The use of the discrete Sugeno integral in decision-making: a survey

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Abstract

An overview of the use of the discrete Sugeno integral as either an aggregation tool or a utility function is presented in the qualitative framework of two decision paradigms: multi-criteria decision-making and decision-making under uncertainty. The parallelism between the representation theorems in both settings is stressed, even if a basic requirement like idempotency should be explicitely stated in multi-criteria decision-making, while its counterpart is implicit in decision under uncertainty by equating the utility of a constant act with the utility of its consequence. Important particular cases of Sugeno integrals such as prioritized minimum and maximum operators, their ordered versions, and Boolean max-min functions are studied.

Keywords: Sugeno integral, aggregation procedures, multi-criteria decision-making, decision-making under uncertainty.

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1 Introduction

In most decision-making problems a global utility function is used to help the decision-maker (DM) make the "best" decision. Of course, the choice of such a global utility function is dictated by the behavior of the DM but also by the nature of the available information, hence by the scale type on which it is represented.

In this paper we deal with the treatment of data given on ordinal scales. Clearly, a meaningful utility function that aggregates ordinal values cannot take advantage of usual arithmetic operations, unless these operations involve only order. In such a context the so-called discrete Sugeno integral appears to be a potential candidate. We intend to discuss its use as a utility function, pointing out its "good" properties but also some of its drawbacks.

The use of the discrete Sugeno integral can be envisaged from two points of view: decision under uncertainty and multi-criteria decision-making. The two problems can be modeled in a very similar way: states of nature in decision under uncertainty correspond to criteria in the other problem. Hence the same mathematical tools apply to both problems, although they differ by particular aspects. For instance, the number of criteria is finite while the number of states of nature is often assumed to be infinite (even a continuum). The analogies between decision-making under uncertainty and multi-criteria evaluation have been noticed for a long time, but their systematic investigation is more recent and may lead to a cross fertilization. Dubois et al. [6] have carried out a comparative study of both problems in the numerical setting of Choquet integral-based evaluations and in the qualitative setting of possibility measures. This has led to obtain a Choquet integral representation theorem in the setting of multi-criteria decision-making [25, 26]. Fargier and Perny [12] have considered the case of purely ordinal theories, stemming from social choice, thus yielding a purely ordinal setting for decision under uncertainty (see also Dubois et al. [1, 4]). Here, the parallel between decision-making under uncertainty and multi-criteria evaluation is highlighted in the finitely-scaled setting, where a monotonic set function, ranging in a finite chain, is used either to qualify the uncertainty of events or the importance of groups of criteria.

The paper is organized as follows. In Section 2 we recall the definition of the Sugeno integral as well as some of its representations. In Section 3 we present the use of the Sugeno integral in two different decision frameworks: decision-making under uncertainty and multi-criteria decision-making. Sections 4 and 5 are devoted to a presentation of some axiomatic characterizations of the Sugeno integral. In Section 6 we present some particular Sugeno integrals, such as the prioritized minimum and maximum. Finally, Section 7 deals with the concept of preferential independence and related properties.

2 The discrete Sugeno integral

We consider a finite set of n elements $N = \{1, \dots, n\}$. Depending on the application, these elements could be players in a cooperative game, criteria in a multi-criteria decision problem, states of nature in a problem of decision under uncertainty, attributes, experts, or voters in an opinion pooling problem, etc.

To define the Sugeno integral we need the concept of fuzzy measure [35]. A fuzzy measure on N is a set function $v : 2^N \to [0, 1]$ satisfying the following conditions:

i) $v(\emptyset) = 0$, $v(N) = 1$,

ii) For any $S, T \subseteq N$, if $S \subseteq T$ then $v(S) \le v(T)$ (monotonicity).

The range of function v is here arbitrarily chosen as the unit interval. In fact any set equipped with a linear ordering relation will do. For instance a finite chain.

In what follows we denote by $\mathcal{F}_N$ the set of all fuzzy measures on N.

We now introduce the concept of discrete Sugeno integral, viewed as an aggregation function from $[0,1]^n$ to $[0,1]$. For theoretical developments, see [16, 35, 36].

The Sugeno integral of a function $x : N \to [0,1]$ with respect to $v \in \mathcal{F}_N$ is defined by

$$\mathcal{S}_v(x) := \bigvee_{i=1}^{n} [x_{(i)} \wedge v(A_{(i)})], \tag{1}$$

where $(\cdot)$ is a permutation on N such that $x_{(1)} \leq \cdots \leq x_{(n)}$. Also, $\wedge := \min$, $\vee := \max$, and $A_{(i)} := \{(i), \ldots, (n)\}$.

We see in this definition that the "coefficient" of each variable x_i is fixed only by the permutation $(\cdot)$. For instance, if $x_3 \leq x_1 \leq x_2$, we have, denoting a function by its image,

$$\mathcal{S}_v(x_1, x_2, x_3) = [x_3 \wedge v(3,1,2)] \vee [x_1 \wedge v(1,2)] \vee [x_2 \wedge v(2)].$$

From the definition we immediately deduce that

$$\mathcal{S}_v(x) \in \{x_1, \ldots, x_n\} \cup \{v(S) \mid S \subseteq N\} \qquad (x \in [0,1]^n).$$

Moreover, denoting by e_S the characteristic vector of $S \subseteq N$ in $\{0,1\}^n$, we have

$$\mathcal{S}_v(e_S) = v(S) \qquad (S \subseteq N), \tag{2}$$

showing that Sugeno integral is completely determined by its values at the vertices of the cube $[0,1]^n$.

It was proved [17, 21, 35] that Sugeno integral can also be put in the following form, which does not need the reordering of the variables:

$$\mathcal{S}_v(x) = \bigvee_{T \subseteq N} \Big[v(T) \wedge \big(\bigwedge_{i \in T} x_i\big) \Big] \qquad (x \in [0,1]^n). \tag{3}$$

However, this formulation involves 2^n terms instead of n.

