

Dichotomy on intervals of strong partial Boolean clones

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ABSTRACT. The following result has been shown recently in the form of a dichotomy: For every total clone C on $2 := \{0, 1\}$, the set $\mathcal{I}(C)$ of all partial clones on 2 whose total component is C , is either finite or of continuum cardinality. In this paper we show that the dichotomy holds, even if only strong partial clones are considered, i.e., partial clones which are closed under taking subfunctions: For every total clone C on 2 , the set $\mathcal{I}_{\text{Str}}(C)$ of all strong partial clones on 2 whose total component is C , is either finite or of continuum cardinality.

1. Introduction

First let A be an arbitrary finite set. Later we concentrate on the Boolean case, i.e., we let $A = 2 := \{0, 1\}$.

A function $f: A^n \rightarrow A$ is called a total function on A . A function $f: S \rightarrow A$ with $S \subseteq A^n$ is called partial function on A and we denote the domain of f by $\text{dom } f := S$. Let $\text{Op}(A)$ be the set of all total functions on A , and let $\text{Par}(A)$ be the set of all partial functions on A .

The function $e_i^n: A^n \rightarrow A$ defined by $e_i^n(x_1, \dots, x_n) := x_i$ is called the n -ary *projection* onto the i -th coordinate. For each $a \in A$ the *constant* function $c_a^n: A^n \rightarrow A$ is defined by $c_a^n(\mathbf{x}) = a$ for all $\mathbf{x} \in A^n$.

Let $f \in \text{Par}(A)$ be n -ary and let $g_1, \dots, g_n \in \text{Par}(A)$ be m -ary. The *composition* $F := f(g_1, \dots, g_n)$ is an m -ary partial function defined by

$$F(x_1, \dots, x_m) := f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

and

$$\text{dom } F := \left\{ x \in \bigcap_{i=1}^n \text{dom } g_i \mid (g_1(x), \dots, g_n(x)) \in \text{dom } f \right\}.$$

Let $C \subseteq \text{Par}(A)$. Then C is called a *partial clone* if it is composition closed and contains the projections. If, additionally, C contains only total functions, i.e., $C \subseteq \text{Op}(A)$, then C is a *total clone*.

Let $f, g \in \text{Par}(A)$. We say that f is a *restriction* (or *subfunction*) of g , written $f \leq g$, if $\text{dom } f \subseteq \text{dom } g$ and $f(x) = g(x)$ for all $x \in \text{dom } f$. Let $X \subseteq \text{Par}(A)$. Then the *strong closure* of X , written $\text{Str}(X)$, is defined by

$$\text{Str}(X) := \{f \in \text{Par}(A) \mid \exists g \in X: f \leq g\}.$$

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If $X = \text{Str}(X)$ then X is called *strong*, or *restriction closed*. That means, a set X of partial functions is strong if it contains all subfunctions of its functions, i.e., $f \in C$ for all $f \in \text{Par}(A)$ and $g \in C$ with $f \leq g$.

Let $\text{Rel}^{(h)}(A)$ be the set of all h -ary relations on A for some $h \geq 1$, i.e., $\text{Rel}^{(h)}(A) := \{X \mid X \subseteq A^h\}$. Furthermore, let $\text{Rel}(A) := \bigcup_{h \geq 1} \text{Rel}^{(h)}(A)$.

Let $\varrho \in \text{Rel}^{(h)}(A)$, and let $f: S \rightarrow A$ with $S \subseteq A^n$ be an n -ary partial function. Then f *preserves* ϱ iff $f(M) \in \varrho$ for any $h \times n$ matrix $M = (m_{ij})$ whose rows belong to the domain of f , i.e. $(m_{i1}, \dots, m_{in}) \in \text{dom } f$ for all i , and whose columns belong to ϱ .

Let $\text{pPol } R$ be the set of all partial functions preserving every relation $\varrho \in R$. Let $\text{Pol } R := (\text{pPol } R) \cap \text{Op}(A)$ be the set of all total functions preserving every relation $\varrho \in R$.

There are three different types of intervals which we consider here. Let C be a total clone of $\text{Op}(A)$. Then we define the three intervals $\mathcal{I}(C)$, $\mathcal{I}_{\text{Str}}(C)$, and $\mathcal{I}_{\text{Str}}^{\subseteq}(C)$ by

$$\begin{aligned} \mathcal{I}(C) &:= \{X \subseteq \text{Par}(A) \mid X \text{ partial clone, and } C = X \cap \text{Op}(A)\} \\ \mathcal{I}_{\text{Str}}(C) &:= \{X \subseteq \text{Par}(A) \mid X \text{ strong partial clone, and } C = X \cap \text{Op}(A)\} \\ \mathcal{I}_{\text{Str}}^{\subseteq}(C) &:= \{X \subseteq \text{Par}(A) \mid X \text{ strong partial clone, and } C \subseteq X\} \\ &= \bigcup_{\substack{D \text{ total clone} \\ C \subseteq D}} \mathcal{I}_{\text{Str}}(D) \end{aligned}$$

Clearly, $\mathcal{I}_{\text{Str}}(C) \subseteq \mathcal{I}(C)$ holds.

