Meaningful aggregation functions mapping ordinal scales into an ordinal scale: a state of the art

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Abstract

We present an overview of the meaningful aggregation functions mapping ordinal scales into an ordinal scale. Three main classes are discussed, namely order invariant functions, comparison meaningful functions on a single ordinal scale, and comparison meaningful functions on independent ordinal scales. It appears that the most prominent meaningful aggregation functions are lattice polynomial functions, that is, functions built only on projections and minimum and maximum operations.

1   Introduction

In many domains we are faced with the problem of aggregating a collection of numerical readings to obtain an average value. Actually, such an aggregation problem is becoming more and more present in an increasing number of areas not only of mathematics or physics, but also of engineering, economical, social, and other sciences. Various aggregation functions and processes have already been proposed in the literature and many others are still to be designed to fulfill newer and newer requirements.

Studies on the aggregation problem have shown that the choice of the aggregation function is far from being arbitrary and should be based upon properties dictated by the framework in which the aggregation is performed. One of the main concerns when choosing an appropriate function is to take into account the scale types of the variables being aggregated. On this issue, Luce [24] observed that the general form of the functional relationship between variables is greatly restricted if we know the scale types of the dependent and independent variables. For instance, if all the variables define a common ordinal scale, it is clear that any relevant aggregation function cannot be constructed from usual arithmetic operations, unless these operations involve only order. Thus, computing the arithmetic mean is forbidden whereas the median or any order statistic is permitted.

Specifically, suppose $x_1, \ldots, x_n, x_{n+1}$ are $n+1$ variables, each x_i having a real interval as a domain, and $x_{n+1} = F(x_1, \ldots, x_n)$ is some unknown aggregation function. The problem is to find the general form of the function F knowing the scale types of the input and output variables. The *scale type* of a variable x_i is defined by the class of *admissible transformations*, transformations, such as that from grams to pounds or degrees Fahrenheit to degrees centigrade, that change the scale into an alternative acceptable scale. In the case of a *ratio scale*, for example, an admissible transformation is a function of the form $x \mapsto rx$, with some $r > 0$, which changes the unit of the scale. Similarly, for an *interval scale*, an admissible transformation is a function $x \mapsto rx + s$, with $r > 0$ and $s \in \mathbb{R}$, which modifies both the origin and the unit of the scale. For an *ordinal scale*, an admissible transformation is a strictly increasing function $x \mapsto \phi(x)$, which changes the values of the scale while preserving their order. For more details on the theory of scale types, see [22, 25, 35, 44, 45].

Luce's principle, called "principle of theory construction", is based on the requirement that admissible transformations of the input variables must lead to an admissible transformation of the output variable. For example, if the input variables are independent scales, then the aggregation function F should satisfy the following condition. For any admissible transformations $\phi_1, \ldots, \phi_n$ of the input variables, there is an admissible transformation $\psi_{\phi_1, \ldots, \phi_n}$ of the output variable so that $F(\phi_1(x_1), \ldots, \phi_n(x_n)) = \psi_{\phi_1, \ldots, \phi_n}(x_{n+1})$ or, equivalently,

$$F(\phi_1(x_1), \ldots, \phi_n(x_n)) = \psi_{\phi_1, \ldots, \phi_n}(F(x_1, \ldots, x_n)). \qquad (1)$$

The solutions of this functional equation constitute the set of the possible aggregation functions, which are "meaningful" in the sense that they do not depend upon the particular scales of measurement chosen for the variables, but only upon their scale types.

We can also assume that the input variables define the same scale, which implies that the same admissible transformation must be applied to all the input variables. In this case, the condition on F is the following. For any common admissible transformation ϕ of the input variables, there is an admissible transformation ψ_ϕ of the output variable so that

$$F(\phi(x_1), \ldots, \phi(x_n)) = \psi_\phi(F(x_1, \ldots, x_n)). \qquad (2)$$

In the extreme case where all the input and output variables define the same scale, then, for any admissible transformation ϕ of the input and output variables, we must have

$$F\big(\phi(x_1),\ldots,\phi(x_n)\big) = \phi\big(F(x_1,\ldots,x_n)\big). \tag{3}$$

Equations (1) and (2) were completely solved in the eighties for ratio scale variables, with domain $\mathbb{R}_+ := [0,\infty)$, and interval scale variables, with domain $\mathbb{R}$, even under some further assumptions such as symmetry, continuity, and nondecreasing monotonicity (in each argument); see [1, 2, 3, 4, 19].

Example 1.1 ([4])**.** If all the input variables are independent ratio scales and the output variable is also a ratio scale then the meaningful aggregation functions $F\colon \mathbb{R}_+^n \to \mathbb{R}_+$ are exactly the solutions of (1), where each admissible transformation is a multiplication by a positive constant. These solutions are given by

$$F(x_1,\ldots,x_n) = a\prod_{i=1}^{n} f_i(x_i) \qquad (a > 0),$$

where the functions $f_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ fulfill the equations $f_i(x_iy_i) = f_i(x_i)f_i(y_i)$ $(i = 1,\ldots,n)$. Under continuity, these solutions are of the form

$$F(x_1,\ldots,x_n) = a\prod_{i=1}^{n} x_i^{c_i} \qquad (a > 0,\ c_1,\ldots,c_n \in \mathbb{R}).$$

For ordinal scales and without any further assumptions, equations (1) and (2) have resisted and have remained unsolved for a long time. At present, the complete description of their solutions is known and has been presented recently in a couple of articles; see [30, 31].

The main purpose of this paper is to present a catalog of all the possible meaningful aggregation functions mapping ordinal scales into an ordinal scale. More precisely, we yield all the possible solutions of each of the functional equations above, where the admissible transformations are strictly increasing functions. We also present the solutions under some further assumptions such as continuity, symmetry, idempotency, and nondecreasing monotonicity (in each argument).

In such an ordinal framework, it is natural to assume that the common domain of the input variables be any open real interval or even the whole real line. However, we consider the more general situation where the domain of the input variables is any real interval E, possibly unbounded, and where the domain of the output variable is the real line, except for equation (3) where this domain must also be the set E. We further assume that the admissible transformations of the input variables are confined to the increasing bijections from E onto E. This latter assumption, which brings no restriction to the solutions of the functional equations, enables us to consider closed domains E whose endpoints remain fixed under any admissible transformation.

We thus provide and discuss all the solutions $F \colon E^n \to \mathbb{R}$ of the functional equations (1) and (2), where $\phi_1, \ldots, \phi_n$, and ϕ are arbitrary increasing bijections from E onto E and where $\psi_{\phi_1, \ldots, \phi_n}$ and ψ_ϕ are strictly increasing functions from $\mathrm{ran}(F)$ into $\mathrm{ran}(F)$. We call those solutions *comparison meaningful functions on independent ordinal scales* and *comparison meaningful functions on a single ordinal scale*, respectively. We also provide all the solutions $F \colon E^n \to E$ of the functional equation (3), where ϕ is an arbitrary increasing bijection from E onto E. We call those solutions *order invariant functions*.

The outline of this paper is as follows. In §2 we introduce the concept of order invariant subsets, which will play a key role in the description of the most general solutions of the three functional equations. In §3 we introduce the lattice polynomial functions, which represent most of the regular (e.g., nondecreasing) solutions of equations (2) and (3). We also recall some of their properties as aggregation functions. In §4 we present and discuss all the order invariant functions. In §§5 and 6 we present respectively the comparison meaningful functions on a single ordinal scale and the comparison meaningful functions on independent ordinal scales. Finally in §7 we provide interpretations of equations (1)–(3) in the setting of aggregation on finite chains.

Throughout this paper we denote by E any real interval, bounded or not, with interior E°. We also denote by $B[E]$ the set of *included boundaries* of E, that is $B[E] := E \setminus E^\circ$. The set of all increasing bijections ϕ of E onto itself is denoted by $\Phi[E]$. As each function $\phi \in \Phi[E]$ preserves the ordinal structure of E, the set $\Phi[E]$ is actually the *order automorphism group*, under composition, of E. Finally, the symbol $[n]$ denotes the index set $\{1, \ldots, n\}$ and, for any $\mathbf{x} \in E^n$ and any $\boldsymbol{\phi} \in \Phi[E]^n$, the symbol $\boldsymbol{\phi}(\mathbf{x})$ denotes the vector $\big(\phi_1(x_1), \ldots, \phi_n(x_n)\big)$.

2   Order invariant subsets

The space E^n can be partitioned into *order invariant subsets*, which are very useful in describing the general solutions of the functional equations introduced above. Those subsets were introduced first by Ovchinnikov (see [38, §3] and [39, §2]) in the general framework of ordered sets and then independently by Bartłomiejczyk and Drewniak [5] for closed real intervals; see also [30, 31, 34]. In this section we introduce them through the concept of group orbit.[1]

Consider the product set $\Phi[E]^n$ and its *diagonal restriction*

$$\Phi_n[E] := \big\{(\underbrace{\phi, \ldots, \phi}_{n}) : \phi \in \Phi[E]\big\}.$$

As $\Phi_n[E]$ is clearly a subgroup of $\Phi[E]^n$, we can define the orbit of any element $\mathbf{x} \in E^n$ under the action of $\Phi_n[E]$, that is, $\Phi_n[E](\mathbf{x}) := \{\boldsymbol{\phi}(\mathbf{x}) : \boldsymbol{\phi} \in \Phi_n[E]\}$. The set of orbits of E^n under $\Phi_n[E]$ forms a partition of E^n into equivalence classes, where $\mathbf{x}, \mathbf{y} \in E^n$ are equivalent if their orbits are the same, that is, if there exists $\boldsymbol{\phi} \in \Phi_n[E]$ such that $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$. The orbits of E^n under $\Phi_n[E]$ are *order invariant subsets* in the following sense (see [5]).

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[1]For definitions and results about the concept of orbit in algebra, see e.g. [16].

Definition 2.1. A nonempty subset I of E^n is called *order invariant* if, for any $\mathbf{x} \in I$, we have $\phi(\mathbf{x}) \in I$ for all $\phi \in \Phi_n[E]$.[2] An order invariant subset of E^n is *minimal* if it has no proper order invariant subset.