The expressions (1) and (3), whose primary operation is $\vee$, are disjunctive forms. Using classical distributivity of $\wedge$ and $\vee$, we can put the Sugeno integral in the following conjunctive forms, see [17, 21]:

$$\begin{aligned}
\mathcal{S}_v(x) &= \bigwedge_{i=1}^{n} [x_{(i)} \vee v(A_{(i+1)})] \qquad (x \in [0,1]^n),\\
\mathcal{S}_v(x) &= \bigwedge_{T \subseteq N} \Big[v(N \setminus T) \vee \big(\bigvee_{i \in T} x_i\big) \Big] \qquad (x \in [0,1]^n).
\end{aligned}$$

It was also proved [19] that Sugeno integral is a kind of "weighted" median. More precisely, we have

$$\mathcal{S}_v(x) = \mathrm{median}[x_1, \ldots, x_n, v(A_{(2)}), v(A_{(3)}), \ldots, v(A_{(n)})] \qquad (x \in [0,1]^n). \tag{4}$$

For instance, if $x_3 \leq x_1 \leq x_2$ then

$$\mathcal{S}_v(x_1, x_2, x_3) = \mathrm{median}[x_1, x_2, x_3, v(1,2), v(2)].$$

As an immediate corollary of (4), Sugeno integral is idempotent and

$$\min_i x_i \leq \mathcal{S}_v(x) \leq \max_i x_i \qquad (x \in [0,1]^n).$$

Moreover, for any $i \geq 2$,

$$x_{(i-1)} < \mathcal{S}_v(x) < x_{(i)} \quad \Rightarrow \quad \mathcal{S}_v(x) = v(A_{(i)}).$$

Another interesting formula is the following [22]. For any $k \in N$, we have

$$\mathcal{S}_v(x) = \mathrm{median}[\mathcal{S}_v(x \,|\, x_k = 1), \mathcal{S}_v(x \,|\, x_k = 0), x_k] \qquad (x \in [0,1]^n), \tag{5}$$

where $x|x_k = 1$ denotes the modification of function x where $k \mapsto x_k$ is changed into $k \mapsto 1$.

For instance, if $n = 2$ and $k = 2$, we have

$$\begin{aligned} \mathcal{S}_v(x_1, x_2) &= \mathrm{median}[\mathcal{S}_v(x_1, 1), \mathcal{S}_v(x_1, 0), x_2] \\ &= \mathrm{median}[x_1 \vee v(2), x_1 \wedge v(1), x_2]. \end{aligned}$$

As a consequence of (5), we have, for any $S \not\ni k$,

$$v(S) < \mathcal{S}_v(x) < v(S \cup \{k\}) \quad \Rightarrow \quad \mathcal{S}_v(x) = x_k\,.$$

Before closing this section, we present an interesting result showing that the Sugeno integral is a very natural concept. First, the unit interval $[0,1]$ can be viewed as a totally ordered lattice with operations $\wedge$ and $\vee$. Next, from the variables $x_1, \ldots, x_n \in [0,1]$ and any constants $r_1, \ldots, r_m \in [0,1]$, we can form a lattice polynomial

$$P_{r_1,\ldots,r_m}(x_1, \ldots, x_n)$$

in the usual manner using $\wedge$, $\vee$, and, of course, parentheses. Now, it can be proved [22] that if such a polynomial fulfills

$$P_{r_1,\ldots,r_m}(0, \ldots, 0) = 0 \quad \text{and} \quad P_{r_1,\ldots,r_m}(1, \ldots, 1) = 1,$$

then it is a Sugeno integral on $[0,1]^n$.

For example,

$$P_{r_1,r_2}(x_1, x_2, x_3) = ((x_1 \vee r_2) \wedge x_3) \vee (x_2 \wedge r_1)$$

is a Sugeno integral on $[0,1]^3$. The corresponding fuzzy measure can be identified by (2).

3 Two decision paradigms

In the present section we present two main classes of decision problems: decision-making under uncertainty and multi-criteria decision-making. These decision paradigms present some remarkable similarities.

3.1 Decision-making under uncertainty

A decision-making problem under uncertainty is a 4-uple $(N, X, A, \succeq)$, where

- $N = \{1, \ldots, n\}$ is the set of the possible *states of nature* (more precisely, descriptions of such states),
- X is the set of the possible *consequences* of acts,
- $A = X^N$ is the set of *potential acts*, that is, the set of functions $x : N \to X$,
- $\succeq$ is a *preference relation* on A, supposed to be a complete preordering.

Note that this description actually corresponds to the framework of Savage [32] but for the finiteness of the setting.

It is generally assumed that there is a utility function that describes the worth of the consequences in X, under the form of a function $u : X \to U$, where U is a totally ordered set (in $\mathbb{R}$, in the usual decision theory). Similarly, it is assumed that there is a set function $v : 2^N \to V$, where V is a totally ordered set as well, that describes the knowledge of the DM regarding the actual state of the world. $v(S)$ is the likelihood of event $S \subseteq N$, according to the DM.

In the following we shall assume that the scales U and V are included in a larger totally ordered set L, so that it is possible to compare $v(S)$ and $u(x)$ for any $S \subseteq N$ and $x \in X$. This assumption seems to be artificial, but it results from the existence of a complete ranking of acts, interpreting the elements of X as constant acts and the events as binary acts with extreme consequences.

Now, given an aggregation function $M_v : L^n \to L$, defined from a fuzzy measure $v \in \mathcal{F}_N$, the question then arises of finding a set of conditions on $\succeq$ for the existence of a fuzzy measure $v \in \mathcal{F}_N$, describing the decision-makers' uncertainty about the state of affairs, and a *utility function* $u : X \to \mathbb{R}$ such that

$$x \succeq y \quad \Leftrightarrow \quad U_v(x) \geq U_v(y) \qquad (x, y \in A),$$

where $U_v : A \to L$ is a global utility function defined by

$$U_v(x) := M_v[u(x_1), \ldots, u(x_n)] \qquad (x \in A).$$

3.2 Multi-criteria decision-making

A multi-criteria decision-making problem is a triple $(N, A, \succeq)$, where

- $N = \{1, \ldots, n\}$ is the set of the *criteria* to satisfy,
- A is the Cartesian product $\Pi_{i \in N} X_i$ that corresponds to the set of *alternatives*, X_i being the evaluation scale related to criterion i $(i \in N)$,
- $\succeq$ is a *preference relation* on A.

Here again, given an aggregation function $M_v : L^n \to L$, defined from a fuzzy measure $v \in \mathcal{F}_N$, one searches for a set of conditions on $\succeq$ for the existence of a fuzzy measure $v \in \mathcal{F}_N$ and n utility functions $u_i : X_i \to \mathbb{R}$ $(i \in N)$ such that

$$x \succeq y \quad \Leftrightarrow \quad U_v(x) \geq U_v(y) \qquad (x, y \in A),$$

where $U_v : A \to \mathbb{R}$ is a global utility function defined by

$$U_v(x) := M_v[u_1(x_1), \ldots, u_n(x_n)] \qquad (x \in A).$$

As we can see this paradigm is formally equivalent [6, 26] to that of decision-making under uncertainty, except that here each weak order on X_i $(i \in N)$ is independent of others, which requires the use of n utility functions $u_1, \ldots, u_n$. In decision under uncertainty we had $X_i = X$ $(i \in N)$ and only one utility function u was needed.

Since we make the hypothesis that the evaluations along each criterion $i \in N$ are qualitative by nature, we will assume that their utilities are evaluated on a finite ordinal scale

$$L_i := \{0 = l_1^{(i)} < \cdots < l_{k_i}^{(i)} = 1\},$$

and that the aggregated values $M_v[u_1(x_1), \ldots, u_n(x_n)]$ also lie in a finite ordinal scale

$$L := \{0 = l_1 < \cdots < l_k = 1\}.$$

Note that axiomatic characterizations of Sugeno integral were presented by Hougaard and Keiding [18] in the framework of a cardinal scale and using a Von Neuman-Morgernstern approach, and by Sabbadin [9, 31] and Marichal [22, 23] in our context.