We need the following families of total Boolean clones. Notice that every total Boolean clone can be written as the intersection of some clones in the list below (see Figure 1 and Section 3.1 [12]).

$$\begin{aligned} T_a &= \text{Pol}\{a\} \text{ for } a \in \{0, 1\} \\ T_{a,\mu} &= \text{Pol}(\{0, 1\}^\mu \setminus \{(b, \dots, b)\}) \text{ for } b \in \{0, 1\}, b \neq a \\ T_{a,\infty} &= \bigcap_{\mu \geq 2} T_{a,\mu} \text{ for } a \in \{0, 1\} \\ M &= \text{Pol} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (\text{set of all monotone functions}) \\ S &= \text{Pol} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{set of all self-dual functions}) \\ L &= \text{Pol}\{(x, x, y, y), (x, y, x, y), (x, y, y, x) \mid x, y \in \{0, 1\}\} \\ &\quad (\text{set of all linear functions}) \\ \Lambda &= \text{Clone}\{\wedge, c_0, c_1\} \\ V &= \text{Clone}\{\vee, c_0, c_1\} \\ \Omega_1 &= \text{Clone}(\text{Op}^{(1)}(2)) \end{aligned}$$

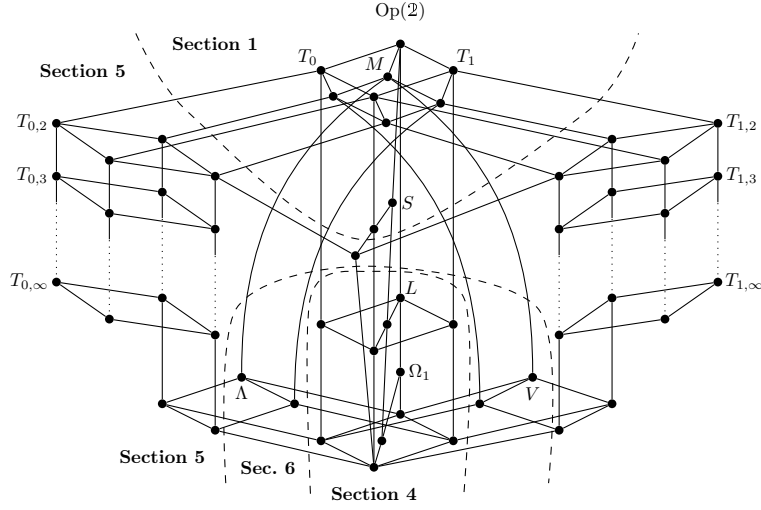


FIGURE 1. Post's lattice (with indications in which sections the corresponding intervals are handled)

The finite intervals $\mathcal{I}(C)$ have been described in [1, 13, 15–17], and the finite intervals of the form $\mathcal{I}_{\text{Str}}^{\subseteq}(C)$ have been described in [5, 8]. These results can be assembled into the following theorem. The finite intervals $\mathcal{I}_{\text{Str}}^{\subseteq}(M \cap T_0 \cap T_1)$ and $\mathcal{I}_{\text{Str}}^{\subseteq}(S \cap T_0 \cap T_1)$ are displayed in Appendix A.

Theorem 1.1. *Let C be a total Boolean clone with*

$$T_0 \cap T_1 \cap M \subseteq C \text{ or } T_0 \cap T_1 \cap S \subseteq C.$$

Then $\mathcal{I}(C)$ and $\mathcal{I}_{\text{Str}}^{\subseteq}(C)$ are finite sets.

C	$ \mathcal{I}(C) $	$ \mathcal{I}_{\text{Str}}^{\subseteq}(C) $	$ \mathcal{I}_{\text{Str}}(C) $
$\text{Op}(2)$	3	1	1
T_a ($a \in \{0, 1\}$)	6	2	1
M	6	2	1
S	6	2	1
$T_0 \cap T_1$	30	7	4
$M \cap T_a$ ($a \in \{0, 1\}$)	15	5	2
$M \cap T_0 \cap T_1$	101	25	13
$S \cap T_0 \cap T_1$	380	33	25

TABLE 1. The sizes of the finite intervals given in Theorem 1.1. The values of $|\mathcal{I}_{\text{Str}}(C)|$ can be deduced from the values of $|\mathcal{I}_{\text{Str}}^{\subseteq}(C)|$ and Post's lattice.

In [1, 18] it was shown that the intervals $\mathcal{I}(C)$ for subclones $C \subseteq B$ with $B \in \{L, \Lambda, V, T_{0,\infty}, T_{1,\infty}\}$ have the size of the continuum. Then in [13] the remaining intervals were determined to be infinite. The authors of [3] then finished the determination of the intervals of the form $\mathcal{I}(C)$ to yield the following theorem.

