On comonotonically modular functions

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1 Preliminaries

The discrete Choquet and the discrete Sugeno integrals are well-known aggregation functions that have been widely investigated due to their many applications in decision making (see the edited book [9]). A convenient way to introduce the discrete Choquet integral is via the concept of Lovász extension. An n-place Lovász extension is a continuous function $L\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each of the $n!$ subdomains

$$\mathbb{R}^n_\sigma = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}, \qquad \sigma \in S_n,$$

is an affine function, where S_n denotes the set of permutations on $[n] = \{1,\ldots,n\}$. Equivalently, Lovász extensions can be defined via the notion of pseudo-Boolean function, i.e., a mapping $\psi\colon \mathbb{B}^n \to \mathbb{R}$; its corresponding set function $v_\psi\colon 2^{[n]} \to \mathbb{R}$ is defined by $v_\psi(A) = \psi(\mathbf{1}_A)$ for every $A \subseteq [n]$, where $\mathbf{1}_A$ denotes the n-tuple whose i-th component is 1 if $i \in A$, and is 0 otherwise. The *Lovász extension* of a pseudo-Boolean function $\psi\colon \mathbb{B}^n \to \mathbb{R}$ is the function $L_\psi\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each subdomain $\mathbb{R}^n_\sigma$ $(\sigma \in S_n)$ is the unique affine function which agrees with ψ at the $n+1$ vertices of the n-simplex $[0,1]^n \cap \mathbb{R}^n_\sigma$ (see [11, 12]). We then have $L_\psi|_{\mathbb{B}^n} = \psi$.

It can be shown (see [8, §5.4.2]) that the Lovász extension of a pseudo-Boolean function $\psi\colon \mathbb{B}^n \to \mathbb{R}$ is the continuous function

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \sum_{i\in[n]} x_{\sigma(i)} \big(L_\psi(\mathbf{1}_{A^\uparrow_\sigma(i)}) - L_\psi(\mathbf{1}_{A^\uparrow_\sigma(i+1)})\big), \qquad \mathbf{x} \in \mathbb{R}^n_\sigma, \tag{1}$$

where $A^\uparrow_\sigma(i) = \{\sigma(i),\ldots,\sigma(n)\}$, with the convention that $A^\uparrow_\sigma(n+1) = \varnothing$. Indeed, for any $k \in [n+1]$, both sides of (1) agree at $\mathbf{x} = \mathbf{1}_{A^\uparrow_\sigma(k)}$. Let ψ^d denote the *dual* of ψ, that is the function $\psi^d\colon \mathbb{B}^n \to \mathbb{R}$ defined by $\psi^d(\mathbf{x}) = \psi(\mathbf{0}) + \psi(\mathbf{1}) - \psi(\mathbf{1}-\mathbf{x})$. Then

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + L_\psi(\mathbf{x}^+) - L_{\psi^d}(\mathbf{x}^-), \tag{2}$$

where $\mathbf{x}^+ = \mathbf{x} \vee \mathbf{0}$ and $\mathbf{x}^- = (-\mathbf{x})^+$. An n-place *Choquet integral* is a nondecreasing Lovász extension $L_\psi\colon \mathbb{R}^n \to \mathbb{R}$ such that $L_\psi(\mathbf{0}) = 0$. It is easy to see that a Lovász extension $L\colon \mathbb{R}^n \to \mathbb{R}$ is an n-place Choquet integral if and only if its underlying pseudo-Boolean function $\psi = L|_{\mathbb{B}^n}$ is nondecreasing and vanishes at the origin (see [8, §5.4]).

Similarly, a convenient way to introduce the discrete Sugeno integral is via the concept of (lattice) polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. It can be shown that polynomial functions are exactly those representable by expressions of the form

$$\bigvee_{A\subseteq[n]} c_A \wedge \bigwedge_{i\in A} x_i, \quad \text{see, e.g., [4, 7].}$$

Over real intervals $I \subseteq \mathbb{R}$, the discrete Sugeno integrals are exactly those polynomial functions $p : I^n \to I$ that are idempotent, i.e., satisfying $p(x,\ldots,x) = x$.