It is easy to see that the set $\mathcal{I}_n[E] := E^n/\Phi_n[E]$ of orbits of E^n under $\Phi_n[E]$ is identical to the set of minimal order invariant subsets of E^n. Moreover, any order invariant subset is a union of those orbits.

The following proposition (for closed E, see [5, 34]) yields a complete description of the orbits.

Proposition 2.1. *We have $I \in \mathcal{I}_n[E]$ if and only if there exists a permutation π on $[n]$ and a sequence $\{\lhd_i\}_{i=0}^n$ of symbols $\lhd_i \in \{<, =\}$, containing at least one symbol $<$ if $\inf E \in E$ and $\sup E \in E$, such that*

$$I = \{\mathbf{x} \in E^n : \inf E \lhd_0 x_{\pi(1)} \lhd_1 \cdots \lhd_{n-1} x_{\pi(n)} \lhd_n \sup E\},$$

where $\lhd_0$ is $<$ if $\inf E \notin E$ and $\lhd_n$ is $<$ if $\sup E \notin E$.

Example 2.1 ([34])**.** The unit square $[0,1]^2$ contains exactly eleven minimal order invariant subsets, namely the open triangles $\{(x_1, x_2) : 0 < x_1 < x_2 < 1\}$ and $\{(x_1, x_2) : 0 < x_2 < x_1 < 1\}$, the open diagonal $\{(x_1, x_2) : 0 < x_1 = x_2 < 1\}$, the four square vertices, and the four open line segments joining neighboring vertices.

Remark 2.1. From Proposition 2.1 we can easily derive an alternative way to characterize the membership of given vectors $\mathbf{x}, \mathbf{y} \in E^n$ in the same orbit. Let Π_n be the set of permutations on $\{0, 1, \ldots, n+1\}$ and, for any $\mathbf{x} \in E^n$, define

$$\Pi(\mathbf{x}) := \{\pi \in \Pi_n : x_{\pi(0)} \leqslant x_{\pi(1)} \leqslant \cdots \leqslant x_{\pi(n+1)}\},$$

where $x_0 := \inf E$ and $x_{n+1} := \sup E$. Then, for any $\mathbf{x}, \mathbf{y} \in E^n$, there exists $I \in \mathcal{I}_n[E]$ such that $\mathbf{x}, \mathbf{y} \in I$ if and only if $\Pi(\mathbf{x}) = \Pi(\mathbf{y})$.[3]

Since $\Phi[E]^n$ is itself a group, we can also define the orbit of any element $\mathbf{x} \in E^n$ under the action of $\Phi[E]^n$, that is, $\Phi[E]^n(\mathbf{x}) := \{\phi(\mathbf{x}) : \phi \in \Phi[E]^n\}$. Just as for the subgroup $\Phi_n[E]$, the set of orbits of E^n under $\Phi[E]^n$ forms a partition of E^n into equivalence classes, where $\mathbf{x}, \mathbf{y} \in E^n$ are equivalent if there exists $\phi \in \Phi[E]^n$ such that $\mathbf{y} = \phi(\mathbf{x})$. The orbits of E^n under $\Phi[E]^n$ are *strongly order invariant subsets* in the following sense (see [31]).

Definition 2.2. A nonempty subset I of E^n is called *strongly order invariant* if, for any $\mathbf{x} \in I$, we have $\phi(\mathbf{x}) \in I$ for all $\phi \in \Phi[E]^n$.[4] A strongly order invariant subset of E^n is *minimal* if it has no proper strongly order invariant subset.

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[2]Equivalently, I is order invariant if $\phi(I) \subseteq I$ for all $\phi \in \Phi_n[E]$. Actually, since $\Phi_n[E]$ is a group, we can even write $\phi(I) = I$.

[3]This condition is more restrictive than *comonotonicity* of vectors $\mathbf{x}$ and $\mathbf{y}$, which simply means that $\Pi(\mathbf{x})$ and $\Pi(\mathbf{y})$ overlap; see [17].

[4]Equivalently, I is strongly order invariant if $\phi(I) \subseteq I$ for all $\phi \in \Phi[E]^n$. Once again, since $\Phi[E]^n$ is a group, we can even write $\phi(I) = I$.

The set $\mathcal{I}_n^*[E] := E^n/\Phi[E]^n$ of orbits of E^n under $\Phi[E]^n$ is identical to the set of minimal strongly order invariant subsets of E^n. Moreover, any strongly order invariant subset is a union of those orbits.

The following proposition [31] yields a complete description of the orbits.

Proposition 2.2. *We have $\mathcal{I}_n^*[E] = \{\times_{i=1}^n I_i : I_i \in \mathcal{I}_1[E]\} = (\mathcal{I}_1[E])^n$, with cardinality $|\mathcal{I}_n^*[E]| = (1 + |B[E]|)^n$.*

Example 2.2 ([34])**.** The unit square $[0,1]^2$ contains exactly nine minimal strongly order invariant subsets, namely the open square $(0,1)^2$, the four square vertices, and the four open line segments joining neighboring vertices.

Let us now show that the set $\mathcal{I}_n^*[E]$ can be described by means of the set $\mathcal{I}_n[E]$. For any $i \in [n]$, let $\mathrm{P}_i \colon E^n \to E$ be the projection operator onto the ith coordinate, that is, $\mathrm{P}_i(\mathbf{x}) := x_i$. We can easily see that, for any $I \in \mathcal{I}_n[E]$, we have $\mathrm{P}_i(I) \in \mathcal{I}_1[E]$. Define an equivalence relation $\sim$ on $\mathcal{I}_n[E]$ as

$$I \sim J \quad \Leftrightarrow \quad \mathrm{P}_i(I) = \mathrm{P}_i(J) \quad (i \in [n]).$$

Then, it is easy to see [31] that

$$\mathcal{I}_n^*[E] = \Big\{ \bigcup_{\substack{J \in \mathcal{I}_n[E] \\ J \sim I}} J : I \in \mathcal{I}_n[E] \Big\} = \Big\{ \underset{i=1}{\overset{n}{\times}} \mathrm{P}_i(I) : I \in \mathcal{I}_n[E] \Big\}.$$

Now, to easily describe certain nondecreasing aggregation functions, it is useful to consider partial orders on $\mathcal{I}_n[E]$ and $\mathcal{I}_n^*[E]$. Starting from the natural order $\{\inf E\} \prec E^\circ \prec \{\sup E\}$ on $\mathcal{I}_1[E]$, we can straightforwardly derive a partial order $\preccurlyeq$ on $\mathcal{I}_n[E]$, namely

$$I \preccurlyeq J \quad \Leftrightarrow \quad \mathrm{P}_i(I) \preccurlyeq \mathrm{P}_i(J) \quad (i \in [n]).$$

The corresponding partial order on $\mathcal{I}_n^*[E]$ is defined similarly.

Remark 2.2. Consider again the set Π_n of permutations on $\{0, 1, \ldots, n+1\}$ (see Remark 2.1). For any $\mathbf{x} \in E^n$, we can define

$$\Pi^*(\mathbf{x}) := \{\pi \in \Pi_n : \pi(i) \leqslant \ell(\mathbf{x}) \;\Leftrightarrow\; x_i = \inf E$$
$$\text{and} \;\; \pi(j) \geqslant n + 1 - u(\mathbf{x}) \;\Leftrightarrow\; x_j = \sup E\},$$

where $x_0 := \inf E$, $x_{n+1} := \sup E$ and $\ell(\mathbf{x}) := \{i \in [n] : x_i = \inf E\}$, $u(\mathbf{x}) := \{j \in [n] : x_j = \sup E\}$. Then, for any $\mathbf{x}, \mathbf{y} \in E^n$, there exists $I \in \mathcal{I}_n^*[E]$ such that $\mathbf{x}, \mathbf{y} \in I$ if and only if $\Pi^*(\mathbf{x}) = \Pi^*(\mathbf{y})$.

3 Lattice polynomial functions and some of their properties

As we will see in the subsequent sections, certain solutions of equations (2) and (3) are constructed from *lattice polynomial functions*. In this section we briefly

recall the basic material about these functions. As we are concerned with aggregation functions defined in real domains, we do not consider lattice polynomial functions on a general lattice, but simply on $\mathbb{R}$, which is a particular lattice. The lattice operations $\wedge$ and $\vee$ then represent the minimum and maximum operations, respectively.

3.1 Lattice polynomial functions

Let us first recall the concept of lattice polynomial function (with real variables); see e.g. Birkhoff [6, §II.5] or Grätzer [15, §I.4].

Definition 3.1. The class of lattice polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ is defined as follows.

(i) For any $k \in [n]$, the projection $P_k \colon \mathbf{x} \mapsto x_k$ is a lattice polynomial function from $\mathbb{R}^n$ to $\mathbb{R}$.

(ii) If p and q are lattice polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$, then $p \wedge q$ and $p \vee q$ are lattice polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$.

(iii) Every lattice polynomial function from $\mathbb{R}^n$ to $\mathbb{R}$ is constructed by finitely many applications of the rules (i) and (ii).

Because $\mathbb{R}$ is a distributive lattice, any lattice polynomial function can be written in *disjunctive* and *conjunctive* forms as follows; see e.g. [6, §II.5].