Theorem 1.2. *Let C be a total Boolean clone such that $C \subseteq B$ and $B \in \{L, \Lambda, V, T_{0,2}, T_{1,2}\}$. Then the set $\mathcal{I}(C)$ has the cardinality of the continuum.*

As stated in [3] this yields a dichotomy on the size of the intervals $\mathcal{I}(C)$ for Boolean clones C .

Theorem 1.3. *Let C be a total Boolean clone.*

Then $\mathcal{I}(C)$ is either finite or has the cardinality of the continuum. Furthermore, $\mathcal{I}(C)$ is finite if and only if $M \cap T_0 \cap T_1 \subseteq C$ or $S \cap T_0 \cap T_1 \subseteq C$.

The aim of this paper is to show that this result can be strengthened in the sense that only strong partial clones are considered. That means $\mathcal{I}(C)$ will be replaced by $\mathcal{I}_{\text{Str}}(C)$ in the statement of the last theorem. Since $\mathcal{I}_{\text{Str}}(C) \subseteq \mathcal{I}(C)$ for every Boolean clone C , we already have that $\mathcal{I}_{\text{Str}}(C)$ is finite if $M \cap T_0 \cap T_1 \subseteq C$ or $S \cap T_0 \cap T_1 \subseteq C$. Thus we will show that the intervals $\mathcal{I}_{\text{Str}}(C)$ have the cardinality of the continuum for all other Boolean clones C .

Although we focus on the case of Boolean clones, there have been some investigations into the general case with $|A| \geq 2$, for example [6] and [7]. Some of these results will be extended with the help of Lemma 3.16.

2. Further definitions

For some natural numbers $n, m \in \mathbb{N}$ with $n \leq m$ we define the sets $[n, m] := \{n, n+1, \dots, m\}$, and $[n] := [1, n]$. Tuples will be written with boldface small letters. For a tuple $\mathbf{x} := (x_1, \dots, x_n) \in A^n$ we define the set $[\mathbf{x}]$ of its entries by $[\mathbf{x}] := \{x_1, \dots, x_n\}$, and let $|\mathbf{x}| := |[x]|$. For $I \subseteq [n]$ we let $\mathbf{x}_I := \{x_i \mid i \in I\}$. For $\mathbf{i} = (i_1, \dots, i_l) \in [n]^l$ with $l \in \mathbb{N}$ we define $\mathbf{x}_{\mathbf{i}} := (x_{i_1}, \dots, x_{i_l}) \in A^l$. We will often use the two special tuples $\mathbf{0} := (0, \dots, 0)$ and $\mathbf{1} := (1, \dots, 1)$; the length of these tuples can be deduced from the context.

2.1. Romov's definability lemma. The statement of Theorem 2.1 proven by Romov in [14] gives a nice characterization of the constructability of relations in the co-clone of a strong partial clone. This enables us to prove the Theorems 5.5 and 6.8 just with relational methods.

The relation $\rho \in \text{Rel}^{(h)}(A)$ is called *irredundant* iff it fulfills the following two conditions:

- (i) ρ has no duplicate rows, i.e., for all i, j with $1 \leq i < j \leq h$, there is a tuple $(a_1, \dots, a_h) \in \rho$ with $a_i \neq a_j$;

- (ii) ρ has no fictitious coordinates, i.e., there is no $i \in \{1, \dots, h\}$, such that $(a_1, \dots, a_h) \in \rho$ implies $(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_h) \in \rho$ for all $x \in A$.

For a relation $\sigma \in \text{Rel}^{(h)}(A)$ we define $\text{Arity } \sigma := h$.

Theorem 2.1. *Let $\Sigma \subseteq \text{Rel}(A)$ and $\rho \in \text{Rel}^{(t)}(A)$ be relations. Furthermore let ρ be irredundant. Then*

$$\bigcap_{\sigma \in \Sigma} \text{pPol } \sigma \subseteq \text{pPol } \rho$$

iff there are some $\gamma_\sigma \subseteq [t]^{\text{Arity } \sigma}$ for all $\sigma \in \Sigma$ such that

$$\rho = \{\mathbf{x} \in A^t \mid \mathbf{x}_i \in \sigma \text{ for all } i \in \gamma_\sigma \text{ and } \sigma \in \Sigma\}$$

and

$$[t] = \bigcup_{\sigma \in \Sigma} \bigcup_{i \in \gamma_\sigma} [i].$$

3. Classes of partial functions

The aim of this section is the introduction of classes of partial functions (or shorter: partial classes) similar to the ideas presented by Harnau in [9–11] for total clones. This concept will prove fruitful in the extension of Theorem 8 [13] as shown in Lemma 3.13. Since we do not need the full power of the Galois connection presented by Harnau we will only prove statements about partial classes relevant to this paper.

