Monograph: Aggregation Functions

Michel Grabisch[a], **Jean-Luc Marichal**[b], **Radko Mesiar**[c], **Endre Pap**[d]

[a]University of Panthéon-Sorbonne, Paris, France
e-mail: michel.grabisch@univ-paris1.fr

[b]University of Luxembourg, Luxembourg
e-mail: jean-luc.marichal@uni.lu

[c]Slovak University of Technology
Bratislava, Slovakia
e-mail: mesiar@math.sk

[d]Department of Mathematics and Informatics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
e-mail: pape@eunet.yu

Abstract: There is given a short overview of the monograph "Aggregation Functions" (M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap), Cambridge University Press (in press) with more details from introductory Chapters 1 and 2.

Keywords: aggregation function, continuity, absolute continuity, Lipschitz condition, conjunction, disjunction, internality

1 Introduction

The central problem we are investigated in this book is the process of combining several numerical values into a single representative, one is called *aggregation*, and the numerical function performing this process is called *aggregation function*, asking also some natural conditions as monotonicity and boundary conditions. There is large field of application of aggregation: applied mathematics (e.g., probability, statistics, decision theory), computer sciences (e.g., artificial intelligence, operations research), as well as many applied fields (economy and finance, pattern recognition and image processing, data fusion, multicriteria decision aid, automated reasoning, etc.). The rapid growing of the above mentioned application fields, largely due to the arrival of computers, has made necessary the establishment of a sound theoretical basis for aggregation functions. Most of the results were disseminated in various journals or specialized books, where usually only one specific class of aggregation functions devoted to one specific domain is discussed. Today, an important amount of literature is already available, many significant results have been found (such as characterizations of various families of aggregation functions), and many connections have been done with either related fields or former works (such as triangular norms in probabilistic metric spaces, theory of means and averages, etc.), see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. For the researcher as well as for the practitioner, this proficiency and abundance of literature, but scattered in many domains, are more a handicap than an advantage, and there is a real lack of a unified and complete view of aggregation functions, where one could find the most important concepts and results, presented in a clear and rigorous way. This book has been written in the intention of filling this gap: it offers a full, comprehensive, rigorous and unified treatment of aggregation functions, and is intended to serve as a reference book in the field. Our main motivation has been to bring a unified viewpoint of the aggregation problem, and to provide an abstract mathematical presentation and analysis of aggregation functions used in various disciplines, without referring explicitly to a given domain. The book also provides to the field a unified terminology and notation.

After giving the contents of the book, we present only the basic definition of the aggregation function with few common examples. As an illustration, we mention only two additional elementary mathematical properties: continuity and position with respect to Min and Max. After that we mention very briefly some topics of the book.

2 Contents of the book

3 Definition and examples

A first natural requirement comes from the fact that the output should be a synthetic value. Then, if inputs are supposed to lie in interval $[a, b]$, the output should also lie in this interval. Moreover, if all input values are identical to the lower bound a, then the output should be also a, and similarly for the case of the upper bound b. This defines a boundary condition. A second natural requirement is nondecreasing monotonicity. It means that if some of the input values increase, the representative output value should reflect this increase, or at worst, stay constant. These two requirements are commonly accepted in the field, and we adopt them as the basic definition of an aggregation function.

We assume throughout that the variables of any aggregation function have a common domain $\mathbb{I}$ which is a nonempty real interval, bounded or not. In some exceptional cases, $\mathbb{I}$ will represent a nonempty interval of the extended real number system $\overline{\mathbb{R}} := [-\infty, \infty]$. This latter case will always be mentioned explicitly and then we will often consider the convention that $0 \cdot \infty = 0$ and $\infty + (-\infty) = -\infty$. Denote by $\mathbb{N}$ the set of strictly positive integers and $[n] := \{1, \ldots, n\}$. Further, $|K|$ is the cardinal number of the set K.