Proposition 3.1. *Let $p \colon \mathbb{R}^n \to \mathbb{R}$ be any lattice polynomial function. Then there are nonconstant set functions $\alpha \colon 2^{[n]} \to \{0,1\}$ and $\beta \colon 2^{[n]} \to \{0,1\}$, with $\alpha(\varnothing) = 0$ and $\beta(\varnothing) = 1$, such that*

$$p(\mathbf{x}) = \bigvee_{\substack{S \subseteq [n] \\ \alpha(S)=1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ \beta(S)=0}} \bigvee_{i \in S} x_i. \tag{4}$$

The set functions α and β that disjunctively and conjunctively define the polynomial function p in Proposition 3.1 are not unique. For example, we have $x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2)$. However, it can be shown [28] that, from among all the possible set functions that disjunctively define a given lattice polynomial function, only one is nondecreasing. Similarly, from among all the possible set functions that conjunctively define a given lattice polynomial function, only one is nonincreasing. These particular set functions are given by $\alpha(S) = p(\mathbf{1}_S)$ and $\beta(S) = p(\mathbf{1}_{[n] \setminus S})$ for all $S \subseteq [n]$, where $\mathbf{1}_S$ denotes the characteristic vector of S in $\{0,1\}^n$. Thus, a lattice polynomial function $p \colon \mathbb{R}^n \to \mathbb{R}$ can always be written as

$$p(\mathbf{x}) = \bigvee_{\substack{S \subseteq [n] \\ p(\mathbf{1}_S)=1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ p(\mathbf{1}_{[n] \setminus S})=0}} \bigvee_{i \in S} x_i.$$

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Remark 3.1. Now it becomes evident that any n-variable lattice polynomial function is a nondecreasing and continuous order invariant function in $\mathbb{R}^n$. We will see in Proposition 4.1 that the converse is also true. A nondecreasing (or continuous) order invariant function in $\mathbb{R}^n$ is a lattice polynomial function.

Denote by $p_\alpha^\vee$ (resp. $p_\beta^\wedge$) the lattice polynomial function disjunctively (resp. conjunctively) defined by a given set function α (resp. β) as defined in Proposition 3.1. Let $f\colon \{0,1\}^n \to \{0,1\}$ be a nonconstant and nondecreasing Boolean function. Then the lattice polynomial function $p_\alpha^\vee$, where $\alpha\colon 2^{[n]} \to \{0,1\}$ is defined by $\alpha(S) := f(\mathbf{1}_S)$ for all $S \subseteq [n]$, is an extension to $\mathbb{R}^n$ of f. Indeed, we immediately have $f(\mathbf{1}_S) = \alpha(S) = p_\alpha^\vee(\mathbf{1}_S)$ for all $S \subseteq [n]$. Consequently, any n-variable lattice polynomial function is an extension to $\mathbb{R}^n$ of a nonconstant and nondecreasing Boolean function.

Throughout we will denote by $\mathcal{C}_n$ the set of $\{0,1\}$-valued nonconstant and nondecreasing set functions on $[n]$. By definition, this set is equipollent to the set of n-variable lattice polynomial functions, as well as to the set of nonconstant and nondecreasing Boolean functions.[5]

Now, regard the lattice polynomial function p as a function from E^n to E. If E is a bounded lattice, we necessarily have $\bigvee_{x\in\varnothing} x := \inf E$ and $\bigwedge_{x\in\varnothing} x := \sup E$. Then, from (4), we immediately see that $p \equiv \inf E$ if $\alpha \equiv 0$, and $p \equiv \sup E$ if $\alpha \equiv 1$. Thus we can extend the definition of lattice polynomial functions by allowing the set function α to be constant.

Let $\mathcal{C}_n[E]$ denote the set $\mathcal{C}_n$ completed with the constant set function $\alpha \equiv 0$, if $\inf E \in E$, and the constant set function $\alpha \equiv 1$, if $\sup E \in E$. Evidently $\mathcal{C}_n[E]$ can be partially ordered by the standard partial order on set functions, namely $\alpha_1 \preccurlyeq \alpha_2$ if and only if $\alpha_1(S) \leqslant \alpha_2(S)$ for all $S \subseteq [n]$. We will refer to this partial order in the subsequent sections.

3.2 Special lattice polynomial functions

We now consider the important special case of symmetric lattice polynomial functions. Denote by $x_{(1)}, \ldots, x_{(n)}$ the *order statistics* resulting from reordering the variables $x_1, \ldots, x_n$ in nondecreasing order, that is, $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$. As Ovchinnikov [38, §7] observed, any order statistic is a symmetric lattice polynomial function. More precisely, for any $k \in [n]$, we have

$$x_{(k)} = \bigvee_{\substack{S \subseteq [n] \\ |S| = n-k+1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ |S| = k}} \bigvee_{i \in S} x_i.$$

Conversely, Marichal [28, §2] showed that any symmetric lattice polynomial function is an order statistic.

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[5]The problem of enumerating the number of distinct nondecreasing Boolean functions of n variables is known as the Dedekind's problem [20, 21] (Sloane's integer sequence A000372). Although Dedekind first considered this question in 1897, there is still no concise closed-form expression for this sequence.

Let us denote by $\mathrm{os}_k \colon \mathbb{R}^n \to \mathbb{R}$ the kth *order statistic function*, that is, $\mathrm{os}_k(\mathbf{x}) := x_{(k)}$. It is then easy to see that, for any $S \subseteq [n]$, we have $\mathrm{os}_k(\mathbf{1}_S) = 1$ if and only if $|S| \geqslant n - k + 1$ and, likewise, we have $\mathrm{os}_k(\mathbf{1}_{[n]\setminus S}) = 0$ if and only if $|S| \geqslant k$. Note that when n is odd, $n = 2k - 1$, the particular order statistic $x_{(k)}$ is the well-known *median* function

$$\mathrm{median}(x_1, \ldots, x_{2k-1}) := x_{(k)}.$$

Another special case of lattice polynomial functions is given by the projection functions, already introduced in §2. Recall that, for any $k \in [n]$, the *projection* function $\mathrm{P}_k \colon \mathbb{R}^n \to \mathbb{R}$ associated with the kth argument is defined by $\mathrm{P}_k(\mathbf{x}) := x_k$. The projection function P_k consists in projecting $\mathbf{x} \in \mathbb{R}^n$ onto the kth coordinate axis. As a particular aggregation function, it corresponds to a dictatorial aggregation.

3.3 Some aggregation properties

Lattice polynomial functions $p \colon E^n \to E$ are clearly continuous and nondecreasing functions. They are also order invariant functions in the sense that they fulfill equation (3) with arbitrary increasing bijections $\phi \colon E \to E$; see e.g. [39].

Lattice polynomial functions also fulfill other properties shared by many aggregation functions. We now examine three of them : internality, idempotency, and discretizability.

The most often encountered functions in aggregation theory are means or averaging functions, such as the weighted arithmetic means. Cauchy [8] considered in 1821 the mean of n independent variables $x_1, \ldots, x_n$ as a function $F(x_1, \ldots, x_n)$ which should be internal to the set of x_i values.

Definition 3.2. $F \colon E^n \to \mathbb{R}$ is an *internal* function if $\bigwedge_{i=1}^n x_i \leqslant F(\mathbf{x}) \leqslant \bigvee_{i=1}^n x_i$ for all $\mathbf{x} \in E^n$.

Such means satisfy trivially the property of *idempotency*, that is, if all x_i are identical, $F(\mathbf{x})$ restitutes the common value.

Definition 3.3. $F \colon E^n \to \mathbb{R}$ is an *idempotent* function if $F(x, \ldots, x) = x$ for all $x \in E$.

Conversely, we can easily see that any nondecreasing and idempotent function $F \colon E^n \to \mathbb{R}$ is internal.

As any lattice polynomial function is clearly internal, it is a mean in the Cauchy sense. Thus, the internality property makes it possible to define means even on ordinal scales (see, e.g. [38]). For example, as a particular lattice polynomial function, the classical median function (see §3.2), which gives the middle value of an odd-length sequence of ordered values, is a continuous, nondecreasing, and symmetric mean defined on ordinal scales. To give a second example,

consider the classical *mode* function, mode: $E^n \to E$, defined by[6]

$$\text{mode}(\mathbf{x}) := \underset{r \in E}{\text{argmax}} \sum_{i=1}^{n} \mathbf{1}_{\{0\}}(x_i - r), \tag{5}$$

where the function $\mathbf{1}_{\{0\}} \colon \mathbb{R} \to \mathbb{R}$ is defined by $\mathbf{1}_{\{0\}}(0) := 1$ and $\mathbf{1}_{\{0\}}(x) := 0$ for all $x \neq 0$ (in case of multiple values for argmax, take the smallest one). This function, which gives the (lowest) most repeated value of a sequence of values, is a symmetric mean defined on ordinal scales, and even on nominal scales.[7] However, since the mode function is not nondecreasing, it is not a lattice polynomial function.

We can also observe that any lattice polynomial function is *discretizable* in the sense that it always yields the value of one of its variables. This property was actually introduced in the framework of triangular norms (see e.g. [9, 11]) but is easily extended to any function as follows.

Definition 3.4. $F \colon E^n \to E$ is a *discretizable* function if $F(\mathbf{x}) \in \{x_1, \ldots, x_n\} \cup B[E]$ for all $\mathbf{x} \in E^n$.

We can readily prove [33] that $F \colon E^n \to E$ is a discretizable function if and only if, for any nonempty finite subset $C \subset E$ and any $\mathbf{x} \in (C \cup B[E])^n$, we have $F(\mathbf{x}) \in C \cup B[E]$. Thus, this property means that the domain and range of F can be restricted to a finite or countable chain.

Another interesting property is *self-duality* (for bounded E, see [13] and the references therein), which is fulfilled for example by the median and the mode functions.

Definition 3.5. Let $\psi \colon E \to E$ be a decreasing and involutive (i.e., $\psi \circ \psi = \text{id}$) bijection (hence necessarily $B[E]$ is not a singleton).

- The ψ-*dual* of a function $F \colon E^n \to E$ is the function $F_\psi \colon E^n \to E$, defined by
$$F_\psi(\mathbf{x}) := \psi^{-1}\big(F\big(\psi(x_1), \ldots, \psi(x_n)\big)\big).$$

- A function $F \colon E^n \to E$ is said to be ψ-*self-dual* if $F_\psi = F$.

- A function $F \colon E^n \to E$ is said to be *weakly self-dual* if it is ψ-self-dual for some decreasing and involutive bijection $\psi \colon E \to E$.

If E is bounded, then the only affine decreasing bijection from E onto itself is given by $\psi^d(x) := \inf E + \sup E - x$, and ψ^d-duality is then called *duality*, with notation $F^d := F_{\psi^d}$. A function $F \colon E^n \to E$ is said to be *self-dual* if $F^d = F$.

Remark 3.2.   (i) By definition, for any function $F \colon E^n \to E$ and any decreasing and involutive bijection $\psi \colon E \to E$, we have $(F_\psi)_\psi = F$.