Natural generalizations of Lovász extensions and polynomial functions are the quasi-Lovász extensions and quasi-polynomial functions, which are best described by

$$f(x_1,\ldots,x_n) = L(\varphi(x_1),\ldots,\varphi(x_n)) \quad \text{and} \quad f(x_1,\ldots,x_n) = p(\varphi(x_1),\ldots,\varphi(x_n)), \text{resp.,}$$

where L is a Lovász extension, p is a polynomial function, and φ a nondecreasing function such that $\varphi(0) = 0$. Such aggregation functions are used in decision under uncertainty, where φ is a utility function and f an overall preference functional. It is also used in multi-criteria decision making where the criteria are commensurate (i.e., expressed in a common scale). For a recent reference, see Bouyssou et al. [1]. In this paper we show that all of these classes of functions can be axiomatized in terms of so-called comonotonic modularity by introducing variants of homogeneity. To simplify our exposition when dealing with these different objects simultaneously in a unified framework, we will assume hereinafter that $I = [-1,1] \subseteq \mathbb{R}$, and we set $I_+ = [0,1]$, $I_- = [-1,0]$ and $I^n_\sigma = I^n \cap \mathbb{R}^n_\sigma$.

2 Comonotonic Modularity

A function $f\colon I^n \to \mathbb{R}$ is said to be *modular* (or a *valuation*) if

$$f(\mathbf{x}) + f(\mathbf{x}') = f(\mathbf{x}\wedge\mathbf{x}') + f(\mathbf{x}\vee\mathbf{x}') \tag{3}$$

for every $\mathbf{x},\mathbf{x}' \in I^n$. It was proved (see Topkis [13, Thm 3.3]) that a function $f\colon I^n \to \mathbb{R}$ is modular if and only if it is *separable*, that is, there exist n functions $f_i\colon I \to \mathbb{R}$, $i \in [n]$, such that $f = \sum_{i\in[n]} f_i$. In particular, any 1-place function $f\colon I \to \mathbb{R}$ is modular.

Two n-tuples $\mathbf{x},\mathbf{x}' \in I^n$ are said to be *comonotonic* if $\mathbf{x},\mathbf{x}' \in I^n_\sigma$ for some $\sigma \in S_n$. A function $f\colon I^n \to \mathbb{R}$ is said to be *comonotonically modular* (or, shortly, *comodular*) if (3) holds for every comonotonic n-tuples $\mathbf{x},\mathbf{x}' \in I^n$. Note that for any function $f\colon I^n \to \mathbb{R}$, condition (3) holds for tuples $\mathbf{x} = x\mathbf{1}_A$ and $\mathbf{x}' = x'\mathbf{1}_A$, where $x,x' \in I$ and $A \subseteq [n]$. Note that if $f\colon I^n \to \mathbb{R}$ is comodular, then by setting $\mathbf{x}' = \mathbf{0}$ in (3) we have

$$f_0(\mathbf{x}) = f_0(\mathbf{x}^+) + f_0(-\mathbf{x}^-) \quad (\text{where } f_0 = f - f(\mathbf{0}).)$$

Theorem 1. ([6]) *For any function $f\colon I^n \to \mathbb{R}$, the following are equivalent.*

(i) *f is comodular.*
(ii) *There are $g\colon I^n_+ \to \mathbb{R}$ and $h\colon I^n_- \to \mathbb{R}$ comodular s.t. $f_0(\mathbf{x}) = g_0(\mathbf{x}^+) + h_0(-\mathbf{x}^-)$ for every $\mathbf{x} \in I^n$. In this case, we can choose $g = f|_{I^n_+}$ and $h = f|_{I^n_-}$.*

(iii) *There are* $g\colon I_+^n \to \mathbb{R}$ *and* $h\colon I_-^n \to \mathbb{R}$ *s.t. for every* $\sigma \in S_n$ *and* $\mathbf{x} \in I_\sigma^n$,

$$f_0(\mathbf{x}) = \sum_{1 \leqslant i \leqslant p} \big(h(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_{\sigma}(i)}) - h(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_{\sigma}(i-1)})\big) + \sum_{p+1 \leqslant i \leqslant n} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) - g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big),$$

where $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$. *In this case, we can choose* $g = f|_{I_+^n}$ *and* $h = f|_{I_-^n}$.