Definition 3.1. *An aggregation function in $\mathbb{I}^n$ is a function $\mathsf{A}^{(n)} : \mathbb{I}^n \to \mathbb{I}$ that*

(i) is nondecreasing (in each variable)

(ii) fulfills the boundary conditions

$$\inf_{\mathbf{x}\in\mathbb{I}^n} \mathsf{A}^{(n)}(\mathbf{x}) = \inf \mathbb{I} \quad \textit{and} \quad \sup_{\mathbf{x}\in\mathbb{I}^n} \mathsf{A}^{(n)}(\mathbf{x}) = \sup \mathbb{I}.$$

To give a first instance, the arithmetic mean, defined by

$$\mathsf{AM}^{(n)}(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1}$$

is clearly an aggregation function in any domain $\mathbb{I}^n$.

The aggregation of a singleton is not an actual aggregation, the following convention is often considered: $\mathsf{A}^{(1)}(x) = x \qquad (x \in \mathbb{I})$.

We now introduce the concept of extended aggregation function.

Definition 3.2. *An extended aggregation function in $\cup_{n\in\mathbb{N}}\mathbb{I}^n$ is a mapping*

$$\mathsf{A} : \bigcup_{n\in\mathbb{N}} \mathbb{I}^n \to \overline{\mathbb{R}}.$$

whose restriction $\mathsf{A}^{(n)} := \mathsf{A}|_{\mathbb{I}^n}$ *to* $\mathbb{I}^n$ *is an aggregation function in* $\mathbb{I}^n$ *for any* $n \in \mathbb{N}$.

For example, the arithmetic mean as an extended aggregation function is the sequence $(\mathsf{AM}^{(n)})_{n\in\mathbb{N}}$, where $\mathsf{AM}^{(n)}$ is defined by (1) for all $n \in \mathbb{N}$.

For illustration we now give a small list of well-known aggregation functions.

(i) The *arithmetic mean* function $\mathsf{AM} : \mathbb{I}^n \to \mathbb{I}$ and the *geometric mean* function $\mathsf{GM} : \mathbb{I}^n \to \mathbb{I}$ are respectively defined by

$$\mathsf{AM}(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \mathsf{GM}(\mathbf{x}) := \Big(\prod_{i=1}^{n} x_i\Big)^{1/n}.$$

Note that, when $n > 1$, the geometric mean is not an aggregation function in any domain. We must consider a domain $\mathbb{I}^n$ such that $\mathbb{I} \subseteq [0, \infty]$.

(ii) For any $k \in [n]$, the *projection* function $\mathsf{P}_k : \mathbb{I}^n \to \mathbb{I}$ and the *order statistic* function $\mathsf{OS}_k : \mathbb{I}^n \to \mathbb{I}$ associated with the kth argument, are respectively defined by

$$\mathsf{P}_k(\mathbf{x}) := x_k, \quad \mathsf{OS}_k(\mathbf{x}) := x_{(k)},$$

where $x_{(k)}$ is the kth lowest coordinate of $\mathbf{x}$, that is,

$$x_{(1)} \leqslant \cdots \leqslant x_{(k)} \leqslant \cdots \leqslant x_{(n)}.$$

The *projection onto the first and the last coordinates* are defined as

$$\mathsf{P}_F(\mathbf{x}) := \mathsf{P}_1(\mathbf{x}) = x_1, \quad \mathsf{P}_L(\mathbf{x}) := \mathsf{P}_n(\mathbf{x}) = x_n.$$

Similarly, the extreme order statistics $x_{(1)}$ and $x_{(n)}$ are respectively the *minimum* and *maximum* functions

$$\mathsf{Min}(\mathbf{x}) := \min(x_1, \ldots, x_n),$$

$$\mathsf{Max}(\mathbf{x}) := \max(x_1, \ldots, x_n),$$

which will sometimes be written by means of the lattice operations $\wedge$ and $\vee$, respectively, that is,

$$\mathsf{Min}(\mathbf{x}) = \bigwedge_{i=1}^{n} x_i \quad \text{and} \quad \mathsf{Max}(\mathbf{x}) = \bigvee_{i=1}^{n} x_i.$$