4 Axiomatic characterizations of Sugeno integral in the MCDM context

In this section we present some axiomatic characterizations of Sugeno integral. The first three are rather technical and have no interpretation as appealing properties in multi-criteria decision-making. Nevertheless, they are very simple to express.

Here and throughout we set

$$\overline{r} := (r, \ldots, r) \qquad (\overline{r} \in [0,1]^n),$$

and

$$xSy := \sum_{i \in S} x_i\, e_i + \sum_{i \notin S} y_i\, e_i \qquad (x, y \in [0,1]^n;\ S \subseteq N),$$

where e_i is the binary n-vector such that only its ith component is one.

Consider an aggregation function $M : [0,1]^n \to \mathbb{R}$ and the following properties:

- *Increasing monotonicity (in the wide sense).* For any $x, x' \in [0,1]^n$, we have

$$x_i \leq x_i' \ \ (i \in N) \quad \Rightarrow \quad M(x) \leq M(x').$$

- *Idempotency.* For any $x \in [0,1]$, we have $M(x, \ldots, x) = x$.

- *Non compensation.* For any $S \subseteq N$ and any $r \in [0,1]$, we have

$$M(\overline{r}S\overline{0}), M(\overline{1}S\overline{r}) \in \{M(\overline{1}S\overline{0}), r\}.$$

- *Weak minitivity and maxitivity.* For any $x \in [0,1]^n$ and any $r \in [0,1]$, we have

$$M(x \wedge \overline{r}) = M(x) \wedge r \quad \text{and} \quad M(x \vee \overline{r}) = M(x) \vee r.$$

- *Comonotonic minitivity and maxitivity.* For any $x, x' \in [0,1]^n$ such that

$$(x_i - x_j)(x'_i - x'_j) \geq 0 \qquad (i, j \in N)$$

  we have

$$M(x \wedge x') = M(x) \wedge M(x') \quad \text{and} \quad M(x \vee x') = M(x) \vee M(x').$$

Then, we have the following three characterizations, see [20, 21].

Theorem 4.1 *Let $M : [0,1]^n \to \mathbb{R}$. The following assertions are equivalent:*
i) M is increasing, idempotent, and non compensatory,
ii) M is increasing and weakly minitive and maxitive,
iii) M is increasing, idempotent, and comonotonic minitive and maxitive,
iv) there exists $v \in \mathcal{F}_N$ such that $M = \mathcal{S}_v$.

In the above characterizations, the existence of a fuzzy measure is not assumed beforehand. We now present a characterization of Sugeno integral as a function which depends upon a set function on N. In multi-criteria decision-making, the values of such a set function can be interpreted as the degrees of importance of the subsets of criteria.

First, let Σ_N be the set of all set functions $\sigma : 2^N \to [0,1]$, with $\sigma(\emptyset) = 0$ and $\sigma(N) = 1$. We do not assume that these set functions are monotone. For any such set function $\sigma \in \Sigma_N$, we define the *weighted max-min function* $\mathrm{W}_\sigma^{\vee\wedge} : [0,1]^n \to \mathbb{R}$ associated to σ by (see [21])

$$\mathrm{W}_\sigma^{\vee\wedge}(x) = \bigvee_{T \subseteq N} \Big[\sigma(T) \wedge (\bigwedge_{i \in T} x_i)\Big] \qquad (x \in [0,1]^n).$$

By Eq. (3), we see that any Sugeno integral is a weighted max-min function. Conversely, for any set function $\sigma \in \Sigma_N$ defining $\mathrm{W}_\sigma^{\vee\wedge}$, we have $\mathrm{W}_\sigma^{\vee\wedge} = \mathrm{W}_\mu^{\vee\wedge} = \mathcal{S}_\mu$, where $\mu \in \mathcal{F}_N$ is defined by

$$\mu(S) = \bigvee_{T \subseteq S} \sigma(T) \qquad (S \subseteq N).$$

Thus any weighted max-min function is also a Sugeno integral. Hence we have

$$\{\mathcal{S}_\mu \mid \mu \in \mathcal{F}_N\} = \{\mathrm{W}_\sigma^{\vee\wedge} \mid \sigma \in \Sigma_N\}.$$

Now, let $L := \{l_1 < \cdots < l_k\}$ be a finite ordinal scale with fixed endpoints $l_1 := 0$ and $l_k := 1$. For each $\sigma \in \Sigma_N$, we consider an aggregation function $M_\sigma : L^n \to L$. Since L is an ordinal scale, the numbers that are assigned to it are defined up to an increasing bijection ϕ from $[0,1]$ onto itself. Thus, each function M_σ should satisfy the following property (see Orlov [28]): A function $F : [0,1]^n \to \mathbb{R}$ is *comparison meaningful* (from an ordinal scale) if for any increasing bijection $\varphi : [0,1] \to [0,1]$ and any n-tuples $x, x' \in [0,1]^n$, we have

$$F(x) \leq F(x') \quad \Leftrightarrow \quad F(\varphi(x)) \leq F(\varphi(x')),$$

where the notation $\varphi(x)$ means $(\varphi(x_1), \ldots, \varphi(x_n))$.

Starting from the idea that $\sigma(S)$ should be equal to $M_\sigma(e_S)$ for all $S \subseteq N$, we will ask σ to range in L. Therefore, the mapping $(x, \sigma) \mapsto M_\sigma(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, should be comparison meaningful. We then have the following characterization (see [22]).

Theorem 4.2 *Let $\mathcal{M}$ be a set of functions $M_\sigma : [0,1]^n \to \mathbb{R}$ ($\sigma \in \Sigma_N$) fulfilling the following three properties:*

- *there exist $\sigma, \sigma' \in \Sigma_N$ and $x, x' \in [0,1]^n$ such that $M_\sigma(x) \neq M_{\sigma'}(x')$,*
- *$M_\sigma(x, \ldots, x) = M_{\sigma'}(x, \ldots, x)$ for all $x \in [0,1]$ and all $\sigma, \sigma' \in \Sigma_N$,*
- *the mapping $(x, \sigma) \mapsto M_\sigma(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, is continuous and comparison meaningful.*

Then there exists a continuous and strictly monotonic function $g : [0,1] \to \mathbb{R}$ such that

$$\mathcal{M} \subseteq \{g \circ \mathcal{S}_v \mid v \in \mathcal{F}_N\} = \{g \circ \mathrm{W}_\sigma^{\vee\wedge} \mid \sigma \in \Sigma_N\}.$$

Conversely, for any such function g, the set $\{g \circ \mathrm{W}_\sigma^{\vee\wedge} \mid \sigma \in \Sigma_N\}$ is a candidate for $\mathcal{M}$.