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[6]As usual, argmax stands for the argument of the maximum, that is to say, the value of the given argument for which the value of the given expression attains its maximum value.

[7]The admissible transformations associated with a nominal scale are one-to-one transformations (injections) of E into itself; see [44, p. 66].

(ii) We note that ψ-duality is an example of ψ-*conjugacy* [23, Chapter 8], whose definition is the same except that it does not require ψ to be decreasing nor involutive. We also observe that the classic notion of duality in ordered sets concerns the converse order relations; see [6, p. 3].

Assume that $B[E]$ is not a singleton and let $\alpha \in \mathcal{C}_n$. We can straightforwardly show that the lattice polynomial function $p_\alpha^\vee \colon E^n \to E$ is weakly self-dual if and only if $\alpha^d = \alpha$, where $\alpha^d \in \mathcal{C}_n$ is the dual of α, defined by $\alpha^d(S) := 1 - \alpha([n] \setminus S)$.

The special case of order statistics is dealt with in the next immediate result (see [28, §5]), which characterizes the median as the only weakly self-dual order statistic.

Proposition 3.2. *Assume that n is odd and that $B[E]$ is not a singleton. The kth order statistic function $\mathrm{os}_k \colon E^n \to E$ is weakly self-dual if and only if $n = 2k - 1$. In this case os_k is the median function.*

4 Order invariant functions

The first meaningful aggregation functions we consider are the *order invariant functions*, which were first investigated (as ordinally stable functions) by Marichal and Roubens [32], and then by many other authors; see [5, 10, 18, 26, 28, 29, 30, 31, 33, 34, 38, 39, 41, 42].

4.1 Definition and first results

Let $x_1, \ldots, x_n$ be independent variables defining the same ordinal scale, with domain E, and suppose that, when aggregating these variables by a function $F \colon E^n \to E$, we require that the dependent variable

$$x_{n+1} = F(x_1, \ldots, x_n) \tag{6}$$

defines the same scale. As equation (6) should represent a meaningful relation between the independent and dependent variables, the aggregation function F should be invariant under actions from $\Phi[E]$. That is, $\phi(x_{n+1}) = F(\phi(x_1), \ldots, \phi(x_n))$ for all $\phi \in \Phi[E]$. Thus, the order invariance property is defined as follows.

Definition 4.1. $F \colon E^n \to E$ is said to be an *order invariant* function if

$$F\big(\phi(x_1), \ldots, \phi(x_n)\big) = \phi\big(F(x_1, \ldots, x_n)\big)$$

for all $\mathbf{x} \in E^n$ and all $\phi \in \Phi[E]$.

The following result (see Propositions 4.5 and 4.6 below) shows that the lattice polynomial functions are the most prominent order invariant functions (however see Theorem 4.1 for a full description of order invariant functions).

Proposition 4.1. *Assume that E is open and consider a function $F \colon E^n \to E$. Then the following three assertions are equivalent:*

(i) *F is a nondecreasing order invariant function.*

(ii) *F is a continuous order invariant function.*

(iii) *F is a lattice polynomial function.*

Proposition 4.1 poses the interesting question of how we can interpret the continuity property for order invariant functions. Let $\Phi'[E]$ be the superset of $\Phi[E]$ consisting of the continuous nondecreasing surjections $\phi\colon E \to E$. The following result [30, §5.2], inspired from [7, Proposition 2], shows that the conjunction of continuity and order invariance is equivalent to requiring that the admissible transformations belong to $\Phi'[E]$.

Proposition 4.2. *$F\colon E^n \to E$ is a continuous order invariant function if and only if $F(\phi(x_1), \ldots, \phi(x_n)) = \phi(F(x_1, \ldots, x_n))$ for all $\mathbf{x} \in E^n$ and all $\phi \in \Phi'[E]$.*

Let $\Phi''[E]$ be the superset of $\Phi[E]$ consisting of all the monotone bijections of E onto itself (assuming that $B[E]$ is not a singleton). It is clear (for bounded E, see e.g. [34, §3]) that the conjunction of weak self-duality (cf. Definition 3.5) and order invariance is equivalent to requiring that the admissible transformations belong to $\Phi''[E]$. The independent and dependent variables then define a *nominal* scale.

Proposition 4.3. *Assume that $B[E]$ is not a singleton. $F\colon E^n \to E$ is a weakly self-dual order invariant function if and only if $F(\phi(x_1), \ldots, \phi(x_n)) = \phi(F(x_1, \ldots, x_n))$ for all $\mathbf{x} \in E^n$ and all $\phi \in \Phi''[E]$.*

4.2   General descriptions

When E is open we have the following description (see [39, Theorem 5.1]).

Proposition 4.4. *Assume that E is open. Then $F\colon E^n \to E$ is an order invariant function if and only if there exists a mapping $\xi\colon \mathcal{I}_n[E] \to [n]$ such that $F|_I = \mathrm{P}_{\xi(I)}|_I$ for all $I \in \mathcal{I}_n[E]$.*

This result shows that, when E is open, the restriction of F to any minimal order invariant subset is a projection function onto one coordinate. That is, for any $I \in \mathcal{I}_n[E]$, there exists $k_I \in [n]$ such that $F|_I = \mathrm{P}_{k_I}|_I$. Clearly, such a function is internal and hence idempotent.

As an example, any nonconstant lattice polynomial function is a continuous, nondecreasing, idempotent, and order invariant function. On the other hand, the mode function (5) is an idempotent and order invariant function that is neither continuous nor nondecreasing.

When E is not open, the restriction of F to any minimal order invariant subset reduces to a constant function or a projection function onto one coordinate (see [31, 34]).

Theorem 4.1. *$F\colon E^n \to E$ is an order invariant function if and only if there exists a mapping $\xi\colon \mathcal{I}_n[E] \to [n]$ such that, for any $I \in \mathcal{I}_n[E]$,*

- *either $F|_I \equiv c \in B[E]$ (assuming $B[E] \neq \varnothing$),*

- *or $F|_I = \mathrm{P}_{\xi(I)}|_I$.*

Remark 4.1. It was proved in [28, Proposition 3.1] (see [18, 38, 39] for preliminary results), that any order invariant function is discretizable, and hence it is internal whenever E is open; it is clear that this follows from Theorem 4.1. For instance, the mode function (5) is order invariant and hence discretizable. The converse is not true. For example, the function $F\colon (0,1)^2 \to (0,1)$ defined by

$$F(x_1, x_2) := \begin{cases} x_1, & \text{if } x_1 + x_2 < 1, \\ x_2, & \text{otherwise,} \end{cases}$$

is discretizable but not order invariant.

When an order invariant function is idempotent, clearly it must be a projection function on the open diagonal of E^n, also on $I = \{(\inf E, \ldots, \inf E)\}$ (if $\inf E \in E$), and on $I = \{(\sup E, \ldots, \sup E)\}$ (if $\sup E \in E$).

4.3 The nondecreasing case

We now present descriptions of order invariant functions which are nondecreasing. The following result (see [28, Corollary 4.4]) shows that, when E is open, the family of nondecreasing order invariant functions in E^n is identical to that of lattice polynomial functions in E^n.

Proposition 4.5. *Assume that E is open. Then $F\colon E^n \to E$ is a nondecreasing order invariant function if and only if it is a lattice polynomial function.*

Corollary 4.1. *Assume that E is open. Then $F\colon E^n \to E$ is a symmetric, nondecreasing, and order invariant function if and only if it is an order statistic function.*

Combining Proposition 3.2 with Corollary 4.1 immediately yields the following axiomatization of the median function.

Corollary 4.2. *Assume that n is odd and that E is open. Then $F\colon E^n \to E$ is a symmetric, weakly self-dual, nondecreasing, and order invariant function if and only if it is the median function.*

A complete description of nondecreasing order invariant functions in E^n, with open or non-open interval E, is given in the following theorem (see [31, 34]). It shows that discontinuities of F may occur only on the border of E^n. Recall that the lattice polynomial function in E^n disjunctively defined by $\alpha \in \mathcal{C}_n[E]$ is denoted $p_\alpha^\vee$ (see §3.1).

Theorem 4.2. *$F\colon E^n \to E$ is a nondecreasing order invariant function if and only if there exists a nondecreasing mapping $\xi\colon \mathcal{I}_n^*[E] \to \mathcal{C}_n[E]$ such that $F|_I = p_{\xi(I)}^\vee|_I$ for all $I \in \mathcal{I}_n^*[E]$.*

Example 4.1. Consider the semiopen interval $E := [a, b)$.[8] The function $F\colon [a, b)^3 \to [a, b)$ defined by

$$F(x_1, x_2, x_3) := \begin{cases} a, & \text{if } x_1 = a, \\ x_3, & \text{if } x_1 \neq a \text{ and } x_2 = a, \\ x_1 \vee x_2 \vee x_3, & \text{otherwise,} \end{cases}$$

is a nondecreasing order invariant function in $[a, b)^3$.

Corollary 4.3. *$F\colon E^n \to E$ is a nondecreasing, idempotent, and order invariant function if and only if there exists a nondecreasing mapping $\xi\colon \mathcal{I}_n^*[E] \to \mathcal{C}_n[E]$, where $\xi[(E^\circ)^n]$ is nonconstant, such that $F|_I = p_{\xi(I)}^\vee|_I$ for all $I \in \mathcal{I}_n^*[E]$.*

4.4 The continuous case

We now consider the family of continuous order invariant functions. It was shown in [28, Corollary 4.2] that, when E is open, this family is identical to the family of lattice polynomial functions in E^n; see also [26, §3.4.2].

Proposition 4.6. *Assume that E is open. Then $F\colon E^n \to E$ is a continuous order invariant function if and only if it is a lattice polynomial function.*

Remark 4.2. Note that this result was independently stated and proved earlier by Ovchinnikov [40, Theorem 5.3] in the more general setting where the range of variables is a doubly homogeneous simple order, that is, a simple order X satisfying the following property:

> For any $x_1, x_2, y_1, y_2 \in X$, with $x_1 < x_2$ and $y_1 < y_2$, there is an automorphism $\phi\colon X \to X$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$.