The *median* of an odd number of values $(x_1, \ldots, x_{2k-1})$ is simply defined by

$$\mathsf{Med}(x_1, \ldots, x_{2k-1}) := x_{(k)},$$

which can be rewritten as

$$\mathsf{Med}(x_1, \ldots, x_{2k-1}) = \bigwedge_{\substack{K\subseteq[2k-1]\\ |K|=k}} \bigvee_{i\in K} x_i.$$

For instance, we have

$$\mathsf{Med}(x_1,x_2,x_3) = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3).$$

For any $\alpha \in \mathbb{I}$, we also define the α-median, $\mathsf{Med}_\alpha : \mathbb{I}^n \to \mathbb{I}$, by

$$\mathsf{Med}_\alpha(\mathbf{x}) = \mathsf{Med}(x_1,\ldots,x_n,\underbrace{\alpha,\ldots,\alpha}_{n-1}).$$

(iii) For any nonempty subset $K \subseteq [n]$, the *partial minimum* $\mathsf{Min}_K : \mathbb{I}^n \to \mathbb{I}$ and the *partial maximum* $\mathsf{Max}_K : \mathbb{I}^n \to \mathbb{I}$, associated with K, are respectively defined by

$$\mathsf{Min}_K(\mathbf{x}) := \bigwedge_{i\in K} x_i, \quad \mathsf{Max}_K(\mathbf{x}) := \bigvee_{i\in K} x_i.$$

(iv) For any weight vector $\mathbf{w} = (w_1,\ldots,w_n) \in [0,1]^n$ such that $\sum_{i=1}^n w_i = 1$, the *weighted arithmetic mean* function $\mathsf{WAM}_\mathbf{w} : \mathbb{I}^n \to \mathbb{I}$ and the *ordered weighted averaging* function $\mathsf{OWA}_\mathbf{w} : \mathbb{I}^n \to \mathbb{I}$ associated with $\mathbf{w}$, are respectively defined by

$$\mathsf{WAM}_\mathbf{w}(\mathbf{x}) := \sum_{i=1}^n w_i\, x_i, \quad \mathsf{OWA}_\mathbf{w}(\mathbf{x}) := \sum_{i=1}^n w_i\, x_{(i)}.$$

(v) The *sum* $\mathsf{\Sigma} : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ and *product* $\mathsf{\Pi} : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ functions are respectively defined by

$$\mathsf{\Sigma}(\mathbf{x}) := \sum_{i=1}^n x_i, \quad \mathsf{\Pi}(\mathbf{x}) := \prod_{i=1}^n x_i.$$

Note that, when $n > 1$, these latter two functions are not aggregation functions in any domain. For the sum, we need to consider the domains $[-\infty,\infty]^n$, $[-\infty,\infty[^n$, $]-\infty,\infty]^n$, $]-\infty,\infty[^n$, $[0,\infty]^n$, $]0,\infty]^n$, $[0,\infty[^n$, $]0,\infty[^n$,$[-\infty,0]^n$, $]-\infty,0]^n$, $[-\infty,0[^n$, $]-\infty,0[^n$. For the product, when restricted to $[0,\infty]^n$, we have to consider the domains $[0,1]^n$, $]0,1]^n$, $[0,1[^n$, $]0,1[^n$, $[1,\infty]^n$, $]1,\infty]^n$, $[1,\infty[^n$, $]1,\infty[^n$, $[0,\infty]^n$, $]0,\infty]^n$, $[0,\infty[^n$, $]0,\infty[^n$.

(vi) Assume $\mathbb{I} = [a,b]$ is a closed interval. The *smallest* and the *greatest* aggregation functions in $[a,b]^n$ are respectively defined by

$$\mathsf{A}_\perp(\mathbf{x}) := \begin{cases} b, & \text{if } x_i = b \text{ for all } i \in [n], \\ a, & \text{otherwise.} \end{cases}$$

$$\mathsf{A}_\top(\mathbf{x}) := \begin{cases} a, & \text{if } x_i = a \text{ for all } i \in [n], \\ b, & \text{otherwise.} \end{cases}$$

By definition, we have $\mathsf{A}_\perp \leqslant \mathsf{A} \leqslant \mathsf{A}_\top$ for any aggregation function $\mathsf{A} : [a,b]^n \to [a,b]$.