The second property mentioned in Theorem 4.2 can be interpreted as follows. When the partial evaluations of a given alternative do not depend on criteria then they do not depend on their importance neither. Note however that this property is used in the proof only at $x = 0$ and $x = 1$.

Regarding idempotent functions, we have the following result, which follows immediately from Theorem 4.2.

Theorem 4.3 *Let $\mathcal{M}$ be a set of functions $M_\sigma : [0,1]^n \to \mathbb{R}$ ($\sigma \in \Sigma_N$) fulfilling the following two properties:*

- *M_σ is idempotent for all $\sigma \in \Sigma_N$,*
- *the mapping $(x, \sigma) \mapsto M_\sigma(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, is continuous and comparison meaningful.*

Then

$$\mathcal{M} \subseteq \{\mathcal{S}_v \mid v \in \mathcal{F}_N\} = \{\mathrm{W}_\sigma^{\vee\wedge} \mid \sigma \in \Sigma_N\}.$$

Conversely, the set $\{\mathrm{W}_\sigma^{\vee\wedge} \mid \sigma \in \Sigma_N\}$ is a candidate for $\mathcal{M}$.

Theorems 4.3 brings a rather natural motivation for the use of the Sugeno integral as an aggregation function. Nevertheless, continuity may seem to be a questionable hypothesis in the sense that its classical definition uses a distance between aggregated values and relies on the cardinal properties of the arguments. Though continuity and comparison meaningfulness are not contradictory, coupling these two axioms sounds somewhat awkward since the latter one implies that the cardinal properties of the partial evaluations should not be used. Suppressing the continuity property or replacing it by a natural property such as increasing monotonicity remains a quite interesting open problem.

Now we point out a property showing that the Sugeno integral can sometimes have a rather unpleasant behavior.

Proposition 4.1 *Let $v \in \mathcal{F}_N$, $S \subseteq N$, $r \in [0,1]$, and $y \in [0,1]^n$. We have*

$$\left.\begin{array}{c} \mathcal{S}_v(\bar{r}Sy) = r \\ y_i > r \ \ \forall\, i \in N \end{array}\right\} \quad \Rightarrow \quad \mathcal{S}_v(\bar{r}Sz) = r \ \text{ for all } z \in [0,1]^n \text{ such that } z_i \geq r \ (i \in N)$$

and

$$\left.\begin{array}{c} \mathcal{S}_v(\bar{r}Sy) = r \\ y_i < r \ \ \forall\, i \in N \end{array}\right\} \quad \Rightarrow \quad \mathcal{S}_v(\bar{r}Sz) = r \ \text{ for all } z \in [0,1]^n \text{ such that } z_i \leq r \ (i \in N).$$

However, this drawback is less due to the use of Sugeno integral proper, than to the use of a finite scale, that creates a limited number of classes of equally rated decisions, hence producing a coarse ranking. A second reason is the lack of compensation. The latter always occurs at some point in a finite scale, since if we consider any function $f : L^2 \to L$ and two consecutive levels l_i and l_{i+1}, the property $f(l_i, l_{i+1}) \in \{l_i, l_{i+1}\}$ will always hold if, as in the case of Sugeno integral, $\min \leq f \leq \max$. An unpleasant consequence of this fact is that some optimal solutions to a problem in the sense of $\mathcal{S}_v$ may fail to be Pareto-optimal. In order to cope with the defect pointed out in Proposition 4.1, a way out is to refine the ordering induced by Sugeno integral in a way similar to the way the rankings induced by the minimum (and the maximum) aggregation have been refined, using the discrimin, and the leximin orderings (see Dubois, Fargier, and Prade [3, 5]). This is a topic for further research.

5 Axiomatic characterizations of Sugeno integral in the context of decision-making under uncertainty

In the framework of expected utility theory several authors have proposed characterizations leading to the use of either the expected value or the so-called Choquet integral, see Savage [32], Schmeidler [33, 34], and Wakker [37]. However such characterizations always presuppose that the set of states and/or the set of consequences is infinite.

Since we are concerned with the treatment of qualitative information, we will assume that both sets are finite and the utility of each consequence is evaluated on a finite ordinal scale

$$L := \{l_1 < \cdots < l_k\} \subset \mathrm{I\!R},$$

that is, a scale which is unique up to order. Without loss of generality, we can embed this scale in the unit interval $[0, 1]$ and fix the endpoints $l_1 := 0$ and $l_k := 1$. This assumption enables us to consider the Sugeno integral as an aggregation function $M_v : L^n \to L$.

In order to better understand the meaning of Sugeno integral in the setting of decision-making under uncertainty, consider a binary act of the form $\overline{r}S\overline{s}$, where $r \in L$, $s \in L$, $r > s$. Thus $\overline{r}S\overline{s}$ is the act which yields a utility level r if S occurs and s otherwise. Then

$$\mathcal{S}_v(\overline{r}S\overline{s}) = s \vee (r \wedge v(S)) = \mathrm{median}(r, s, v(S)).$$

If we had chosen $r < s$ then $\mathcal{S}_v(\overline{r}S\overline{s}) = \mathrm{median}(r, s, v(N \setminus S))$. This evaluation of act $\overline{r}S\overline{s}$ means the following. If the DM is confident enough in the occurrence of S ($v(S) \geq r > s$) then $\mathcal{S}_v(\overline{r}S\overline{s}) = r$, the utility in case S occurs. If the confidence in S is not high enough ($r > s \geq v(S)$) then the DM presupposes S will not occur and $\mathcal{S}_v(\overline{r}S\overline{s}) = s$, the utility in case S does not occur. If the confidence level is mild ($r \geq v(S) \geq s$) then the utility of $\overline{r}S\overline{s}$ exactly reflects this confidence level.

It implies that the DM's attitude in front of uncertainty is entirely captured by the confidence function v. This is very different from expected utility where the attitude of the DM in front of risk is modeled by the shape of the utility function (concave if the DM is cautious).

A DM is said to be uncertainty-averse with respect to event S iff $v(S) < n(v(N \setminus S))$ where n is the order reversing map on L ($n(t) = 1 - t$ if $L = [0, 1]$). Indeed the confidence relative to the occurrence of $N \setminus S$ is evaluated by $v(N \setminus S)$, but then $n(v(N \setminus S))$ is a natural evaluation of the confidence in S, as much as $v(S)$ itself (in probability theory these evaluations coincide). The cautious evaluation of this confidence in S is $v(S) \wedge n(v(N \setminus S))$, generally. By convention, we assume that $v(S)$, not $n(v(N \setminus S))$, qualifies the uncertainty relative to S, and the uncertainty averse DM relative to S selects a set function v such that $v(S) < n(v(N \setminus S))$. The uncertainty-aversion effect is easily understood by the fact that it is more difficult to get a high utility value with $v(S)$ than with $n(v(N \setminus S))$. Note that if v expresses uncertainty aversion relative to S, it expresses the same relative to $N \setminus S$, namely

$$v(S) < n(v(N \setminus S)) \quad \Leftrightarrow \quad v(N \setminus S) < n(v(S)),$$

since n is an involutive function.