For the definition of a partial class we need to define the following Maltsev-operations ζ , τ , Δ , ∇ , and \star . Let $f \in \text{Par}^{(n)}(A)$ and $g \in \text{Par}^{(m)}(A)$. Then we define

$$\begin{aligned} (\zeta f)(x_1, \dots, x_n) &:= f(x_2, x_3, \dots, x_n, x_1), \\ (\tau f)(x_1, \dots, x_n) &:= f(x_2, x_1, x_3, \dots, x_n), \\ (\Delta f)(x_1, \dots, x_{n-1}) &:= f(x_1, x_1, x_2, \dots, x_{n-1}), \\ \zeta f &= \tau f = \Delta f = f \text{ if } n = 1, \\ (\nabla f)(x_1, \dots, x_{n+1}) &:= f(x_2, \dots, x_{n+1}), \\ (f \star g)(x_1, \dots, x_{n+m-1}) &:= f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{n+m-1}). \end{aligned}$$

Definition 3.1. Let $X \subseteq \text{Par}(A)$. Then X is called a *partial class* if it closed under the operations \star , ζ , τ , ∇ , and Δ .

Remark 3.2. Partial classes can also be defined similar to the definition of partial clones in the beginning of this paper. A set X of partial functions on A is a partial class if $f(g_1, \dots, g_n) \in X$ for all $f \in X$ and $g_1, \dots, g_n \in X \cup \text{Proj}(A)$, where $\text{Proj}(A)$ is the set of projections on A .

Lemma 3.3. *Let $X, Y \subseteq \text{Par}(A)$ be two partial classes. Then $X \cap Y$ is also a partial class.*

The partial classes containing the projections are exactly the partial clones.
If $X, Y \subseteq \text{Par}(A)$, then we define the set $X \star Y \subseteq \text{Par}(A)$ by

$$X \star Y := \{f \star g \mid f \in X, g \in Y\}.$$

3.1. Relation pairs. Similar to the work done by Harnau in [9–11] we introduce relation pairs to characterize strong partial classes.

For each $h \geq 1$ let $\text{Pair}^{(h)}(A)$ be the set of all pairs (ρ, ρ') with $\rho' \subseteq \rho \subseteq A^h$, and $\text{Pair}(A) := \bigcup_{h \geq 1} \text{Pair}^{(h)}(A)$.

Let $(\rho, \rho') \in \text{Pair}^{(h)}(A)$ for some $h \geq 1$, and $f \in \text{Par}^{(n)}(A)$ for some $n \geq 1$. Then f *preserves* the relation pair (ρ, ρ') , if for all matrices M with columns in ρ , and rows in $\text{dom } f$ the tuple $f(M)$ belongs to ρ' . We write $f \in \text{cPol}(\rho, \rho')$, or $(\rho, \rho') \in \text{cInv } f$.

If $\rho = \rho'$ then the preservation of the relation pair (ρ, ρ') coincides with the preservation of the relation ρ , i.e., $\text{cPol}(\rho, \rho) = \text{pPol } \rho$.

Let $X \subseteq \text{Par}(A)$ and $Q \subseteq \text{Pair}(A)$. We define $\text{cPol } Q$ and $\text{cInv } X$ by

$$\begin{aligned} \text{cPol } Q &:= \bigcap_{q \in Q} \text{cPol } q, \\ \text{cInv } X &:= \bigcap_{f \in X} \text{cInv } f. \end{aligned}$$

Lemma 3.4. *Let $f \in \text{cPol}(\rho, \rho')$ and $g \in \text{cPol}(\sigma, \sigma')$ with $\sigma' \subseteq \rho \subseteq \sigma$.*

Then $f \star g \in \text{cPol}(\rho, \rho')$.

Proof. Let $f \in \text{cPol}^{(n)}(\rho, \rho')$ and $g \in \text{cPol}^{(m)}(\sigma, \sigma')$ with $\sigma' \subseteq \rho \subseteq \sigma \in \text{Rel}^{(h)}(A)$. Let M be an $(h, m+n-1)$ -matrix with columns $\mathbf{x}_1, \dots, \mathbf{x}_{m+n-1} \in \rho$, and rows $\mathbf{y}_1, \dots, \mathbf{y}_h \in \text{dom}(f \star g)$. Let $\mathbf{y}'_j := (\mathbf{y}_j)_{(1, \dots, m)}$ for each $j \in [h]$.

Then $\mathbf{y}'_1, \dots, \mathbf{y}'_h \in \text{dom } g$ by the definition of \star , and $\mathbf{x}_1, \dots, \mathbf{x}_m \in \sigma$. Thus $\mathbf{x} := g(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \sigma' \subseteq \rho$. From this $(f \star g)(\mathbf{x}_1, \dots, \mathbf{x}_{m+n-1}) = f(\mathbf{x}, \mathbf{x}_{m+1}, \dots, \mathbf{x}_{m+n-1}) \in \rho'$ and thus $f \star g \in \text{cPol}(\rho, \rho')$. \square

Lemma 3.5. *Let $q \in \text{Pair}(A)$. Then $\text{cPol } q$ is a non-empty strong partial class of $\text{Par}(A)$.*

Proof. Let $(\rho, \rho') := q \in \text{Pair}(A)$.

We first show that $\text{cPol}(\rho, \rho')$ is a partial class. Let $f, g \in \text{cPol}(\rho, \rho')$.