4 Elementary mathematical properties

The basic properties for aggregation functions are divided into elementary mathematical properties (monotonicity, continuity, symmetry, etc.), grouping based properties (associativity, decomposability, etc.), invariance to a change of scale (ratio, difference, interval, ordinal scales), and various other properties (neutral and annihilator elements, additivity, etc.). We also introduce invariance properties related to the scale types of the variables being aggregated.

We restrict here only on an analytical property: continuity, and the position on real axe.

4.1 Continuity

Standard continuity

Definition 4.1. $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *is a continuous function if*

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^* \\ \mathbf{x}\in\mathbb{I}^n}} \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{x}^*) \qquad (\mathbf{x}^* \in \mathbb{I}^n).$$

The continuity property essentially means that any small changes in the arguments (possible minor errors) should not entail a big change in the aggregated value (output error).

Figure 1. provides a Venn diagram showing the relationship between some classes of functions defined on bounded closed intervals.

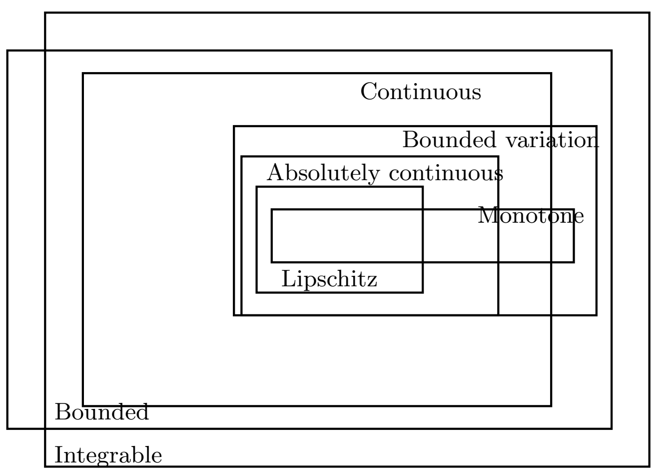

Figure 1: Relations between some classes of functions

A stronger form: Uniform continuity

Definition 4.2. *Let $\|\cdot\| : \mathbb{R}^n \to [0, \infty[$ be a norm and let $D \subseteq \mathbb{I}^n$. A function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ is uniformly continuous in D (with respect to $\|\cdot\|$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathsf{F}(\mathbf{x}) - \mathsf{F}(\mathbf{y})| < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{y}\| < \delta$ and $\mathbf{x}, \mathbf{y} \in D$.*

A function $\mathsf{F} : [a, b]^n \to \mathbb{R}$ is uniformly continuous in $[a, b]^n$ if and only if it is continuous in $[a, b]^n$.

A stronger form: Absolute continuity

Definition 4.3. *We say that the unary function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any finite system of pairwise nonintersecting intervals $]a_i, b_i[\subset]a, b[$, $i \in [n]$, for which $\sum_{i=1}^n (b_i - a_i) < \delta$ the inequality $\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon$ holds.*

Every absolutely continuous function on a closed interval is continuous on this interval.

We extend the notion of absolutely continuity to functions of two variables. For that purpose we introduce, for a subrectangle $R = [a', b'] \times [c', d']$ of $[a, b] \times [c, d]$ and a function $\mathsf{F} : [a, b] \times [c, d] \to \mathbb{R}$, the following notation

$$\Delta_R(\mathsf{F}) = \mathsf{F}(a', c') - \mathsf{F}(b', c') - \mathsf{F}(a', d') + \mathsf{F}(b', d').$$

Definition 4.4. *We say that $\mathsf{F} : [a, b] \times [c, d] \to \mathbb{R}$ is absolutely continuous if the following two conditions are satisfied*

(i) *given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{R \in \mathcal{R}} |\Delta_R(\mathsf{F})| < \varepsilon$$

whenever $\mathcal{R}$ is a finite collection of pairwise nonoverlapping subrectangles $R = [a', b'] \times [c', d']$ of $[a, b] \times [c, d]$ with

$$\sum_{R \in \mathcal{R}} \lambda(R) < \delta\,,$$

where λ is the Lebesgue measure;

(ii) *the marginal functions $\mathsf{F}(\cdot, d)$ and $\mathsf{F}(b, \cdot)$ are absolutely continuous functions of a single variable on $[a, b]$ and $[c, d]$, respectively.*

A stronger form: Lipschitz condition The continuity property can be strengthened into the well-known Lipschitz condition.