In the next section we will propose variants of homogeneity, which will show that the class of comodular functions subsumes important aggregation functions (such as Sugeno and Choquet integrals) as well as several extensions that are pertaining to decision making under uncertainty. We finish this section with a noteworthy consequence of Theorem 1 that provides a "comonotonic" analogue of Topkis' characterization [13] of modular functions as separable functions, and which provides an alternative description of comodular functions.

Corollary 1. *A function* $f\colon I^n \to \mathbb{R}$ *is comodular if and only if it is comonotonically separable, that is, for every* $\sigma \in S_n$*, there exist functions* $f_i^\sigma\colon I \to \mathbb{R}$*,* $i \in [n]$*, such that*

$$f(\mathbf{x}) = \sum_{i=1}^{n} f_i^\sigma(x_{\sigma(i)}) = \sum_{i=1}^{n} f^\sigma_{\sigma^{-1}(i)}(x_i), \qquad \mathbf{x} \in I^n \cap \mathbb{R}^n_\sigma.$$

Remark 1. (i) Quasi-polynomial functions were axiomatized in [2] in terms of two well-known conditions in aggregation theory, namely, comonotonic maxitivity and comonotonic minitivity. It is not difficult to verify that both properties imply comonotonic modularity, and hence quasi-polynomial functions are comodular or, equiv., comonotonically separable.

(ii) The discrete Shilkret integral can be seen as an aggregation function $f\colon I^n \to \mathbb{R}$, $I \subseteq \mathbb{R}$, that can be represented by an expression of the form

$$f(\mathbf{x}) = \bigvee_{A \subseteq [n]} s_A \cdot \bigwedge_{i \in A} x_i, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Essentially the Shilkret integral differs from the Choquet integral in the fact that meet-terms are aggregated by join rather than by sum, and from the Sugeno integral in the fact that each meet-term is transformed by scalar multiplication rather than by scalar meet.
Surprisingly and despite these similarities, unlike the Choquet and Sugeno integrals, the Shilkret integral is not comodular, and hence not comonotonically separable: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the Shilkret integral $f(x_1,x_2) = 0.2 \cdot x_1 \vee 0.4 \cdot x_2$, and let $\mathbf{x} = (0.1, 0.1)$ and $\mathbf{x}' = (0.2, 0)$.

3 Homogeneity variants

Despite the negative result concerning the Shilkret integral, the class of comodular functions subsumes a wide variety of integral-like functions, such as quasi-polynomial functions (see Remark 1). The next theorem introduces different variants of homogeneity which, together with comonotonic modularity, provide axiomatizations for the various classes of (extended) integrals we consider in this paper.

Theorem 2. *A function* $f\colon I^n \to \mathbb{R}$ *is a*

1. *quasi-Lovász extension* **iff** *f is comodular and there is* $\varphi\colon I \to \mathbb{R}$ *nondec. s.t.*

$$f(x\mathbf{1}_A) = sign(x)\varphi(x)\, f(sign(x)\, \mathbf{1}_A) \tag{4}$$

2. *Lovász extension* **iff** *f is comodular and there is* $\varphi\colon I \to \mathbb{R}$ *nondecreasing s.t.*

$$f(x\mathbf{1}_A) = sign(x)\,x\,f(sign(x)\,\mathbf{1}_A) \tag{5}$$

3. *quasi-polynomial function* **iff** *it is comodular and there is* $\varphi\colon I \to \mathbb{R}$ *nondec. s.t.*

$$f(x \wedge \mathbf{1}_A) = \varphi(x) \wedge f(\mathbf{1}_A) \quad and \quad f(x \vee \mathbf{1}_A) = \varphi(x) \vee f(\mathbf{1}_A) \tag{6}$$