As any open interval E of the real line is clearly a doubly homogeneous simple order, Ovchinnikov's result encompasses that of Proposition 4.6.[9]

A complete description of continuous order invariant function in E^n was stated in [28, Corollary 4.3] as follows (see also [31]).

Theorem 4.3. *$F\colon E^n \to E$ is a continuous order invariant function if and only if there exists $\alpha \in \mathcal{C}_n[E]$ such that $F = p_\alpha^\vee$.*

Theorem 4.3 actually says that a continuous order invariant function $F\colon E^n \to E$ is either the constant function $F \equiv \inf E$ if $\inf E \in E$, or the constant function $F \equiv \sup E$ if $\sup E \in E$, or any lattice polynomial function in E^n (any order statistic function in E^n if F is symmetric).

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[8]Here the poset $\mathcal{I}_n^*[E]$ contains 8 elements (a point, three open line segments, three open square facets, and an open cube).

[9]Note that the extension of this result to the (infinite) case of functional operators was described in [41]; see also [42].

Remark 4.3. From Theorems 4.1 and 4.3 it follows that a function $F\colon E^n \to E$ is a lattice polynomial function if and only if its restriction to each closed simplex of the standard triangulation of E^n is a projection function onto one coordinate (see also [28, Proposition 2.1]).

Corollary 4.4. *$F\colon E^n \to E$ is a continuous, idempotent, and order invariant function if and only if it is a lattice polynomial function.*

Corollary 4.5. *$F\colon E^n \to E$ is a symmetric, continuous, idempotent, and order invariant function if and only if it is an order statistic function.*

Corollary 4.6. *Assume that n is odd and that $B[E]$ is not a singleton. Then $F\colon E^n \to E$ is a symmetric, weakly self-dual, continuous, idempotent, and order invariant function if and only if it is the median function.*

Remark 4.4. By combining Proposition 3.2 with Theorem 4.3, we immediately see that idempotency is not necessary in Corollary 4.6. Indeed, a weak self-dual lattice polynomial function cannot be constant and hence it is necessarily idempotent.

5 Comparison meaningful functions on a single ordinal scale

We now present the class of *comparison meaningful functions on a single ordinal scale*. These functions were introduced first by Orlov [36] and then investigated by many other authors; see [18, 26, 27, 28, 29, 30, 31, 38, 46].

5.1 Definition and first results

Let $x_1, \ldots, x_n$ be independent variables defining the same ordinal scale, with domain E, and suppose that, when aggregating these variables by a function $F\colon E^n \to \mathbb{R}$, we require that the dependent variable $x_{n+1} = F(x_1, \ldots, x_n)$ defines an ordinal scale, with an arbitrary domain in $\mathbb{R}$. According to Luce's principle [24], any admissible transformation of the independent variables must lead to an admissible transformation of the dependent variable. This condition can be formulated as follows.

Definition 5.1. *$F\colon E^n \to \mathbb{R}$ is said to be a comparison meaningful function on a single ordinal scale if, for any $\phi \in \Phi_n[E]$, there is a strictly increasing mapping $\psi_\phi\colon \mathrm{ran}(F) \to \mathrm{ran}(F)$ such that $F[\phi(\mathbf{x})] = \psi_\phi[F(\mathbf{x})]$ for all $\mathbf{x} \in E^n$.*

Comparison meaningful functions on a single ordinal scale were first introduced by Orlov [36] as those functions preserving the comparison of aggregated values when changing the scale defined by the independent variables.[10] We paraphrase from Orlov:

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[10]A general study on meaningfulness of ordinal comparisons can be found in [43].

When one compares two sets of objects according to a criterion, it is sometimes required to evaluate each object on the same ordinal scale (e.g., by means of measurement or expert estimate). The aggregated values of the evaluations corresponding to each set of objects are computed by a certain aggregation function, and then compared together. It is natural to require that the inferences made from this comparison are meaningful, that is, depend only on the initial information, but not on the scale used.[11]

The equivalence between Definition 5.1 and Orlov's definition can be formulated mathematically as follows (see [30]).

Proposition 5.1. $F\colon E^n \to \mathbb{R}$ *is a comparison meaningful function on a single ordinal scale if and only if*

$$F(\mathbf{x}) \left\{ {< \atop =} \right\} F(\mathbf{x}') \quad \Rightarrow \quad F\big(\phi(\mathbf{x})\big) \left\{ {< \atop =} \right\} F\big(\phi(\mathbf{x}')\big)$$

for all $\mathbf{x}, \mathbf{x}' \in E^n$ *and all* $\phi \in \Phi_n[E]$.[12]

Remark 5.1. Although the condition in Proposition 5.1 is natural and even mandatory to aggregate ordinal values, it severely restricts the allowable operations for defining a meaningful aggregation function. For example, the comparison of two arithmetic means is meaningless on an ordinal scale. Indeed, considering the pairs of values $(3, 5)$ and $(1, 8)$, we have $\frac{1}{2}(3+5) < \frac{1}{2}(1+8)$ and, using any admissible transformation ϕ such that $\phi(1) = 1$, $\phi(3) = 4$, $\phi(5) = 7$, and $\phi(8) = 8$, we have $\frac{1}{2}(\phi(3) + \phi(5)) > \frac{1}{2}(\phi(1) + \phi(8))$.

Order invariant functions and comparison meaningful functions on a single ordinal scale can actually be related through the idempotency property. Indeed, when a comparison meaningful function on a single ordinal scale is idempotent then the output scale must coincide with the input scale. This result is stated in the next proposition (see [28, Proposition 3.3] and preliminary work in [18, 38]).

Proposition 5.2. *Consider a function* $F\colon E^n \to E$.

- *If F is idempotent and comparison meaningful on a single ordinal scale then it is order invariant.*

- *If F is order invariant then it is comparison meaningful on a single ordinal scale.*

- *If E is open then F is idempotent and comparison meaningful on a single ordinal scale if and only if it is order invariant.*

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[11]More generally, a statement using scales of measurement is said to be *meaningful* if its truth or falsity is invariant when every scale is replaced by another acceptable version of it; see [44, p. 59].

[12]Equivalently, $F\colon E^n \to \mathbb{R}$ is a comparison meaningful function on a single ordinal scale if and only if

$$F(\mathbf{x}) \leqslant F(\mathbf{x}') \quad \Leftrightarrow \quad F\big(\phi(\mathbf{x})\big) \leqslant F\big(\phi(\mathbf{x}')\big)$$

for all $\mathbf{x}, \mathbf{x}' \in E^n$ and all $\phi \in \Phi_n[E]$.

Just as for order invariant functions, continuity of comparison meaningful functions on a single ordinal scale can be interpreted by means of the set $\Phi'[E]$ of continuous nondecreasing surjections from E onto E; see [30, §5.2]. Denote by $\Phi'_n[E]$ the diagonal restriction of $\Phi'[E]^n$ (see §2).

Proposition 5.3. $F \colon E^n \to \mathbb{R}$ *is a continuous and comparison meaningful function on a single ordinal scale if and only if, for any $\phi \in \Phi'_n[E]$, there is a continuous and nondecreasing mapping $\psi_\phi \colon \mathrm{ran}(F) \to \mathrm{ran}(F)$ such that $F(\phi(\mathbf{x})) = \psi_\phi(F(\mathbf{x}))$ for all $\mathbf{x} \in E^n$.*

5.2 General descriptions

The class of comparison meaningful functions on a single ordinal scale can be described as follows (see [31, Theorem 3.1]).

Theorem 5.1. $F \colon E^n \to \mathbb{R}$ *is a comparison meaningful function on a single ordinal scale if and only if, for any $I \in \mathcal{I}_n[E]$, there exist an index $k_I \in [n]$ and a strictly monotonic or constant function $g_I \colon \mathrm{P}_{k_I}(I) \to \mathbb{R}$ such that $F|_I = (g_I \circ \mathrm{P}_{k_I})|_I$, where, for any $I, I' \in \mathcal{I}_n[E]$,*

- *either $g_I = g_{I'}$,*

- *or $\mathrm{ran}(g_I) = \mathrm{ran}(g_{I'})$ is a singleton,*

- *or $\mathrm{ran}(g_I) < \mathrm{ran}(g_{I'})$,*

- *or $\mathrm{ran}(g_I) > \mathrm{ran}(g_{I'})$.*[13]

Thus, a comparison meaningful function on a single ordinal scale reduces, on each minimal order invariant subset of E^n, to a constant or a transformed projection function onto one coordinate.

Example 5.1. We have seen in Example 2.1 that there are eleven minimal order invariant subsets in the unit square $[0,1]^2$, namely

- $I_1 := \{(0,0)\}$, $I_2 := \{(1,0)\}$, $I_3 := \{(1,1)\}$, $I_4 := \{(0,1)\}$,

- $I_5 := (0,1) \times \{0\}$, $I_6 := \{1\} \times (0,1)$, $I_7 := (0,1) \times \{1\}$, $I_8 := \{0\} \times (0,1)$,

- $I_9 := \{(x_1, x_2) \mid 0 < x_1 = x_2 < 1\}$, $I_{10} := \{(x_1, x_2) \mid 0 < x_1 < x_2 < 1\}$, $I_{11} := \{(x_1, x_2) \mid 0 < x_2 < x_1 < 1\}$.

Let $k_{I_j} := 1$ and $g_{I_j}(x) := 1 - x$ if $j \in \{1,2,3,5,6,9,11\}$, and $k_{I_j} := 2$ and $g_{I_j}(x) := 2x - 3$ if $j \in \{4,7,8,10\}$, where always $x \in \mathrm{P}_{k_{I_j}}(I_j)$. Then the corresponding comparison meaningful function $F \colon [0,1]^2 \to \mathbb{R}$ is given by

$$F(x_1, x_2) := \begin{cases} 1 - x_1, & \text{if } x_1 \geqslant x_2, \\ 2x_2 - 3, & \text{if } x_1 < x_2. \end{cases}$$

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[13]Note that $\mathrm{ran}(g_I) < \mathrm{ran}(g_{I'})$ means that for all $r \in \mathrm{ran}(g_I)$ and all $r' \in \mathrm{ran}(g_{I'})$, we have $r < r'$.

