On three properties of the discrete Choquet integral

Miguel Couceiro and Jean-Luc Marichal

Mathematics Research Unit, FSTC
University of Luxembourg, Luxembourg
{miguel.couceiro, jean-luc.marichal}@uni.lu

Abstract. Three important properties in aggregation theory are investigated, namely horizontal min-additivity, horizontal max-additivity, and comonotonic additivity, which are defined by certain relaxations of the Cauchy functional equation in several variables. We show that these properties are equivalent and we completely describe the functions characterized by them. By adding some regularity conditions, these functions coincide with the Lovász extensions vanishing at the origin, which subsume the discrete Choquet integrals.

1 Introduction

A noteworthy aggregation function is the so-called discrete Choquet integral, which has been widely investigated in aggregation theory, due to its many applications for instance in decision making. A convenient way to introduce the discrete Choquet integral is via the concept of Lovász extension. An n-place Lovász extension is a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each of the $n!$ subdomains

$$\mathbb{R}^n_\sigma = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\} \qquad (\sigma \in S_n)$$

is an affine function, where S_n denotes the set of permutations on $[n] = \{1, \ldots, n\}$. An n-place Choquet integral is simply a nondecreasing (in each variable) n-place Lovász extension which vanishes at the origin. For general background, see [5, §5.4].

In this paper we investigate three properties of the discrete Choquet integral, namely, comonotonic additivity, horizontal min-additivity, and horizontal max-additivity. After recalling the definitions of Lovász extensions and discrete Choquet integrals (Section 2), we show that the three properties above are actually equivalent. We describe the function class axiomatized by these properties and we show that, up to certain regularity conditions (based on those we usually add to the Cauchy functional equation to get linear solutions only), these properties completely characterize those n-place Lovász extensions which vanish at the origin. Nondecreasing monotonicity is then added to characterize the class of n-place Choquet integrals (Section 3).

We employ the following notation throughout the paper. Let $\mathbb{R}_+ = [0, \infty[$ and $\mathbb{R}_- =]-\infty, 0]$. For every $A \subseteq [n]$, the symbol $\mathbf{1}_A$ denotes the n-tuple whose ith component is 1, if $i \in A$, and 0, otherwise. Let also $\mathbf{1} = \mathbf{1}_{[n]}$ and $\mathbf{0} = \mathbf{1}_{\varnothing}$. The symbols $\wedge$ and $\vee$ denote the minimum and maximum functions, respectively. For every function $f\colon \mathbb{R}^n \to \mathbb{R}$, we define its diagonal section $\delta_f\colon \mathbb{R} \to \mathbb{R}$ by $\delta_f(x) = f(x\mathbf{1})$. More generally, for every $A \subseteq [n]$, we define the function $\delta_f^A\colon \mathbb{R} \to \mathbb{R}$ by $\delta_f^A(x) = f(x\mathbf{1}_A)$.

It is important to notice that comonotonic additivity as well as horizontal min-additivity and horizontal max-additivity extend the classical additivity property defined by the Cauchy functional equation for n-place functions

$$f(\mathbf{x}+\mathbf{x}') = f(\mathbf{x})+f(\mathbf{x}') \qquad (\mathbf{x},\mathbf{x}'\in\mathbb{R}^n). \tag{1}$$

In this regard, recall that the general solution $f\colon\mathbb{R}^n\to\mathbb{R}$ of the Cauchy equation (1) is given by $f(\mathbf{x})=\sum_{k=1}^n f_k(x_k)$, where the $f_k\colon\mathbb{R}\to\mathbb{R}$ ($k\in[n]$) are arbitrary solutions of the basic Cauchy equation $f_k(x+x')=f_k(x)+f_k(x')$ (see [1, §2–4]). If f_k is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure, then f_k is necessarily a linear function ([1]).

2 Lovász extensions

Consider a *pseudo-Boolean function*, that is, a function $\phi\colon\{0,1\}^n\to\mathbb{R}$, and define the set function $v_\phi\colon 2^{[n]}\to\mathbb{R}$ by $v_\phi(A)=\phi(\mathbf{1}_A)$ for every $A\subseteq[n]$. Hammer and Rudeanu [6] showed that such a function has a unique representation as a multilinear polynomial of n variables

$$\phi(\mathbf{x}) = \sum_{A\subseteq[n]} a_\phi(A)\prod_{i\in A}x_i\,,$$

where the set function $a_\phi\colon 2^{[n]}\to\mathbb{R}$, called the *Möbius transform* of v_ϕ, is defined by

$$a_\phi(A) = \sum_{B\subseteq A}(-1)^{|A|-|B|}\,v_\phi(B).$$

The *Lovász extension* of a pseudo-Boolean function $\phi\colon\{0,1\}^n\to\mathbb{R}$ is the function $f_\phi\colon\mathbb{R}^n\to\mathbb{R}$ whose restriction to each subdomain $\mathbb{R}^n_\sigma$ ($\sigma\in S_n$) is the unique affine function which agrees with ϕ at the $n+1$ vertices of the n-simplex $[0,1]^n\cap\mathbb{R}^n_\sigma$ (see [7, 9]). We then have $f_\phi|_{\{0,1\}^n}=\phi$.