Definition 4.5. *Let* $\|\cdot\| : \mathbb{R}^n \to [0,\infty[$ *be a norm. If a function* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *satisfies the inequality*

$$|\mathsf{F}(\mathbf{x}) - \mathsf{F}(\mathbf{y})| \leqslant c\,\|\mathbf{x} - \mathbf{y}\| \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{I}^n), \tag{2}$$

for some constant $c \in\,]0,\infty[$, *then we say that* F *satisfies the Lipschitz condition or is Lipschitzian (with respect to* $\|\cdot\|$*). More precisely, any function* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *satisfying (2) is said to be c-Lipschitzian. The greatest lower bound of constants* $c > 0$ *in (2) is called the Lipschitz constant.*

Important examples of norms are given by the Minkowski norm of order $p \in [1,\infty[$, namely

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p},$$

also called the L_p-norm, and its limiting case $\|\mathbf{x}\|_\infty := \max_i |x_i|$, which is the Chebyshev norm.

The c-Lipschitz condition has an interesting interpretation when applied in aggregation. It allows us to estimate the relative output error in comparison with input errors

$$|\mathsf{F}(\mathbf{x}) - \mathsf{F}(\mathbf{y})| \leqslant c\,\varepsilon$$

whenever $\|\mathbf{x} - \mathbf{y}\| \leqslant \varepsilon$ for some $\varepsilon > 0$.

The Lipschitz property of functions is defined standardly on domains where the norm cannot achieve ∞. Formally it can be defined also on $\mathbb{I}^n$ for an unbounded interval $\mathbb{I}$. However, if $\mathbb{I}$ contains $-\infty$ or ∞, then the Lipschitz property does not imply continuity, in general. For example, the smallest aggregation function $\mathsf{A}_\perp$ on $[0,\infty]^n$ is Lipschitzian for any norm but it is not continuous.

In all cases when $\mathbb{I} \subseteq \mathbb{R}$, the Lipschitz property implies the absolute continuity. However, the converse does not hold. For instance, $\log x$ on $]0,1]$ is absolutely continuous but not Lipschitzian.

Example 4.6. *(i) The product* $\Pi : [0,1]^2 \to [0,1]$ *is* 1*-Lipschitzian. Therefore it is absolutely continuous.*

(ii) The geometric mean $\mathsf{GM} : [0,1]^2 \to [0,1]$ *is not Lipschitzian. However, it is continuous and therefore uniformly continuous.*

Proposition 4.7. *The smallest and the greatest aggregation functions defined in* $[a,b]^n$ *being* 1*-Lipschitzian with respect to the norm* $\|\cdot\|_p$ *are respectively given by* $\mathsf{A}_*^{(n)} : [a,b]^n \to [a,b]$, *with*

$$\mathsf{A}_*^{(n)}(\mathbf{x}) := \mathsf{Max}(b - \|n \cdot b - \mathbf{x}\|_p, a),$$

and $\mathsf{A}^{*(n)} : [a,b]^n \to [a,b]$, *with*

$$\mathsf{A}^{*(n)}(\mathbf{x}) := \mathsf{Min}(a + \|\mathbf{x} - n \cdot a\|_p, b).$$

4.2 Conjunction, disjunction, and internality

Considering the functions Min and Max as dominating or dominated functions gives rise to three main classes of aggregation functions: *conjunctive* functions, *disjunctive* functions and *internal* functions.