It is easy to see that $\zeta f, \tau f, \Delta f, \nabla f \in \text{cPol}(\rho, \rho')$. From Lemma 3.4 with $\sigma = \rho$ and $\sigma' = \rho'$ follows $f \star g \in \text{cPol}(\rho, \rho')$. Thus $\text{cPol}(\rho, \rho')$ is a partial class of $\text{Par}(A)$.

We now show that $\text{cPol}(\rho, \rho')$ is strong. Let $f \in \text{cPol}(\rho, \rho')$ and $g \leq f$, and assume to the contrary that $g \notin \text{cPol}(\rho, \rho')$. Then there is a matrix M with columns $\mathbf{x}_1, \dots, \mathbf{x}_n \in \rho$ and rows $\mathbf{y}_1, \dots, \mathbf{y}_h \in \text{dom } g$, such that $g(M) \notin \rho'$. Since $\text{dom } g \subseteq \text{dom } f$ we have $f(M) = g(M) \notin \rho'$. Thus $f \notin \text{cPol}(\rho, \rho')$ contradicting the assumption. Thus $\text{cPol}(\rho, \rho')$ is strong.

It is non-empty since the partial function c_\emptyset with empty domain preserves any relation pair q . \square

Lemma 3.6. *Let $Q \subseteq \text{Pair}(A)$. Then $\text{cPol } Q$ is a non-empty strong partial class of $\text{Par}(A)$.*

Proof. By Lemma 3.5 we have that $\text{cPol } q$ is a strong partial class for all $q \in Q$. Then by Lemma 3.3 and the definition of $\text{cPol } Q$, we see that $\text{cPol } Q$ is a partial class. Furthermore, the intersection of two strong sets is also strong. It is non-empty since $c_\emptyset \in \text{cPol } Q$. \square

Remark 3.7. It is possible to show, that for every non-empty strong partial class $X \subseteq \text{Par}(A)$, there is some $Q \subseteq \text{Pair}(A)$ with $X = \text{cPol } Q$. Since this and other further properties of the operators cPol and cInv are not needed in this paper, they will not be proven here.

Lemma 3.8. *Let $\rho \in \text{Rel}(A)$ with $\rho \neq \emptyset$.*

Then $\text{cPol}(\rho, \emptyset) \cap \text{Op}(A) = \emptyset$.

Proof. Let $f \in \text{Op}^{(n)}(A)$, $\rho \in \text{Rel}^{(h)}(A)$, and $\mathbf{x} \in \rho$. Let M be the matrix formed by n -fold repetition of the column \mathbf{x} . Let the rows of M be called $\mathbf{y}_1, \dots, \mathbf{y}_h$. Clearly, $\mathbf{y}_i \in \text{dom } f$ for all $i \in [h]$ since f is a total function. But $f(M) \notin \emptyset$, and thus $f \notin \text{cPol}(\rho, \emptyset)$. \square

Lemma 3.9. *Let $\rho \in \text{Rel}(A)$, $f \in \text{Par}(A)$ and $g \in \text{cPol}(\rho, \emptyset)$.*

Then $f \star g \in \text{cPol}(\rho, \emptyset)$.

Proof. Let $\rho \in \text{Rel}^{(h)}(A)$, $f \in \text{Par}^{(n)}(A)$ and $g \in \text{cPol}^{(m)}(\rho, \emptyset)$.

If $\rho = \emptyset$, then $\text{cPol}(\rho, \emptyset) = \text{cPol}(\emptyset, \emptyset) = \text{Par}(A)$. Thus $f \star g \in \text{cPol}(\rho, \emptyset)$.

Let $\rho \neq \emptyset$. Assume to the contrary, that $f \star g \notin \text{cPol}(\rho, \emptyset)$. Then there is a matrix M with columns $\mathbf{x}_1, \dots, \mathbf{x}_{m+n-1} \in \rho$, and rows $\mathbf{y}_1, \dots, \mathbf{y}_h \in \text{dom}(f \star g)$. We can now look at the matrix M' formed by the first m columns, and with rows $\mathbf{y}'_1, \dots, \mathbf{y}'_h$. Then $\mathbf{y}_i \in \text{dom}(f \star g)$ implies $\mathbf{y}'_i \in \text{dom } g$ for all $i \in [h]$. But since $\mathbf{x}_1, \dots, \mathbf{x}_m \in \rho$ we get $g \notin \text{cPol}(\rho, \emptyset)$ in contradiction to the assumption. \square

Corollary 3.10. *Let $X \subseteq \text{Par}(A)$ and $\rho \in \text{Rel}(A)$.*

Then $X \star \text{cPol}(\rho, \emptyset) \subseteq \text{cPol}(\rho, \emptyset)$.

The following corollary follows from Lemma 3.4.

Corollary 3.11. *Let $\rho \in \text{Rel}(A)$.*

Then $\text{cPol}(\rho, \emptyset) \star \text{pPol } \rho \subseteq \text{cPol}(\rho, \emptyset)$.

The last two corollaries can now be combined into the final statement of this subsection.