When a comparison meaningful function on a single ordinal scale is idempotent, it must satisfy $g_I(x) = x$, for all $x \in \mathrm{P}_{k_I}(I)$, whenever either I is the open diagonal of E^n, or $I = \{(\inf E, \ldots, \inf E)\}$ (if $\inf E \in E$), or $I = \{(\sup E, \ldots, \sup E)\}$ (if $\sup E \in E$).

5.3 The nondecreasing case

The following result [31] yields, when E is open, a description of all nondecreasing comparison meaningful functions $F: E^n \to \mathbb{R}$ on a single ordinal scale.

Proposition 5.4. *Assume that E is open. Then $F: E^n \to \mathbb{R}$ is a nondecreasing comparison meaningful function on a single ordinal scale if and only if there exist $\alpha \in \mathcal{C}_n$ and a strictly increasing or constant function $g: E \to \mathbb{R}$ such that $F = g \circ p_\alpha^\vee$.*

As we can see, all the functions described in Proposition 5.4 are continuous up to possible discontinuities of the function g.

The following corollaries [28, Theorem 4.4] (see [29, Theorem 3.1] for preliminary results) immediately follow from Proposition 5.4.

Corollary 5.1. *Assume that E is open. Then $F: E^n \to \mathbb{R}$ is a nondecreasing, idempotent, and comparison meaningful function on a single ordinal scale if and only if it is a lattice polynomial function.*

Corollary 5.2. *Assume that E is open. Then $F: E^n \to \mathbb{R}$ is a symmetric, nondecreasing, idempotent, and comparison meaningful function on a single ordinal scale if and only if it is an order statistic function.*

Corollary 5.3. *Assume that n is odd and that E is open. Then $F: E^n \to \mathbb{R}$ is a symmetric, weakly self-dual, nondecreasing, idempotent, and comparison meaningful function on a single ordinal scale if and only if it is the median function.*

A complete description of nondecreasing comparison meaningful functions $F: E^n \to \mathbb{R}$ on a single ordinal scale is given in the next theorem [31, Corollary 4.1]. Let $\mathcal{G}(E)$ be the set of all strictly increasing or constant real functions g defined either on E°, or on the singleton $\{\inf E\} \cap E$, or on the singleton $\{\sup E\} \cap E$ (if these singletons exist). This set is partially ordered as follows: $g_1 \preccurlyeq g_2$ if either $g_1 = g_2$, or $\mathrm{ran}(g_1) = \mathrm{ran}(g_2)$ is a singleton, or $\mathrm{ran}(g_1) < \mathrm{ran}(g_2)$.

Theorem 5.2. *$F: E^n \to \mathbb{R}$ is a nondecreasing comparison meaningful function on a single ordinal scale if and only if there exist nondecreasing mappings $\gamma: \mathcal{I}_n^*[E] \to \mathcal{G}(E)$ and $\xi: \mathcal{I}_n^*[E] \to \mathcal{C}_n[E]$ such that $F|_I = (\gamma(I) \circ p_{\xi(I)}^\vee)|_I$ for all $I \in \mathcal{I}_n^*[E]$.*

If furthermore F is idempotent, then by nondecreasing monotonicity, we have $\mathrm{ran}(F) = E$ and, by Proposition 5.2, F is order invariant. Hence we have the following corollary.

Corollary 5.4. $F \colon E^n \to \mathbb{R}$ *is a nondecreasing, idempotent, and comparison meaningful function on a single ordinal scale if and only if there exists a nondecreasing mapping $\xi \colon \mathcal{I}_n^*[E] \to \mathcal{C}_n[E]$, where $\xi[(E^\circ)^n]$ is nonconstant, such that $F|_I = p_{\xi(I)}^\vee|_I$ for all $I \in \mathcal{I}_n^*[E]$.*

5.4 The continuous case

Based on a preliminary result [29, §4] (see also [26, §3.4.2]), a full description of continuous comparison meaningful functions on a single ordinal scale was given in [28, Theorem 4.2] as follows.

Theorem 5.3. $F \colon E^n \to \mathbb{R}$ *is a continuous comparison meaningful function on a single ordinal scale if and only if there exist $\alpha \in \mathcal{C}_n$ and a continuous and strictly monotonic or constant function $g \colon E \to \mathbb{R}$ such that $F = g \circ p_\alpha^\vee$.*

Corollary 5.5. $F \colon E^n \to \mathbb{R}$ *is a continuous, idempotent, and comparison meaningful function on a single ordinal scale if and only if it is a lattice polynomial function.*

Remark 5.2. The result in Corollary 5.5 was stated and proved first in social choice theory by Yanovskaya [46, Theorem 1] when $E = \mathbb{R}$.

Corollary 5.6. $F \colon E^n \to \mathbb{R}$ *is a symmetric, continuous, and comparison meaningful function on a single ordinal scale if and only if there exist $k \in [n]$ and a continuous strictly monotonic or constant function $g \colon E \to \mathbb{R}$ such that $F = g \circ \mathrm{os}_k$.*

Corollary 5.7. $F \colon E^n \to \mathbb{R}$ *is a symmetric, continuous, idempotent, and comparison meaningful function on a single ordinal scale if and only if it is an order statistic function.*

Remark 5.3. A slightly stronger version of the result in Corollary 5.7, consisting in replacing idempotency with internality, was actually proved first by Orlov [36] in $\mathbb{R}^n$, then by Marichal and Roubens [32, Theorem 1] in E^n (see also [26, Theorem 3.4.13]), and finally by Ovchinnikov [38, Theorem 4.3] in the more general framework where the range of variables is a simple order X whose open intervals are homogeneous and nonempty (see also [40, §6]).

Corollary 5.8. *Assume that n is odd and that $B[E]$ is not a singleton. Then $F \colon E^n \to \mathbb{R}$ is a symmetric, weakly self-dual, continuous, idempotent, and comparison meaningful function on a single ordinal scale if and only if it is the median function.*

6 Comparison meaningful functions on independent ordinal scales

In this section we present the class of *comparison meaningful functions on independent ordinal scales*, which were introduced by Aczél and Roberts [3,

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Case #21] and studied by Kim [19] (see preliminary work in Osborne [37]) and then investigated by some other authors; see [28, 29, 30, 31].

6.1 Definition and first results

Let $x_1, \ldots, x_n$ be independent variables defining independent ordinal scales, with a common domain E, and suppose that, when aggregating these variables by a function $F \colon E^n \to \mathbb{R}$, we require that the dependent variable $x_{n+1} = F(x_1, \ldots, x_n)$ defines an ordinal scale, with an arbitrary domain in $\mathbb{R}$. This condition can be formulated as follows.

Definition 6.1. $F \colon E^n \to \mathbb{R}$ is said to be a *comparison meaningful function on independent ordinal scales* if, for any $\boldsymbol{\phi} \in \Phi[E]^n$, there is a strictly increasing mapping $\psi_{\boldsymbol{\phi}} \colon \mathrm{ran}(F) \to \mathrm{ran}(F)$ such that $F\big(\boldsymbol{\phi}(\mathbf{x})\big) = \psi_{\boldsymbol{\phi}}\big(F(\mathbf{x})\big)$ for all $\mathbf{x} \in E^n$.

Comparison meaningful functions on independent ordinal scales can also be defined as those functions preserving the comparison of aggregated values when changing the scales defined by the independent variables (see [30]).

Proposition 6.1. $F \colon E^n \to \mathbb{R}$ *is a comparison meaningful function on independent ordinal scales if and only if*

$$F(\mathbf{x}) \left\{ \begin{matrix} < \\ = \end{matrix} \right\} F(\mathbf{x}') \quad \Rightarrow \quad F\big(\boldsymbol{\phi}(\mathbf{x})\big) \left\{ \begin{matrix} < \\ = \end{matrix} \right\} F\big(\boldsymbol{\phi}(\mathbf{x}')\big)$$

for all $\mathbf{x}, \mathbf{x}' \in E^n$ *and all* $\boldsymbol{\phi} \in \Phi[E]^n$.[14]

Comparison meaningfulness on independent ordinal scales is a very strong condition, much stronger than comparison meaningfulness on a single ordinal scale. For example, it was proved [28, Lemma 5.2] that this condition reduces any lattice polynomial function to a projection function onto one coordinate.

Regarding continuity of comparison meaningful function on independent ordinal scales, it can be interpreted in the same way as for comparison meaningful function on a single ordinal scale; see [30, §5.2]. Consider again the set $\Phi'[E]$ of continuous nondecreasing surjections from E onto E.

Proposition 6.2. $F \colon E^n \to \mathbb{R}$ *is a continuous and comparison meaningful function on independent ordinal scales if and only if, for any* $\boldsymbol{\phi} \in \Phi'[E]^n$, *there is a continuous and nondecreasing mapping* $\psi_{\boldsymbol{\phi}} \colon \mathrm{ran}(F) \to \mathrm{ran}(F)$ *such that* $F\big(\boldsymbol{\phi}(\mathbf{x})\big) = \psi_{\boldsymbol{\phi}}\big(F(\mathbf{x})\big)$ *for all* $\mathbf{x} \in E^n$.

6.2 General descriptions

The description of comparison meaningful functions on independent ordinal scales is very similar to that of comparison meaningful functions on a single ordinal scale. The result can be formulated as follows [31, Corollary 3.1].

---

[14]Equivalently, $F \colon E^n \to \mathbb{R}$ is a comparison meaningful function on independent ordinal scales if and only if

$$F(\mathbf{x}) \leqslant F(\mathbf{x}') \quad \Leftrightarrow \quad F\big(\boldsymbol{\phi}(\mathbf{x})\big) \leqslant F\big(\boldsymbol{\phi}(\mathbf{x}')\big)$$

for all $\mathbf{x}, \mathbf{x}' \in E^n$ and all $\boldsymbol{\phi} \in \Phi[E]^n$.