Definition 4.8. $\mathsf{A} : \mathbb{I}^n \to \overline{\mathbb{R}}$ *is conjunctive if*

$$\inf \mathbb{I} \leqslant \mathsf{A} \leqslant \mathsf{Min}.$$

Conjunctive functions combine values as if they were related by a logical "and" operator. That is, the result of combination can be high only if all the values are high. Thus, a low value can never be compensated by a high value. The most common conjunctive functions defined on $[0, 1]^n$ are t-norms.

Definition 4.9. $\mathsf{A} : \mathbb{I}^n \to \overline{\mathbb{R}}$ *is disjunctive if*

$$\mathsf{Max} \leqslant \mathsf{A} \leqslant \sup \mathbb{I}.$$

Disjunctive functions combine values as an "or" operator, so that the result of combination is high if at least one value is high. Thus, a high value cannot be compensated by a low value. Such functions are, in this sense, dual of conjunctive functions. The most common disjunctive functions defined on $[0, 1]^n$ are t-conorms.

Definition 4.10. *A function* $\mathsf{A} : \mathbb{I}^n \to \overline{\mathbb{R}}$ *is internal if*

$$\mathsf{Min} \leqslant \mathsf{A} \leqslant \mathsf{Max}.$$

Internality is a property shared by all the means and averaging functions, which are the most often encountered functions in the literature on aggregation. In multicriteria decision making, these functions are also called *compensative* functions. In fact, in this kind of functions, a bad (respectively good) score on one criterion can generally be compensated by a good (respectively bad) one on another criterion, so that the result of the aggregation will be medium.

5 Some topics of the book

The presentation is mathematical and rigorous, avoiding jargon and inherent imprecision from the various applied domains where aggregation functions are used (often under different names such aggregation operators, merging functions, connectives, etc.), but keeping as far as possible the standard terminology of mathematics. As far as possible, every result is given with its proof, unless the proof is long and requires extra material. In this case, a reference to the proof is always given.

A large section is devoted in chapter on disjunctive aggregation functions to triangular norms (t-norms for short): different families, continuous Archimedean t-norms, additively generated t-norms, ordinal sums, etc. There are presented important class of copulas, well known in probability theory, as well as uninorms and nullnorms (as hybrid of t-norms and t-conorms). The concepts of means and averages functions, as well as their relationships, are presented in full generality. Then main subclasses of means, such as quasi-arithmetic ones, some of their special cases (e.g., root-mean-power functions, exponential means), and some of their generalizations (e.g., quasi-linear means, quasi-arithmetic means with weight functions, non-strict quasi-arithmetic means) are presented. Considering nonadditive integrals in the discrete finite case defines a new important class of aggregation functions. Nonadditive integrals are defined with respect to capacities (nonadditive monotone measures), and in particular generalize the notion of expected value. Special care is taken on the Choquet integral, since this is the most representative of nonadditive integrals, and the Sugeno integral, and finally other families of nonadditive integrals. There are given some operations to create new aggregation functions from existing ones, as transformation, composition, introduction of weights on variables, ordinal sums, and various other means (idempotization, symmetrization, etc.). We describe the aggregation functions that are meaningful when considering ratio, difference, and interval scales. We analyze how to extend aggregation functions to the interval $[-1, 1]$ (bipolar scale), that is, to perform a kind a symmetrization with respect to 0 while keeping properties of the aggregation function. Behavioral analysis of aggregation functions is done through various indices and values (like the expected value), which in some sense constitutes the identity card of the aggregation function.

An important topic in practice is how to choose a suitable aggregation function. There are given various ways to identify aggregation functions from data. There are presented the rather unexpected consequences of defining an aggregation function with an infinite (either countable or uncountable) number of arguments. There is given a short description with references of the main fields of applications of aggregation functions, essentially in decision making, data fusion, and artificial intelligence, and with more details on an application to the mixture of uncertainty measures.

6 Conclusion

We present here only few of basic definitions contained in Chapters 1 and 2 of the book. The book is intended primarily to researchers and graduate students in applied mathematics and computer sciences, secondarily to practitioners in, e.g., decision making, optimization, economy and finance, artificial intelligence, data fusion, computer vision, etc. It could also be used as a textbook for graduate students in applied mathematics and computer sciences.

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