Corollary 3.12. *Let $\rho \in \text{Rel}(A)$, $T := \text{cPol}(\rho, \emptyset)$ and $D \subseteq \text{pPol } \rho$.*

Then $T \star D \subseteq T$ and $D \star T \subseteq T$.

3.2. Classes to intervals. In the proof that the interval $\mathcal{I}_{\text{Str}}(D)$ is of continuum cardinality for some total clone D , we try to make as few constructions as possible. This can be achieved if we find some clone C with $D \subseteq C$, construct

a set $I \subseteq \mathcal{I}_{\text{Str}}(C)$ of continuum cardinality, and then find restrictions of the partial clones in I , such that these restricted partial clones lie in $\mathcal{I}_{\text{Str}}(D)$, and I does not collapse.

For this purpose we prove a stronger version of Theorem 8 [13] as follows.

Lemma 3.13. *Let C and D be clones of $\text{Op}(A)$ with $D \subseteq C$, let T be a strong partial class of $\text{Par}(A)$, and let $I \subseteq \mathcal{I}_{\text{Str}}(C)$, such that the following conditions hold*

- (i) $T \cap \text{Op}(A) \subseteq D$,
- (ii) $T \star \text{Str}(D) \subseteq \text{Str}(D) \cup T$, and $\text{Str}(D) \star T \subseteq \text{Str}(D) \cup T$,
- (iii) $X \cap T \neq Y \cap T$ for all $X, Y \in I$ with $X \neq Y$.

Then

$$|\mathcal{I}_{\text{Str}}(D)| \geq |I|.$$

Proof. For each $X \in I$ we define X_D by

$$X_D := \text{Str}(D) \cup (X \cap T).$$

We let $I_D := \{X_D \mid X \in I\}$, and show that $I_D \subseteq \mathcal{I}_{\text{Str}}(D)$. By (iii) and since $X \cap \text{Str}(D) = Y \cap \text{Str}(D)$ for all $X, Y \in \mathcal{I}_{\text{Str}}(C)$ we have that $|I_D| = |I|$.

Let $X \in I$ be arbitrary. By (i) we have that

$$\begin{aligned} X_D \cap \text{Op}(A) &= (\text{Str}(D) \cup (X \cap T)) \cap \text{Op}(A) \\ &= \underbrace{(\text{Str}(D) \cap \text{Op}(A))}_D \cup \underbrace{(X \cap (T \cap \text{Op}(A)))}_{\subseteq D} \\ &= D. \end{aligned}$$

Thus we only have to show that X_D is a strong partial clone.

Since $\text{Str}(D)$, X , and T are strong partial classes, we see that $\text{Str}(X_D) = X_D$, and that X_D is closed with respect to ζ , τ , ∇ and Δ . Furthermore, X_D contains the projections, since $\text{Str}(D) \subseteq X_D$, and D is a clone.

It remains to show that X_D is closed with respect to \star . Let $f, g \in X_D$. We want to show that $f \star g \in X_D$.

There are several cases:

- $f, g \in \text{Str}(D)$. Then $f \star g \in \text{Str}(D) \subseteq X_D$, since $\text{Str}(D)$ is a strong partial clone.
- $f, g \in X \cap T$. Then $f \star g \in X \cap T \subseteq X_D$, since $X \cap T$ is a strong partial class.
- $f \in \text{Str}(D)$, and $g \in X \cap T$; or $g \in \text{Str}(D)$, and $f \in X \cap T$. Since $D \subseteq C \subseteq X$, $X \cap T \subseteq X$ and X is a partial clone, we have $f \star g \in X$. By (ii) we have $f \star g \in \text{Str}(D) \cup T$. Thus

$$\begin{aligned} f \star g &\in (\text{Str}(D) \cup T) \cap X \\ &= (\text{Str}(D) \cap X) \cup (X \cap T) \\ &= \text{Str}(D) \cup (X \cap T) \\ &= X_D. \end{aligned}$$

Thus X_D is a strong partial clone with $X_D \cap \text{Op}(A) = D$. This implies $X_D \in \mathcal{I}_{\text{Str}}(D)$. Therefore $I_D \subseteq \mathcal{I}_{\text{Str}}(D)$, and consequently $|\mathcal{I}_{\text{Str}}(D)| \geq |I|$. \square

One example of a strong partial class T needed in the preceding lemma is the partial class $\text{cPol}(\{0\}, \emptyset)$ of all partial functions not defined on $(0, \dots, 0)$. This was implicitly used in [13] and [3].

Each of the sets I defined in this paper will be indexed by the subsets of a countable infinite set $N \subseteq \mathbb{N}$. As such the set I has the same cardinality as the powerset of \mathbb{N} , which has the cardinality of the continuum, and therefore I has the cardinality of the continuum.

