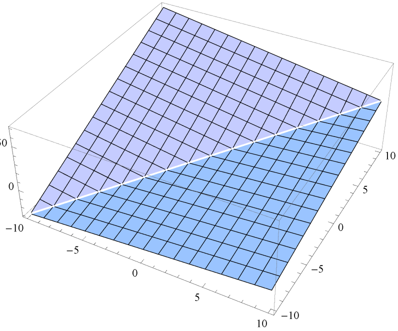
Immediate consequences :

- $\check{f}_\phi$ is piecewise linear and continuous
- $\check{f}_\phi(-\mathbf{x}) = -\check{f}_\phi(\mathbf{x})$
- $\check{f}_\phi(c\mathbf{x}) = c\,\check{f}_\phi(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$

Symmetric Lovász extension

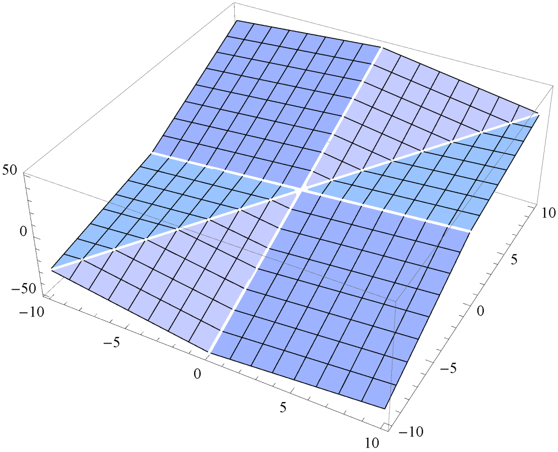
Example :





$$f_\phi(x_1, x_2) = x_1 + 5x_2 - 3\min(x_1, x_2)$$

Symmetric Lovász extension



$$\check{f}_\phi(\mathbf{x}) = f_\phi(\mathbf{x}^+) - f_\phi(\mathbf{x}^-)$$

Symmetric Lovász extension

We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a *symmetric Lovász extension* if there exists a function $\phi\colon \{0,1\}^n \to \mathbb{R}$ such that $f = \check{f}_\phi$.

Definition (Šipoš, 1979)

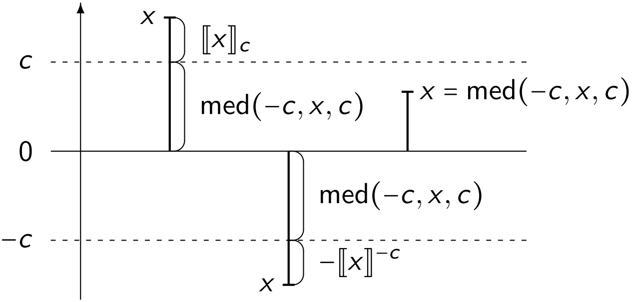
An n-variable *symmetric Choquet integral* is a nondecreasing (in each variable) symmetric Lovász extension vanishing at the origin

New results

Horizontal median-additivity

We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *horizontally median-additive* if, for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \geqslant 0$, we have

$$f(\mathbf{x}) = f\big(\mathrm{med}(-c, \mathbf{x}, c)\big) + f\big([\![\mathbf{x}]\!]_c\big) + f\big([\![\mathbf{x}]\!]^{-c}\big)$$



New results

Horizontal median-additivity

A function $f : \mathbb{R}^n \to \mathbb{R}$ *is* **horizontally median-additive** *if and only if the following conditions hold :*

(i) $f|_{\mathbb{R}^n_+}$ *and* $f|_{\mathbb{R}^n_-}$ *are comonotonically additive*

(ii) $f(\mathbf{x}) = f(\mathbf{x}^+) + f(-\mathbf{x}^-)$ *for every* $\mathbf{x} \in \mathbb{R}^n$

New results

Symmetric Lovász extensions

A function $f : \mathbb{R}^n \to \mathbb{R}$ *is a* **symmetric Lovász extension** *if and only if the following conditions hold :*

(*i*) f *is horizontally median-additive*

(*ii*) f *is homogeneous,*
i.e., $f(c\mathbf{x}) = c\, f(\mathbf{x})$ *for every* $\mathbf{x} \in \mathbb{R}^n$ *and every* $c \in \mathbb{R}$

Remark :

1. Condition (*ii*) can be relaxed into :
   $x \mapsto f(x\mathbf{1}_S)$ is homogeneous for every $S \subseteq [n]$
2. First axiomatizations of the symmetric Choquet integral :
   → just add monotonicity

New results

Condition (*ii*) can be further relaxed :

Each of the maps

$$x \mapsto f(x\mathbf{1}_S) \text{ on } \mathbb{R}_+ \text{ and } \mathbb{R}_- \quad (S \subseteq [n])$$

is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure

and $f(-\mathbf{1}_S) = -f(\mathbf{1}_S)$ for every $S \subseteq [n]$

Final remark

Our results can be easily extended to functions $f\colon J^n \to \mathbb{R}$, where J is a nontrivial real interval containing the origin 0

Also, the condition $f(\mathbf{0}) = 0$ can be easily dropped off

Thank you for your attention!

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