The contrary of an uncertainty-averse DM relative to S is an uncertainty-prone DM, for whom $v(S) > n(v(N \setminus S))$. If $v(S) = n(v(N \setminus S))$ then the DM is uncertainty-neutral w.r.t. S. Choosing a set function v such that $v(S) \leq n(v(N \setminus S))$ for all S models a DM who is uncertainty-averse (for all events). Examples of set functions modeling such attitudes are lower probabilities, belief functions, or necessity measures. Set functions modeling a systematic decision-prone attitude are for instance upper probabilities, plausibility functions, and possibility measures. Among them, only possibility and necessity functions make sense in the ordinal setting.

In the following we assume that

- The set X^N of acts is equipped with a complete preordering structure $\succeq$. (Savage's first axiom). Then the scale L can be viewed as the quotient set $X^N/\sim$ w.r.t. the equivalence relation induced by $\succeq$. The use of an L-valued aggregation function M_v ensures this.

- This ordering is not trivial, that is: $\exists\, x, y, x \succ y$ (Savage's fifth axiom), that is $M_v(x) > M_v(y)$.

It is easy to check that Sugeno's integral verifies the following property: let S be any subset of N and $z \in X^N$. Let $\overline{r}$ and $\overline{s}$ be constant acts, then Sugeno integral satisfies

$$\text{(CCA)}: \quad r \geq s \quad \Rightarrow \quad \mathcal{S}_v(\overline{r}Sz) \geq \mathcal{S}_v(\overline{s}Sz) \quad \forall\, z \in X^N,$$

(coherence w.r.t. constant acts). (CCA) is a weak version of Savage's third axiom.

The following result can be obtained [9]:

Proposition 5.1 *If an aggregation function M_v, which ensures a non trivial complete preordering of acts verifies CCA then it is increasingly monotonic. Moreover the set function is also monotonic w.r.t. set inclusion (letting $v(S) = M_v(e_S)$, where $e_S = \overline{1}S\overline{0}$.)*

Now we shall describe two properties that sound natural in the face of a one-shot act, that is an act that is not supposed to be repeated. For such acts, the evaluation of utility of a decision hardly admits any form of compensation since the actual utility is the utility of the consequence actually obtained in the state of affairs that prevailed when the decision was applied. Especially in the face of total ignorance (modeled by $v(S) = v(N \setminus S) = 0\ \forall\ S \neq N$ for an uncertainty-averse DM, and $v(S) = v(N \setminus S) = 1\ \forall S \neq \emptyset$ for an uncertainty-prone DM) the utility of an act x will be $\min_i u(x_i)$ (uncertainty-aversion) or $\max_i u(x_i)$ (uncertainty-prone).

For instance, suppose you are asked to toss a coin only once and that obtaining heads makes you win 100 Euros while getting tails makes you lose 100 Euros. Suppose the die is fair. Since you play only once, it is strange to assume that you can get anything but win or lose 100 Euros out of that game. Assuming the utility of +100 equals 1 while that of −100 is 0, it would be strange to claim that, in the front of ignorance the utility of that single-shot game is anything but 0 (uncertainty averse) or 1 (uncertainty-prone).

Suppose a constant act $\overline{r}$ and two acts $y, z \in X^N$. Suppose $y \succ z$ and $\overline{r} \succ z$. Let $y \wedge \overline{r}$ be the act such that its ith component is $y_i \wedge r$. In the face of a single-shot act, it is very natural to admit that $y \wedge \overline{r} > z$ (restricted conjunctive dominance). Indeed in any state of nature, the DM expectations if choosing act $y \wedge \overline{r}$ will always be greater than z in a one-shot setting. Hence the aggregation function should satisfy

$$\text{(RCD)}: \quad M_v(\overline{r}) > M_v(z) \quad \text{and} \quad M_v(y) > M_v(z) \quad \text{implies} \quad M_v(\overline{r} \wedge y) > M_v(z)\,.$$

A similar reasoning leads to find it natural to obey the dual property (restricted disjunctive dominance)

$$\text{(RDD)}: \quad M_v(z) > M_v(\overline{r}) \quad \text{and} \quad M_v(z) > M_v(y) \quad \text{implies} \quad M_v(z) > M_v(\overline{r} \vee y)\,,$$

where $\overline{r} \vee y$ has its ith component equal to $r \vee y_i$. Indeed selecting the best consequences of two acts dominated by z cannot lead to overrule this act z in a one-shot setting.

Now we can prove the following proposition [9, 31]:

Proposition 5.2 *If M_v is increasing and satisfies both (RCD) and (RDD) then it is both weakly minitive and maxitive.*

Dubois, Prade, and Sabbadin [9] have proved the following theorem.

Theorem 5.1 (Axiomatization of Sugeno integral as a qualitative utility) *Let $\succeq$ be a preference relation between acts which is complete, transitive, non trivial, and satisfies (CCA), (RCD), (RDD). Then there is a finite chain L, a utility function u on X , and a L-valued monotonic set function v on N such that*

$$x \succeq y \quad \Leftrightarrow \quad \mathcal{S}_v(x) \geq \mathcal{S}_v(y)$$

where

$$\mathcal{S}_v(x) = \bigvee_{x_i \in X} [u(x_{(i)}) \wedge v(A_{(i)})].$$

Clearly, due to the above discussion, this result is the exact counterpart of Theorem 4.1 reduced to *ii*) and *iv*).

Another axiomatization of Sugeno integral as a qualitative utility has been proposed by Sabbadin [31], replacing (RCD) and (RDD) by two simpler, if actually more restrictive, axioms. The first axiom is non compensation. It expresses the requirement that the utility of a binary act $\overline{r}S\overline{s}$ with $r > s$ is either $u(r)$, $u(s)$, or $v(S)$. As explained earlier, this is very natural in the scope of one-shot decision under uncertainty. Then the DM can only expect from act $\overline{r}S\overline{s}$ to receive r, or s, and if the DM uncertainty is not extreme, the utility of $\overline{r}S\overline{s}$ reflects the confidence in S. However this property goes along with a presupposition that $v(S)$ and $u(r)$, $u(s)$ can be compared and are evalued on the same scale. In the scope of decision under uncertainty this is a trivial consequence of the assumption that all acts can be compared, since comparing $v(S)$ and $u(r)$ comes down to comparing a constant act $\overline{r}$ and a binary act $\overline{1}S\overline{0} = e_S$, that is, comparing a sure gain with a simple bet with extreme consequences.