Theorem 6.1. *$F\colon E^n \to \mathbb{R}$ is a comparison meaningful function on independent ordinal scales if and only if, for any $I \in \mathcal{I}_n^*[E]$, there exist an index $k_I \in [n]$ and a strictly monotonic or constant function $g_I\colon \mathrm{P}_{k_I}(I) \to \mathbb{R}$ such that $F|_I = (g_I \circ \mathrm{P}_{k_I})|_I$, where, for any $I, I' \in \mathcal{I}_n^*[E]$,*

- *either $g_I = g_{I'}$,*

- *or $\mathrm{ran}(g_I) = \mathrm{ran}(g_{I'})$ is a singleton,*

- *or $\mathrm{ran}(g_I) < \mathrm{ran}(g_{I'})$,*

- *or $\mathrm{ran}(g_I) > \mathrm{ran}(g_{I'})$.*

Thus, a comparison meaningful function on independent ordinal scales reduces, on each minimal strongly order invariant subset of E^n, to a constant or a transformed projection function onto one coordinate.

When a comparison meaningful function on independent ordinal scales is idempotent, it must satisfy $g_I(x) = x$, for all $x \in \mathrm{P}_{k_I}(I)$, whenever either $I = (E^\circ)^n$, or $I = \{(\inf E, \ldots, \inf E)\}$ (if $\inf E \in E$), or $I = \{(\sup E, \ldots, \sup E)\}$ (if $\sup E \in E$).

When E is open, the family $\mathcal{I}_n^*[E]$ reduces to $\{E^\circ\}$, thus considerably simplifying Theorem 6.1 as follows.

Proposition 6.3. *Assume that E is open. Then $F\colon E^n \to \mathbb{R}$ is a comparison meaningful function on independent ordinal scales if and only if there exist $k \in [n]$ and a strictly monotonic or constant function $g\colon E \to \mathbb{R}$ such that $F = g \circ \mathrm{P}_k$.*

Corollary 6.1. *Assume that E is open. Then $F\colon E^n \to \mathbb{R}$ is an idempotent and comparison meaningful function on independent ordinal scales if and only if it is a projection function.*

It follows from Proposition 6.3 that, when E is open and $n \geqslant 2$, any symmetric and comparison meaningful function on independent ordinal scales is necessarily a constant function. In this case, it cannot be idempotent.

6.3 The nondecreasing case

Starting from Proposition 6.3 we deduce immediately the following characterizations.

Proposition 6.4. *Assume that E is open. Then $F\colon E^n \to \mathbb{R}$ is a nondecreasing comparison meaningful function on independent ordinal scales if and only if there exist $k \in [n]$ and a strictly increasing or constant function $g\colon E \to \mathbb{R}$ such that $F = g \circ \mathrm{P}_k$.*

When E is not open, we have the following (see [31, Corollary 4.2]).

Theorem 6.2. *$F\colon E^n \to \mathbb{R}$ is a nondecreasing comparison meaningful function on independent ordinal scales if and only if there exists a mapping $\xi\colon \mathcal{I}_n^*[E] \to [n]$ and a nondecreasing mapping $\gamma\colon \mathcal{I}_n^*[E] \to \mathcal{G}(E)$ such that $F|_I = (\gamma(I) \circ \mathrm{P}_{\xi(I)})|_I$ for all $I \in \mathcal{I}_n^*[E]$, where if $\gamma(I) = \gamma(I')$ then also $\xi(I) = \xi(I')$ (unless $\gamma(I) = \gamma(I')$ is constant).*

6.4 The continuous case

As we already mentioned in §6.1, comparison meaningfulness on independent ordinal scales reduces any lattice polynomial function to a projection function onto one coordinate. From this result we deduce immediately the following characterizations; see [28, §5].

Theorem 6.3. $F\colon E^n \to \mathbb{R}$ *is a continuous and comparison meaningful function on independent ordinal scales if and only if there exist $k \in [n]$ and a continuous and strictly monotonic or constant function $g\colon E \to \mathbb{R}$ such that $F = g \circ \mathrm{P}_k$.*

Corollary 6.2. $F\colon E^n \to \mathbb{R}$ *is a continuous, idempotent, and comparison meaningful function on independent ordinal scales if and only if it is a projection function.*

Remark 6.1. The result in Theorem 6.3 was proved first by Kim [19, Corollary 1.2] in $\mathbb{R}^n$ (see Osborne [37] for preliminary results).

It follows from Theorem 6.3 that, if $n \geqslant 2$, any symmetric, continuous, and comparison meaningful function on independent ordinal scales is necessarily a constant function.

7 Aggregation on finite chains by chain independent functions

In this final section, mainly based on a paper by the authors [30], we give interpretations of order invariance and comparison meaningfulness properties in the setting of aggregation on finite chains (i.e., totally ordered finite sets). These interpretations show that the order invariant functions and comparison meaningful functions always have isomorphic discrete representatives defined on finite chains. These discrete functions do not depend on the chains on which they are defined.

7.1 Introduction

Let A be a set of *alternatives* (objects, individuals, etc.) and consider an open real interval E, possibly unbounded.[15] In representational measurement theory [44, 45], a *scale of measurement* can be seen as a mapping $h\colon A \to E$ that assigns a real number to each element of A according to some attribute or criterion.[16] As already mentioned in the introduction, such a scale is an ordinal scale if any other acceptable version of it is of the form $\phi \circ h$ for some strictly increasing function $\phi\colon E \to E$.

An ordinal scale is finite if $\mathrm{ran}(h)$ is a finite subset of E, that is of the form

$$\mathrm{ran}(h) = \{e_1 < e_2 < \cdots < e_k\},$$

---

[15] Without loss of generality, we can assume that $E = (0,1)$ or $E = \mathbb{R}$.

[16] A criterion is an attribute defined in a preference-ordered domain.

where the values $e_1, e_2, \ldots, e_k$ represent the possible rating benchmarks defined along some ordinal criterion. We shall assume throughout that $|\mathrm{ran}(h)| = k \geqslant 2$.

Since the values $e_1, e_2, \ldots, e_k$ of the scale are defined up to order, that is, within a strictly increasing function $\phi \colon E \to E$, we can simply replace $\mathrm{ran}(h)$ with a finite chain $(S, \preccurlyeq)$ of k elements, that is,

$$S = \{s_1 \prec s_2 \prec \cdots \prec s_k\},$$

where $\preccurlyeq$ represents a total order on S and $\prec$ represents its asymmetric part. In this representation we denote by $s_* := s_1$ (resp. $s^* := s_k$) the bottom element (resp. top element) of the chain.

Example 7.1. Consider the problem of evaluating a commodity by a consumer according to a given ordinal criterion. Typically this evaluation is done by rating the product on a finite ordinal scale. For instance we could consider the following rating benchmarks:

$$1 = \text{Bad}, \ 2 = \text{Weak}, \ 3 = \text{Fair}, \ 4 = \text{Good}, \ 5 = \text{Excellent}.$$

Since the scale values are determined only up to order, this scale can be replaced with a finite chain $S = \{\text{B} \prec \text{W} \prec \text{F} \prec \text{G} \prec \text{E}\}$ whose elements B, W, F, G, E refer to the following linguistic terms: *bad, weak, fair, good, excellent.*

It is well known (see [22, Chapter 1]) that the total order $\preccurlyeq$ defined on S can always be numerically represented in E by means of an isomorphism (a strictly increasing function) $f \colon S \to E$ such that

$$s_i \preccurlyeq s_j \quad \Leftrightarrow \quad f(s_i) \leqslant f(s_j) \qquad (s_i, s_j \in S).$$

Moreover, just as for the mapping h, the isomorphism f is defined up to a strictly increasing function $\phi \colon E \to E$. That is, with f all isomorphisms $f' = \phi \circ f$ (and only these) represent the same order on S.

By choosing f so that $f(s_i) = e_i$ for $i = 1, \ldots, k$, we immediately see that the elements of A can be ordinally evaluated not only by means of the numerical mapping $h \colon A \to \mathrm{ran}(h)$ but also by the non-numerical mapping $h_S \colon A \to S$, defined by $h_S := f^{-1} \circ h$. The following diagram illustrates the relationship among the mappings, where h and f are defined within a strictly increasing function $\phi \colon E \to E$:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \phi \circ h\ } & E \\
& h_S \searrow & \Big\uparrow {\scriptstyle \phi \circ f} \\
& & S
\end{array}
$$

We may also consider non-open intervals E with the natural condition that if $f(s) = \inf E \in E$ for some $s \in S$ then $s = s_*$, and similarly, if $f(s) = \sup E \in E$ for some $s \in S$ then $s = s^*$. In that case, the isomorphism f is required to be *endpoint preserving*, that is, if $\inf E \in E$ (resp. $\sup E \in E$) then $f(s_*) = \inf E$

(resp. $f(s^*) = \sup E$), regardless of the chain $(S, \preccurlyeq)$ considered.[17] Consequently also all the functions $\phi\colon E \to E$ must be endpoint preserving in the sense that $\phi(x) = x$ for all $x \in B[E]$. Due to the finiteness of the ordinal scales, we may even assume that the functions ϕ are continuous, which amounts to assuming that they all belong to $\Phi[E]$.

The endpoint preservation assumption of f (and hence of ϕ) clarifies why we consider numerical representations in an interval E of $\mathbb{R}$, possibly non-open, rather than $\mathbb{R}$ itself.[18]

In this section, the set of all endpoint preserving isomorphisms $f\colon S \to E$ is denoted $\mathrm{F}[S, E]$. The diagonal restriction of $\mathrm{F}[S, E]^n$ is the set

$$\mathrm{F}_n[S, E] := \{\underbrace{(f, \ldots, f)}_{n} : f \in \mathrm{F}[S, E]\}.$$

Finally, for any $\mathbf{a} \in S^n$ and any $\mathbf{f} \in \mathrm{F}[S, E]^n$, the symbol $\mathbf{f}(\mathbf{a})$ denotes the vector $\big(f_1(a_1), \ldots, f_n(a_n)\big)$.

7.2 Aggregation by order invariant functions

Suppose we have n evaluations expressed in a finite chain $(S, \preccurlyeq)$, with $|S| = k \geqslant 2$. To aggregate these evaluations and obtain an overall evaluation in the same chain, we can use a discrete aggregation function $G\colon S^n \to S$, which is a ranking function sorting k^n n-tuples into k classes. (Here, "discrete" means that the domain of the function G is a discrete set.)