3.3. Subclones missing a constant. First we use Lemma 3.13 in a general setting, involving two clones C and D in $\text{Op}(A)$ with $D \subseteq C$ and $c_a \in C \setminus D$ for some $a \in A$. For a partial function $f \in \text{Par}^{(n)}(A)$ and some $a \in A$ we define the $(n+1)$ -ary partial function $f_a \in \text{Par}(A)$ by

$$\begin{aligned} \text{dom } f_a &:= \{(a, \mathbf{x}) \mid \mathbf{x} \in \text{dom } f\}, \\ f_a(a, \mathbf{x}) &:= f(\mathbf{x}) \text{ for all } \mathbf{x} \in \text{dom } f. \end{aligned}$$

Lemma 3.14. *Let $C \subseteq \text{Op}(A)$ be a clone with $c_a \in C$, and $X \in \mathcal{I}_{\text{Str}}(C)$. Then $f \in X$ if and only if $f_a \in X$.*

Proof. Assume $f \in X$. Then $f_a \leq \nabla f \in X = \text{Str}(X)$, and thus $f_a \in X$.

Now assume that $f_a \in X$. Additionally, we have $c_a \in C \subseteq X$. Thus $f = \Delta(f_a \star c_a) \in X$. \square

Lemma 3.15. *Let $D \subseteq \text{Op}(A)$ be a clone with $c_a \notin D$. Then there is some $\rho \in \text{Inv } D$ with $(a, \dots, a) \notin \rho$.*

Proof. Assume to the contrary, that $(a, \dots, a) \in \rho$ for all $\rho \in \text{Inv } D$. Then $c_a \in \text{Pol } \rho$ for all $\rho \in \text{Inv } D$, and thus $c_a \in D$. Contradiction. \square

Lemma 3.16. *Let $C, D \subseteq \text{Op}(A)$ be clones with $c_a \in C \setminus D$ and $D \subseteq C$. Then $|\mathcal{I}_{\text{Str}}(D)| \geq |\mathcal{I}_{\text{Str}}(C)|$.*

Proof. By Lemma 3.15 there is some relation ρ with $(a, \dots, a) \notin \rho$ and $D \subseteq \text{Pol } \rho$. Let $T := \text{cPol}(\rho, \emptyset)$, and $I := \mathcal{I}_{\text{Str}}(C)$. We want to use Lemma 3.13.

Since $T \cap \text{Op}(A) = \emptyset \subseteq D$ condition (i) is fulfilled, and by Corollary 3.12 condition (ii) is fulfilled.

Now we show that condition (iii) is fulfilled as well. Let $X, Y \in \mathcal{I}_{\text{Str}}(C)$ with $X \neq Y$; w.l.o.g. there is some $f \in X \setminus Y$. By Lemma 3.14 we have $f_a \in X \setminus Y$. We just need to show that $f_a \in T$.

Assume to the contrary that $f_a \notin T$. Let f_a be n -ary, and ρ be h -ary. Then there is a matrix M such that

- its row $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_h \in \text{dom } f_a$, and
- its columns $\mathbf{y}_1, \dots, \mathbf{y}_n \in \rho$.

By the definition of f_a and choice of ρ we see that $\mathbf{y}_1 = (a, \dots, a) \notin \rho$. This is a contradiction. Thus $f_a \in T$, and consequently $X \cap T \neq Y \cap T$.

Therefore all conditions of Lemma 3.13 are fulfilled, and we get $|\mathcal{I}_{\text{Str}}(D)| \geq |\mathcal{I}_{\text{Str}}(C)|$. \square

This lemma can be applied to the main results of Theorems 10 and 19 in [6]. Let \mathcal{B}_A be the set of all h -universal relations ($3 \leq h \leq |A| - 1$), and let \mathcal{L}_A be the set of all prime affine relations on A (the definition of these relations can be found in [6]). Then for each $\rho \in \mathcal{B}_A \cup \mathcal{L}_A$ the following properties hold

- $\text{Pol } \rho$ is a maximal clone of $\text{Op}(A)$,
- $c_a \in \text{Pol } \rho$ for all $a \in A$,
- $\mathcal{I}_{\text{Str}}(\text{Pol } \rho)$ has the cardinality of the continuum.

With Lemma 3.16 we obtain the following statement.

Theorem 3.17. *Let $D \subseteq \text{Op}(A)$ a clone with $D \subseteq \text{Pol } \rho$ for some $\rho \in \mathcal{B}_A \cup \mathcal{L}_A$, and $c_a \notin D$ for some $a \in A$. Then $\mathcal{I}_{\text{Str}}(D)$ has the cardinality of the continuum.*

4. The subclones of L

In this section we use the results from [1] to show that the interval $\mathcal{I}_{\text{Str}}(D)$ has continuum cardinality for all clones $D \subseteq L$.