The second axiom is that of the existence of a certainty equivalent to any simple bet $\overline{1}S\overline{0}$, namely

$$\text{CE}: \quad \forall S\ \exists\, x_i \in X,\ \overline{1}S\overline{0} \sim x_i,$$

which means $v(S) = u(x_i)$. Then the representation theorem for qualitative utility described by Sugeno integral is still valid if the conditions (RCD) and (RDD) are changed into non-compensation and the existence of a certainty-equivalent for each binary act $\overline{1}S\overline{0}$ (CE). However the latter axiom, $v(S) = u(x_i)$ for some i, is maybe more restrictive than necessary, since commensurateness between uncertainty and utility just means $\forall S, \exists i, u(x_{i-1}) \leq v(S) \leq u(x_i)$. Hence the (CE) assumption implies that the scale of utilities has not more levels than elements in X. On the contrary the former axiomatization allows for as many as $|X \cup 2^N|$ levels.

Actually, it can be shown that the (CE) axiom can be dropped and Sugeno integral can be axiomatized using the non trivial complete ordering assumption on acts, (CCA), and non compensation only. Comparing to Theorem 4.1, notice that the idempotence assumption is not explicitly made. However, it is always implicit in the setting of Savage where the utility of a constant act is equated to the utility of the corresponding consequence ($M_\sigma(\overline{r}) = r$).

6 Particular Sugeno integrals

In this section we present some subfamilies of the class of Sugeno integrals, namely the prioritized maximum and minimum, the ordered prioritized maximum and minimum, and the Boolean max-min functions. We also discuss their relevance for the two decision-making paradigms.

6.1 Prioritized maximum and minimum operators

The minimum and maximum operators have been extended by Dubois and Prade [8] in a way which is consistent with possibility theory: the prioritized minimum (pmin) and maximum (pmax).

Using the concept of possibility and necessity of fuzzy events [7, 39], one can evaluate the possibility that a relevant goal is attained, and the necessity that all the relevant goals are attained by the help of pmin and pmax operators. The formal analogy with the weighted arithmetic mean is obvious.

For any vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ such that

$$\bigvee_{i=1}^{n} \omega_i = 1,$$

the prioritized maximum operator pmax_ω associated to ω is defined by

$$\text{pmax}_\omega(x) = \bigvee_{i=1}^{n} (\omega_i \wedge x_i), \quad x \in [0,1]^n.$$

For any vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ such that

$$\bigwedge_{i=1}^{n} \omega_i = 0,$$

the prioritized minimum operator pmin_ω associated to ω is defined by

$$\mathrm{pmin}_\omega(x) = \bigwedge_{i=1}^{n} (\omega_i \vee x_i), \quad x \in [0,1]^n.$$

The operator pmax_ω is a Sugeno integral $\mathcal{S}_v$ such that

$$v(T) = \bigvee_{i \in T} \omega_i \qquad (T \subseteq N).$$

In this case, v represents a *possibility measure* Π, which is characterized by the following property:

$$\Pi(S \cup T) = \Pi(S) \vee \Pi(T) \qquad (S, T \subseteq N).$$

Similarly, pmin_ω is a Sugeno integral $\mathcal{S}_v$ such that

$$v(T) = \bigwedge_{i \in N \setminus T} \omega_i \qquad (T \subseteq N).$$

In this case, v represents a *necessity measure* $\mathcal{N}$, which is characterized by the following property:

$$\mathcal{N}(S \cap T) = \mathcal{N}(S) \wedge \mathcal{N}(T) \qquad (S, T \subseteq N).$$

As particular Sugeno integrals, the prioritized minimum and maximum operators can be characterized as follows, see [8, 20, 30] .

Proposition 6.1 *Let $v \in \mathcal{F}_N$. The following three assertions are equivalent:*

i) *v is a possibility measure*

ii) *there exists $\omega \in [0,1]^n$ such that $\mathcal{S}_v = \mathrm{pmax}_\omega$*

iii) *$\mathcal{S}_v(x \vee x') = \mathcal{S}_v(x) \vee \mathcal{S}_v(x')$ for all $x, x' \in [0,1]^n$.*

The following three assertions are equivalent:

iv) *v is a necessity measure*

v) *there exists $\omega \in [0,1]^n$ such that $\mathcal{S}_v = \mathrm{pmin}_\omega$*

vi) *$\mathcal{S}_v(x \wedge x') = \mathcal{S}_v(x) \wedge \mathcal{S}_v(x')$ for all $x, x' \in [0,1]^n$.*

Properties *iii*) and *vi*) can be justified indirectly in the setting of decision under uncertainty. When the set function is a necessity measure (resp. a possibility measure), the DM is systematically uncertainty-averse (resp. uncertainty-prone). Uncertainty aversion is modeled in terms of acts by means of a pessimism axiom that reads as follows [10, 31]:

$$(\mathrm{PES}): \quad \forall x, y, \quad ySx \succ x \quad \Rightarrow \quad x \succeq xSy\,.$$

This axiom sounds reasonable for a systematic uncertainty-averse DM. Indeed, if $ySx \succ x$ it means that changing x_i into y_i for $i \in S$ improves the situation for the DM. The reason why it improves the situation is that overall y is better than x if S does occur, and moreover, the DM considers it sure enough that S will occur, otherwise DM would neglect y as being implausibly obtained. Now when DM considers xSy, then y will be neglected. Indeed since DM considers the occurrence of S sure enough, (s)he considers the occurrence of its complement rather impossible, and will not focus on consequences of states where S does not occur. So xSy does not improve the situation w.r.t. x.

Now it can be established [10, 31] that if axiom (PES) is satisfied by an increasing aggregation M_v then if $z = x \wedge y$ then $M_v(z) = M_v(x)$ or $M_v(z) = M_v(y)$ which obviously implies the minitivity property for M_v. Hence the representation theorem for non trivial complete preorderings on acts obeying (CCA), (RDD), and (PES) in terms of a prioritized minimum (involving a necessity measure for capturing the attitude of the DM in front of uncertainty) is closely related to the second part of Proposition 6.1.

A similar remark for the prioritized maximum can be made. In this case the uncertainty-prone DM may use a preference relation on acts satisfying an optimism axiom

$$(\mathrm{OPT}): \quad \forall x, y, \quad x \succ ySx \quad \Rightarrow \quad xSy \succeq x\,.$$

Here, changing x into y when S occurs depreciates the act. For the optimistic DM, it means that y is less attractive than x and that s(he) considers S as fully possible; now when evaluating xSy, DM considers that since S is possible, (s)he still expects the benefits offered by consequence x when S occurs, regardless of y which is obtained if S does not occur. Indeed either y is less attractive than x, and DM relies on S occuring, so $xSy \succeq x$ holds, or y is more attractive than x; hence $xSy \succeq x$ in any case. Similarly to the pessimistic case, if (OPT) is satisfied by M_v, then if $z = x \vee y$, it follows that $M_v(z) = M_v(x)$ or $M_v(z) = M_v(y)$ [10, 31], hence M_v is maxitive, hence v is a possibility measure and the prioritized maximum is retrieved with the assumptions of Proposition 6.1.