It can be shown (see [5, §5.4.2]) that the Lovász extension of a pseudo-Boolean function $\phi\colon\{0,1\}^n\to\mathbb{R}$ is the continuous function $f_\phi(\mathbf{x})=\sum_{A\subseteq[n]}a_\phi(A)\bigwedge_{i\in A}x_i$. Its restriction to $\mathbb{R}^n_\sigma$ is the affine function

$$f_\phi(\mathbf{x}) = (1-x_{\sigma(n)})\,\phi(\mathbf{0})+x_{\sigma(1)}\,v_\phi(A^{\uparrow}_\sigma(1))+\sum_{i=2}^n(x_{\sigma(i)}-x_{\sigma(i-1)})\,v_\phi(A^{\uparrow}_\sigma(i)), \tag{2}$$

where $A^{\uparrow}_\sigma(i)=\{\sigma(i),\ldots,\sigma(n)\}$. We say that a function $f\colon\mathbb{R}^n\to\mathbb{R}$ is a *Lovász extension* if there is a pseudo-Boolean function $\phi\colon\{0,1\}^n\to\mathbb{R}$ such that $f=f_\phi$.

An n-place *Choquet integral* is a nondecreasing Lovász extension $f_\phi\colon\mathbb{R}^n\to\mathbb{R}$ such that $f_\phi(\mathbf{0})=0$. It is easy to see that a Lovász extension $f\colon\mathbb{R}^n\to\mathbb{R}$ is an n-place Choquet integral if and only if its underlying pseudo-Boolean function $\phi=f|_{\{0,1\}^n}$ is nondecreasing and vanishes at the origin (see [5, §5.4]).

3 Axiomatizations of Lovász extensions and discrete Choquet integrals

Two n-tuples $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$ are said to be *comonotonic* if there exists $\sigma \in S_n$ such that $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n_\sigma$. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *comonotonically additive* if, for every comonotonic n-tuples $\mathbf{x},\mathbf{x}' \in \mathbb{R}^n$, we have

$$f(\mathbf{x}+\mathbf{x}') = f(\mathbf{x}) + f(\mathbf{x}'). \tag{3}$$

Given $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, let $[\![\mathbf{x}]\!]_c = \mathbf{x} - \mathbf{x} \wedge c$ and $[\![\mathbf{x}]\!]^c = \mathbf{x} - \mathbf{x} \vee c$. We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is

- *horizontally min-additive* if, for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) + f([\![\mathbf{x}]\!]_c). \tag{4}$$

- *horizontally max-additive* if, for every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \vee c) + f([\![\mathbf{x}]\!]^c). \tag{5}$$

We now describe the function classes axiomatized by these two properties. To this extent, we let $A^{\downarrow}_{\sigma}(i) = \{\sigma(1),\ldots,\sigma(i)\}$.

Theorem 1. *A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is horizontally min-additive if and only if there exists $g\colon \mathbb{R}^n \to \mathbb{R}$, with δ_g and $\delta_g^A|_{\mathbb{R}_+}$ additive for every $A \subseteq [n]$, such that, for every $\sigma \in S_n$,*

$$f(\mathbf{x}) = \delta_g(x_{\sigma(1)}) + \sum_{i=2}^{n} \delta_g^{A^{\uparrow}_{\sigma}(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma). \tag{6}$$

In this case, we can choose $g = f$.

Theorem 2. *A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is horizontally max-additive if and only if there exists $h\colon \mathbb{R}^n \to \mathbb{R}$, with δ_h and $\delta_h^A|_{\mathbb{R}_-}$ additive for every $A \subseteq [n]$, such that, for every $\sigma \in S_n$,*

$$f(\mathbf{x}) = \delta_h(x_{\sigma(n)}) + \sum_{i=1}^{n-1} \delta_h^{A^{\downarrow}_{\sigma}(i)}(x_{\sigma(i)} - x_{\sigma(i+1)}) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma).$$

In this case, we can choose $h = f$.

Using Theorems 1 and 2, one can show that each of the two horizontal additivity properties is equivalent to comonotonic additivity. Thus we have the following result.

Theorem 3. *For any function $f\colon \mathbb{R}^n \to \mathbb{R}$, the following assertions are equivalent.*

(i) *f is comonotonically additive.*
(ii) *f is horizontally min-additive.*
(iii) *f is horizontally max-additive.*

If any of these conditions is fulfilled, then δ_f, $\delta_f^A|_{\mathbb{R}_+}$, and $\delta_f^A|_{\mathbb{R}_-}$ are additive $\forall\, A \subseteq [n]$.

We now axiomatize the class of n-place Lovász extensions. To this extent, a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *positively homogeneous of degree one* if $f(c\mathbf{x}) = c\, f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ and every $c > 0$.

Theorem 4. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function and let $f_0 = f - f(\mathbf{0})$. Then f is a Lovász extension if and only if the following conditions hold:*

(i) *f_0 is comonotonically additive or horizontally min-additive or horizontally max-additive.*

(ii) *Each of the maps δ_{f_0} and $\delta_{f_0}^A|_{\mathbb{R}_+}$ $(A \subseteq [n])$ is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure.*

The set $\mathbb{R}_+$ can be replaced by $\mathbb{R}_-$ in (ii). Condition (ii) holds whenever Condition (i) holds and $\delta_{f_0}^A$ is positively homogeneous of degree one for every $A \subseteq [n]$.

Remark 1. (a) Since any Lovász extension vanishing at the origin is positively homogeneous of degree one, Condition (ii) of Theorem 4 can be replaced by the stronger condition: f_0 is positively homogeneous of degree one.

(b) Axiomatizations of the class of n-place Choquet integrals can be immediately derived from Theorem 4 by adding nondecreasing monotonicity. Similar axiomatizations using comonotone additivity (resp. horizontal min-additivity) were obtained by de Campos and Bolaños [3] (resp. by Benvenuti et al. [2, §2.5]).

(c) The concept of comonotonic additivity appeared first in Dellacherie [4] and then in Schmeidler [8]. The concept of horizontal min-additivity was previously considered by Šipoš [10] and then by Benvenuti et al. [2, §2.3] where it was called "horizontal additivity".

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