Among all the possible aggregation functions, we could choose one that is "independent" of the chain used.[19] Such a *chain independent* aggregation function is necessarily based on a numerical function $F\colon E^n \to E$ that can be represented in any finite chain $(S, \preccurlyeq)$ by a discrete analog $G\colon S^n \to S$ in the sense that the following identity

$$F(x_1, \ldots, x_n) = f\big(G\big(f^{-1}(x_1), \ldots, f^{-1}(x_n)\big)\big) \qquad (\mathbf{x} \in E^n)$$

holds for all isomorphisms $f \in \mathrm{F}[S, E]$.

As the following theorem shows [30, Proposition 4.1], this condition completely characterizes the order invariant functions.

Theorem 7.1. *$F\colon E^n \to E$ is an order invariant function if and only if, for any finite chain $(S, \preccurlyeq)$, there exists an aggregation function $G\colon S^n \to S$ such that, for any $f \in \mathrm{F}[S, E]$, we have*

$$F\big(f(a_1), \ldots, f(a_n)\big) = f\big(G(a_1, \ldots, a_n)\big) \qquad (\mathbf{a} \in S^n). \tag{7}$$

---

[17]This amounts to assuming that all the chains considered have a common bottom element s_* (resp. a common top element s^*) whose numerical representation is $\inf E$ (resp. $\sup E$).

[18]If E is closed, one typically chooses $E = [0, 1]$ or $E = \mathbb{R} \cup \{-\infty, \infty\}$.

[19]For example, we could use any lattice polynomial function, which does not depend on the chain used.

Thus, an order invariant function is characterized by the fact that it can always be represented by a discrete aggregation function $G\colon S^n \to S$ on any finite chain $(S, \preccurlyeq)$, regardless of the cardinality of this chain.[20] It is informative to represent the identity (7) by the following commutative diagram, where $\mathbf{f} := (f, \ldots, f)$:

$$
\begin{array}{ccc}
E^n & \xrightarrow{\ F\ } & E \\
\uparrow{\scriptstyle \mathbf{f}} & & \uparrow{\scriptstyle f} \\
S^n & \xrightarrow{\ G\ } & S
\end{array}
$$

It is clear from (7) that the discrete function G representing F in $(S, \preccurlyeq)$ is uniquely determined and, in some sense, is isomorphic to the "restriction" of F to S^n. For example, if $n = 2$ and $F(\mathbf{x}) = x_1 \wedge x_2$ (resp. $F(\mathbf{x}) = \inf E$) then the unique representative G of F is defined by $G(\mathbf{a}) = a_1 \wedge a_2$ (resp. $G(\mathbf{a}) = s_*$).

Evidently an order invariant function is nondecreasing if and only if its discrete representative is nondecreasing. Another property that might be required on order invariant functions is continuity (see §4.4) whose discrete counterpart, called *smoothness*, is defined as follows (see [14]).[21]

Definition 7.1. Consider $(n+1)$ finite chains $(S_0, \preccurlyeq_{S_0}), \ldots, (S_n, \preccurlyeq_{S_n})$. A discrete function $G\colon \times_{i=1}^n S_i \to S_0$ is said to be *smooth* if, for any $\mathbf{a}, \mathbf{b} \in \times_{i=1}^n S_i$, the elements $G(\mathbf{a})$ and $G(\mathbf{b})$ are equal or neighboring whenever there exists $j \in [n]$ such that a_j and b_j are neighboring and $a_i = b_i$ for all $i \neq j$.

The following important result [30, Proposition 5.1] relates the continuity property of order invariant functions to the smoothness condition of its discrete representatives, thus making continuity sensible and even appealing for order invariant functions.

Proposition 7.1. *An order invariant function $F\colon E^n \to E$ is continuous if and only if it is represented only by smooth discrete aggregation functions.*

7.3 Aggregation by comparison meaningful functions on a single ordinal scale

Consider the more general situation where the evaluations to be aggregated are expressed in the same finite chain $(S, \preccurlyeq_S)$ and the overall evaluation is expressed in a finite chain $(T, \preccurlyeq_T)$, possibly different from $(S, \preccurlyeq_S)$. Again, we can consider aggregation functions $G\colon S^n \to T$ and, among them, we might want to choose aggregation functions that are independent of the chains used.

---

[20]It is important to remember that considering a discrete function $G\colon S^n \to S$, where $(S, \preccurlyeq)$ is a given chain, is not equivalent to considering an order invariant function $F\colon E^n \to E$. Indeed, defining an order invariant function is much more restrictive since such a function should be independent of any scale. For instance, if $n = 2$ and E is open, we see by Proposition 4.4 that there are only four order invariant functions (namely x_1, x_2, $x_1 \wedge x_2$, and $x_1 \vee x_2$) while the number of possible discrete functions $G\colon S^2 \to S$ is clearly k^{k^2}, where $k = |S|$.

[21]Fodor [11, Theorem 2] (see [30] for the general case) showed that the smoothness condition is equivalent to the discrete version of the intermediate value theorem [12, Lemma 1].

As the following theorem shows [30, Proposition 4.3], such chain independent functions are constructed from numerical functions $F\colon E^n \to \mathbb{R}$ that are exactly the comparison meaningful functions on a single ordinal scale.

Theorem 7.2. *$F\colon E^n \to \mathbb{R}$ is a comparison meaningful function on a single ordinal scale if and only if, for any finite chain $(S, \preccurlyeq_S)$, there exists a finite chain $(T, \preccurlyeq_T)$ and a surjective aggregation function $G\colon S^n \to T$ such that, for any $\mathbf{f} \in \mathrm{F}_n[S, E]$, there is an isomorphism $g_\mathbf{f}\colon T \to \mathbb{R}$ such that*

$$F\big(\mathbf{f}(\mathbf{a})\big) = g_\mathbf{f}\big(G(\mathbf{a})\big) \qquad (\mathbf{a} \in S^n). \tag{8}$$

Thus, a comparison meaningful function on a single ordinal scale is characterized by the fact that it can always be represented by a discrete aggregation function $G\colon S^n \to T$ on any finite chain $(S, \preccurlyeq)$, regardless of the cardinality of this chain. The identity (8) can be graphically represented by the following commutative diagram

$$
\begin{array}{ccc}
E^n & \xrightarrow{\ F\ } & \mathbb{R} \\
\mathbf{f} \big\uparrow & & \big\uparrow g_\mathbf{f} \\
S^n & \xrightarrow[\ G\ ]{} & T
\end{array}
$$

It can be easily shown [30, §4.2] that, given a comparison meaningful function $F\colon E^n \to \mathbb{R}$ on a single ordinal scale and a finite chain $(S, \preccurlyeq_S)$, the output chain $(T, \preccurlyeq_T)$ and the functions $G\colon S^n \to T$ and $g_\mathbf{f}\colon T \to \mathbb{R}$ are uniquely determined.

The analog of Proposition 7.1 can be stated as follows [30, Proposition 5.2]. Unfortunately here we no longer have a necessary and sufficient condition.

Proposition 7.2. *A continuous comparison meaningful function $F\colon E^n \to \mathbb{R}$ on a single ordinal scale is represented only by smooth discrete aggregation functions.*

7.4 Aggregation by comparison meaningful functions on independent ordinal scales

We now assume that the n evaluations are expressed in independent finite chains $(S_i, \preccurlyeq_{S_i})$, $i = 1, \dots, n$, and that the overall evaluation is expressed in a finite chain $(T, \preccurlyeq_T)$. We can consider aggregation functions $G\colon \times_{i=1}^n S_i \to T$ and, among them, we might want to choose aggregation functions that are independent of the chains used.

As the following theorem shows [30, Proposition 4.6], such chain independent functions are constructed from numerical functions $F\colon E^n \to \mathbb{R}$ that are exactly the comparison meaningful functions on independent ordinal scales.

Theorem 7.3. *$F\colon E^n \to \mathbb{R}$ is a comparison meaningful function on independent ordinal scales if and only if, for any finite chains $(S_i, \preccurlyeq_{S_i})$, $i = 1, \dots, n$,*

there exists a finite chain $(T, \preccurlyeq_T)$ and a surjective aggregation function $G \colon \times_{i=1}^{n} S_i \to T$ such that, for any $\mathbf{f} \in \mathrm{F}[S, E]^n$, there is a isomorphism $g_{\mathbf{f}} \colon T \to \mathbb{R}$ such that

$$F\bigl(\mathbf{f}(\mathbf{a})\bigr) = g_{\mathbf{f}}\bigl(G(\mathbf{a})\bigr) \qquad (\mathbf{a} \in \underset{i=1}{\overset{n}{\times}} S_i).$$

Thus, a comparison meaningful function on independent ordinal scales is characterized by the fact that it can always be represented by a discrete aggregation function $G \colon \times_{i=1}^{n} S_i \to T$, regardless of the cardinality of the chains considered. Here the commutative diagram is given by

$$
\begin{array}{ccc}
E^n & \overset{F}{\longrightarrow} & \mathbb{R} \\[2pt]
\mathbf{f} \Big\uparrow & & \Big\uparrow g_{\mathbf{f}} \\[2pt]
\underset{i=1}{\overset{n}{\times}} S_i & \underset{G}{\longrightarrow} & T
\end{array}
$$

Here again, it can be easily shown [30, §4.3] that, given a comparison meaningful function $F \colon E^n \to \mathbb{R}$ on independent ordinal scales and n finite chains $(S_i, \preccurlyeq_{S_i})$, $i = 1, \ldots, n$, the output chain $(T, \preccurlyeq_T)$ and the functions $G \colon \times_{i=1}^{n} S_i \to T$ and $g_{\mathbf{f}} \colon T \to \mathbb{R}$ are uniquely determined.

Regarding continuous comparison meaningful functions, we have the following result [30, Proposition 5.3].

Proposition 7.3. *A continuous comparison meaningful function $F \colon E^n \to \mathbb{R}$ on independent ordinal scales is represented only by smooth discrete aggregation functions.*

Acknowledgments

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