It should be noted that the prioritized maximum and minimum were originally introduced as "weighted maximum and minimum", where the term "weighted" is used by analogy with the weighted average. However, it is more natural to call these operations "prioritized maximum and minimum". Indeed they are useful in the handling of constraint priority in fuzzy constraint satisfaction problems (see [2]).

6.2 Ordered prioritized maximum and minimum operators

For any vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ such that

$$1 = \omega_1 \geq \ldots \geq \omega_n,$$

the ordered prioritized maximum operator [11] opmax_ω associated to ω is defined by

$$\mathrm{opmax}_\omega(x) = \bigvee_{i=1}^{n} (\omega_i \wedge x_{(i)}), \quad x \in [0,1]^n.$$

For any vector $\omega' = (\omega'_1, \ldots, \omega'_n) \in [0,1]^n$ such that

$$\omega'_1 \geq \ldots \geq \omega'_n = 0,$$

the ordered prioritized minimum operator $\mathrm{opmin}_{\omega'}$ associated to ω' is defined by

$$\mathrm{opmin}_{\omega'}(x) = \bigwedge_{i=1}^{n} (\omega'_i \vee x_{(i)}), \quad x \in [0,1]^n.$$

In this definition the inequalities $\omega_1 \geq \ldots \geq \omega_n$ and $\omega'_1 \geq \ldots \geq \omega'_n$ are not restrictive. Indeed, if there exists $i \in \{1, \ldots, n-1\}$ such that $\omega_i \leq \omega_{i+1}$ and $\omega'_i \leq \omega'_{i+1}$ then we have

$$\begin{aligned}
(\omega_i \wedge x_{(i)}) \vee (\omega_{i+1} \wedge x_{(i+1)}) &= \omega_{i+1} \wedge x_{(i+1)}, \\
(\omega'_i \vee x_{(i)}) \wedge (\omega'_{i+1} \vee x_{(i+1)}) &= \omega'_i \vee x_{(i)}.
\end{aligned}$$

This means that ω_i can be replaced by ω_{i+1} in opmax_ω and ω'_{i+1} by ω'_i in $\mathrm{opmin}_{\omega'}$.

Dubois *et al.* [11] used the ordered prioritized maximum (opmax) and minimum (opmin) for modeling soft partial matching. The basic idea of opmax (and opmin) is the same as in the OWA operator introduced by Yager [38]. That is, in both papers coefficients are associated with a particular rank rather than a particular element. The main difference between OWA and opmax (and opmin) is in the underlying non-ordered aggregation operation. OWA uses weighted arithmetic mean while opmax and opmin apply prioritized maximum and minimum. At first glance, this does not seem to be an essential difference. However, Dubois and Prade [8] proved that opmax and opmin are equivalent to the median of the ordered values and some appropriately chosen additional numbers used instead of the original weights.

The operator opmax_ω is a Sugeno integral $\mathcal{S}_v$ such that

$$v(T) = \omega_{n-|T|+1} \qquad (T \subseteq N, T \neq \emptyset).$$

Thus, $v(T)$ only depends on the cardinality of T.

Similarly, the operator $\mathrm{opmin}_{\omega'}$ operator is a Sugeno integral $\mathcal{S}_v$ such that

$$v(T) = \omega'_{n-|T|} \qquad (T \subseteq N, T \neq N).$$

The next proposition [20, 21] shows that any ordered prioritized maximum operator can be put in the form of an ordered prioritized minimum operator and conversely.

Proposition 6.2 *Let ω and ω' be weight vectors defining* opmax_ω *and* $\mathrm{opmin}_{\omega'}$ *respectively. We have*

$$\mathrm{opmin}_{\omega'} = \mathrm{opmax}_\omega \quad \Leftrightarrow \quad \omega'_i = \omega_{i+1} \quad \forall i \in \{1, \ldots, n-1\}.$$

Using (4), we can also see that, for all $x \in [0,1]^n$,

$$\begin{aligned}
\mathrm{opmax}_\omega(x) &= \mathrm{median}(x_1, \ldots, x_n, \omega_2, \ldots, \omega_n), \\
\mathrm{opmin}_{\omega'}(x) &= \mathrm{median}(x_1, \ldots, x_n, \omega'_1, \ldots, \omega'_{n-1}).
\end{aligned}$$

Finally, we have the following characterization [15, 20].

Proposition 6.3 *Let $v \in \mathcal{F}_N$. The following assertions are equivalent:*

i) *v depends only on the cardinality of subsets*
ii) *there exists $\omega \in [0,1]^n$ such that* $\mathcal{S}_v = \mathrm{opmax}_\omega$
iii) *there exists $\omega' \in [0,1]^n$ such that* $\mathcal{S}_v = \mathrm{opmin}_{\omega'}$
iv) *$\mathcal{S}_v$ is a symmetric function.*

Ordered prioritized maximum and minimum have not been very much considered for decision-making under uncertainty. However it is interesting to consider what this utility function may mean. Clearly the uncertainty function depends only on the cardinality of events. This form of uncertainty function expresses some kind of ignorance, since all states will be equally plausible, as well as each pair of states, each k-uple of states. So the symmetry of the utility function w.r.t. the states reflects a situation where the DM cannot discriminate among states. Since there are many such set functions (contrary to the probabilistic and the possibilistic cases where v is then respectively the uniform probability or possibility) it may be interesting to try and describe the type of uncertainty they capture. This is a topic of further research.

6.3 Boolean max-min functions

When the fuzzy measure v is $\{0,1\}$-valued, the Sugeno integral $\mathcal{S}_v$ becomes a Boolean max-min function [21], also called a lattice polynomial [29]. Thus, its definition is the following.

For any non-constant set function $c : 2^N \to \{0,1\}$ such that $c(\emptyset) = 0$, the Boolean max-min function $\mathrm{B}_c^{\vee\wedge} : [0,1]^n \to [0,1]$, associated to c, is defined by

$$\mathrm{B}_c^{\vee\wedge}(x) := \bigvee_{T\subseteq N} \Big[c(T) \bigwedge_{i\in T} x_i\Big] = \bigvee_{\substack{T\subseteq N\\ c(T)=1}} \bigwedge_{i\in T} x_i.$$

In this subsection we investigate this particular Sugeno integral. First, we can readily see that any Boolean max-min function always provides one of its arguments. Moreover, we have the following results [20, 22].

Proposition 6.4 *Consider a function $M : [0,1]^n \to \mathbb{R}$. The following assertions are equivalent:*

i) there exists a set function $c : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_c^{\vee\wedge}$

ii) there exists a $\{0,1\}$-valued $v \in \mathcal{F}_N$ such that $M = \mathcal{S}_v$

iii) there exists $v \in \mathcal{F}_N$ such that $M = \mathcal{S}_v$ and $M(x) \in \{x_1, \ldots, x_n\}$ $(x \in [0,1]^n)$

iv) M is continuous, idempotent, and comparison meaningful.