4. *polynomial function* **iff** *it is comodular and for every x in the range of f*

$$f(x \wedge \mathbf{1}_A) = x \wedge f(\mathbf{1}_A) \quad and \quad f(x \vee \mathbf{1}_A) = x \vee f(\mathbf{1}_A)$$

Proof. The first two assertions follow immediatly from (2) and Theorem 1. Necessity in the last two assertions follows from Remark 1 and the fact that quasi-polynomials and polynomial functions are quasi-min and quasi-max homogeneous, and range-min and range-max homogeneous, resp. (see [2, 3]). For sufficiency in the third, note that from (6) and Theorem 1 (by applying the left identity on I_+ and the right on I_-), it follows that f is nondecreasing and quasi-min and quasi-max homogeneous, and thus it is a quasi-polynomial function (see Theorem 17 in [2]). Sufficiency in the fourth assertion follows similarly but using results from [3]. □

Remark 2. (i) For the symmetric variants of quasi-Lovász extensions and Lovász extensions replace (4) and (5) by $f(x\mathbf{1}_A) = \varphi(x)f(\mathbf{1}_A)$ (φ odd) and $f(x\mathbf{1}_A) = xf(\mathbf{1}_A)$, resp. (see [6]).

(ii) For Choquet integrals add nondecreasing monotonicity, and for Sugeno integrals replace "for every x in the range of f" by "for every $x \in I$".

References

1. D. Bouyssou, D. Dubois, H. Prade, and M. Pirlot, editors. *Decision-Making Process - Concepts and Methods*. ISTE/John Wiley, London, 2009.
2. M. Couceiro and J.-L. Marichal. Axiomatizations of quasi-polynomial functions on bounded chains, *Aequationes Mathematicae* **396**:1 (2009) 195–213.
3. M. Couceiro and J.-L. Marichal. Representations and characterizations of polynomial functions on chains, *Journal of Multiple-Valued Logic and Soft Computing* **16**:1-2 (2010) 65–86.
4. M. Couceiro and J.-L. Marichal. Quasi-polynomial functions over bounded distributive lattices, *Aequationes Mathematicae*, **80**:3 (2010) 319–334.
5. M. Couceiro and J.-L. Marichal. Axiomatizations of Lovász extensions and symmetric Lovász extensions of pseudo-Boolean functions, *Fuzzy sets and Systems* **181**:1 (2011) 28–38.
6. M. Couceiro and J.-L. Marichal. Axiomatizations of quasi-Lovász extensions of pseudo-Boolean functions, *Aequationes Mathematicae* **82** (2011) 213–231.
7. M. Couceiro and J.-L. Marichal. Polynomial functions over bounded distributive lattices, *Journal of Multiple-Valued Logic and Soft Computing*. In press.
8. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Encyclopedia of Mathematics and its Applications 127. Cambridge University Press, Cambridge, UK, 2009.
9. M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy measures and integrals - Theory and applications*, volume 40 of *Studies in Fuzziness and Soft Computing*. Physica-Verlag, Heidelberg, 2000.
10. P. Hammer and S. Rudeanu. *Boolean methods in operations research and related areas*. Berlin-Heidelberg-New York: Springer-Verlag, 1968.
11. L. Lovász. Submodular functions and convexity. In *Mathematical programming, 11th int. Symp., Bonn 1982, 235–257*. 1983.
12. I. Singer. Extensions of functions of 0-1 variables and applications to combinatorial optimization. *Numer. Funct. Anal. Optimization*, 7:23–62, 1984.
13. D. M. Topkis. Minimizing a submodular function on a lattice. *Operations Research*, 26(2):305–321, 1978.