Theorem 6.1 *The function $M : [0,1]^n \to \mathbb{R}$ is non-constant, continuous, and comparison meaningful if and only if there exists a set function $c : 2^N \to \{0,1\}$ and a continuous and strictly monotonic function $g : [0,1] \to \mathbb{R}$ such that $M = g \circ \mathrm{B}_c^{\vee\wedge}$.*

For any $k \in N$, the Sugeno integral on $[0,1]^n$ defined from the Dirac measure associated with k, that is, the $\{0,1\}$-valued fuzzy measure $v \in \mathcal{F}_N$ defined by $v(T) = 1$ if and only if $T \ni k$, is called a "dictatorial" Sugeno integral. In that case, we have

$$\mathcal{S}_v(x) = x_k \qquad (x \in [0,1]^n).$$

This particular type of Boolean max-min functions will play an important role in the next section.

7 Preferential independence and related properties

The key property in the classical decision theory under uncertainty is the "independence" condition that requires separability of preferences across disjoint states of nature. It was originally introduced by Marschak [24] and Nash [27] as clearly pointed out by Fishburn and Wakker [14]. This property can be expressed in two equivalent ways:

- *Conjoint independence* or *mutual independence* (MI) is satisfied for a binary relation $\succeq$ on the Cartesian product $A = \Pi_{i \in N} X_i$ if, for any alternatives $x, y, z, t \in A$,

$$xSz \succeq ySz \quad \Leftrightarrow \quad xSt \succeq ySt \tag{6}$$

  for any $S \subseteq N$.

- *Coordinate independence* (CI) is satisfied for $\succeq$ if relation (6) is restricted to $S = N \setminus \{k\}$, for any $k \in N$.

These properties are obviously violated if a Sugeno integral is used as a representation of $\succeq$.

Weak separability (WS) is a weaker concept of independence and corresponds to the restriction of (6) to $S = \{k\}$ for any $k \in S$. With the use of relation (5), we get the following result [22]:

Theorem 7.1 *If $\succeq$ is a weakly separable weak order then the Sugeno integral that represents $\succeq$ is dictatorial, namely it is based on a Dirac measure.*

Properties (MI)≡(CI) and (WS) will now be presented in weaker terms as follows:

- *Directional mutual independence* (DMI) is satisfied for $\succeq$ if, for any alternatives $x, y, z, t \in A$,

$$xSz \succ ySz \quad \Rightarrow \quad xSt \succeq ySt \tag{7}$$

  for any $S \subseteq N$, where $\succ$ represents the asymmetric part of $\succeq$ (see also [10] and [31, p.78]).

- *Directional coordinate independence* (DCI) corresponds to (7) where S is restricted to $N \setminus \{k\}$ for any $k \in N$.

- *Directional weak separability* (DWS) if (7) is restricted to $S = \{k\}$ for any $k \in N$.

It is clear that (DMI)⇒(DCI) and (DMI)⇒(DWS).
We also obtain an interesting result:

Proposition 7.1 *If a Sugeno integral represents $\succeq$ then the preference relation is directionally weakly separable (DWS) but violates directional coordinate independence (DCI).*

For particular Sugeno integrals as the prioritized maximum and minimum operators and the ordered prioritized maximum and minimum operators we obtain stronger results expressed in the following proposition:

Proposition 7.2 *i) If* opmax_ω *or* opmin_ω *represents $\succeq$, the preference relation is directionally coordinate independent (DCI) but violates directional mutual independence (DMI). ii) If* pmax_ω *or* pmin_ω *represents $\succeq$, the preference relation satisfies the directional mutual independence (DMI) but violates mutual independence (MI).*

These results may sound paradoxical or counter-intuitive. However the main reason for the violation of independence by Sugeno integral is due to the fact that for monotonic set functions one may have

$$v(S) > v(S') \quad \text{and} \quad v(S \cup T) < v(S' \cup T)\,, \tag{8}$$

where $T \cap (S \cup S') = \emptyset$. Note that this situation can easily be encountered for belief functions and plausibility functions of Shafer. So one must be careful before discarding set functions for which (8) can be observed.

The failure of preferential independence of Sugeno integral in the MCDM context is the counterpart of the failure of the Sure thing principle in the decision under uncertainty framework. This lack of independence is much more drastic than for Choquet integral, since in the latter, independence is only restricted to special situations.

In decision under uncertainty frameworks, the failure of independence for Sugeno integral has two consequences. First, conditional preference is difficult to study, which may make the modeling of rational decisions more difficult in a dynamic context when new information is obtained by the DM. Moreover, given constant acts $\overline{r}$, $\overline{s}$, $\overline{r'}$, $\overline{s'}$, with $r > s$ and $r' > s'$, only the weak stability property holds for Sugeno integral :

$$\mathcal{S}_v(\overline{r}S\overline{s}) > \mathcal{S}_v(\overline{r}T\overline{s}) \quad \Rightarrow \quad \mathcal{S}_v(\overline{r'}S\overline{s'}) \geq \mathcal{S}_v(\overline{r'}T\overline{s'}), \tag{9}$$

for two events S, T. Hence there is ambiguity as to how the uncertainty relation on acts can be defined from binary acts. Fortunately, if $r' > r > s > s'$, then (9) holds with a strict inequality on the right-hand side. It leads us to define $v(S)$ as the utility of the act $\overline{1}S\overline{0}$, with extreme consequences, because it induces the most refined confidence relation between events understood as binary acts $\overline{r}S\overline{s}$. How to define the conditioning of confidence relations (hence of fuzzy measures) in this context is still an open problem.

8 Conclusions

Sugeno integral really appears as the natural counterpart of the Lebesgue and Choquet integrals in the ordinal setting.

This paper strongly suggests that it may be useful as a tool for decision under uncertainty as well as for multi-criteria decision-making. However more insight is needed in order to actually use such a tool in concrete situations.

From the point of view of multi-criteria decision-making, it would be useful to characterize in a transparent way the range of aggregation operations it covers, so as to figure out the expressive power of the ordinal approach. Moreover, the practical use of this ordinal approach is totally conditioned on the possibility of laying bare suitable finite scales for each criterion and their commensurate union into a single value-scale. How to achieve this in practice is far from obvious and requires that suitable questions be asked to the DM so as to get some commensurability landmarks relating the criteria.

In the scope of decision-making under uncertainty the major issue is the proper definition of conditioning, for the purpose of dynamic decision-making.

Lastly, in both settings, the potential lack of Pareto-optimality of best solutions, due to the coarseness of the finite scale setting, should be addressed through a suitable refinement of classes of equivalent decisions [3, 13].

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