We need to define some functions first as given in [1]. Let $n(k, p) := (2k - 1)p + 1$, $k \geq 2$ and $p \geq 1$. Let $\#_1 \mathbf{x}$ be the number of 1's in the tuple \mathbf{x} . Define the $n(k, p)$ -ary partial function τ_p^k by

$$\begin{aligned} \text{dom } \tau_p^k &:= \{\mathbf{1}\} \cup \{\mathbf{x} \in 2^{n(k, p)} \mid \#_1 \mathbf{x} \leq p\}, \\ \tau_p^k(\mathbf{x}) &:= \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{1}, \\ 0 & \text{if } \mathbf{x} \in \text{dom } \tau_p^k \setminus \{\mathbf{1}\}. \end{cases} \end{aligned}$$

We define p_j by $p_1 := 1$ and $p_j := n(j, p_{j-1})$ for all $j \geq 2$. Set $\xi_j := \tau_{p_j}^{j+1}$ for all $j \geq 1$.

Lemma 4.1 (Lemma 11 [1]). *Let $j \geq 1$.*

Then $\xi_j \notin [\{\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots\} \cup \text{Str}(L)]$.

As a consequence we get the following theorem.

Theorem 4.2. *The interval $\mathcal{I}_{\text{Str}}(L)$ has the cardinality of the continuum.*

Proof. Let $X_J := [\{\xi_j \mid j \in J\} \cup \text{Str}(L)]$ for every $J \subseteq \mathbb{N} \setminus \{0\}$. By Lemma 4.1 we see that $X_J \neq X_{J'}$ if $J \neq J'$, and thus the set $I := \{X_J \mid J \subseteq \mathbb{N} \setminus \{0\}\}$ has the cardinality of the continuum. Furthermore, $I \subseteq \mathcal{I}_{\text{Str}}(L)$. Since L is a maximal clone and $|\mathcal{I}_{\text{Str}}(\text{Op}(2))| = 1$, we conclude that $\mathcal{I}_{\text{Str}}(L)$ has the cardinality of the continuum. \square

Lemma 4.3. *Let $D \subseteq L$ be a clone with $C \subseteq D$ with $D \in \{T_0, T_1, S\}$. Then $\mathcal{I}_{\text{Str}}(D)$ has the cardinality of the continuum.*

Proof. We have $c_0, c_1 \in L$, and $c_1 \notin T_0$, $c_0 \notin T_1$, $c_0 \notin S$. Thus Lemma 3.16 is applicable with $C = L$, and by Theorem 4.2 follows that $\mathcal{I}_{\text{Str}}(D)$ has the cardinality of the continuum. \square

4.1. The remaining two subclones of L . The only two subclones of L not covered yet are $C_{01} := [c_0, c_1]$ and $\Omega_1 := [\text{Op}^{(1)}(\mathfrak{D})]$. Let ρ_C , ρ_1 and ρ_L be the three 4-ary relations defined by

$$\rho_C := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \rho_1 := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \rho_L := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Although the fact $\rho_C \subseteq \rho_1 \subseteq \rho_L$ is not used directly, the similar structure of these relations makes the proof of Lemma 4.4 a bit easier.

As shown by Blochina in [2] (see also Section 10.2 [12]) the relations ρ_C , ρ_1 and ρ_L characterize the clones C_{01} , Ω_1 , and L , respectively. That means the following equalities hold:

$$\begin{aligned} C_{01} &= \text{Pol } \rho_C, \\ \Omega_1 &= \text{Pol } \rho_1, \\ L &= \text{Pol } \rho_L. \end{aligned}$$

Lemma 4.4. *Let $j \geq 1$.*

Then $\xi_j \in \text{pPol } \rho_1$ and $\xi_j \in \text{pPol } \rho_C$.

Proof. Let $\rho \in \{\rho_1, \rho_C\}$. Assume to the contrary, that ξ_j does not preserve ρ .

Then there is a matrix M such that

- its rows $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \text{dom } \xi_j$,
- its columns $\mathbf{y}_1, \dots, \mathbf{y}_{p_j+1} \in \rho$, and
- $\mathbf{z} := (z_1, z_2, z_3, z_4) := (\xi_j(\mathbf{x}_1), \xi_j(\mathbf{x}_2), \xi_j(\mathbf{x}_3), \xi_j(\mathbf{x}_4)) \notin \rho$.

We will show that $\mathbf{z} = \mathbf{y}_l$ for some $l \in [p_j+1]$.

Let $K := \{k \in [4] \mid \mathbf{x}_k \neq \mathbf{1}\}$. Clearly, $z_k = 0$ iff $k \in K$. Since $\mathbf{0}, \mathbf{1} \in \rho$, we have that $\mathbf{z} \notin \{\mathbf{0}, \mathbf{1}\}$ and thus there are $i, i' \in [4]$ with $\mathbf{x}_i = \mathbf{1}$ and $\mathbf{x}_{i'} \neq \mathbf{1}$. This implies $1 \leq |K| \leq 3$.

By the construction of ξ_j each row \mathbf{x}_k for $k \in K$ has at most p_j -many 1's. But ξ_j has an arity of

$$p_{j+1} = (2(j+1) - 1)p_j + 1 = (2j+1)p_j + 1 \geq 3p_j + 1.$$

Thus there is some column \mathbf{y}_l with $(\mathbf{y}_l)_k = 0$ for all $k \in K$. Furthermore, $(\mathbf{y}_l)_{k'} = 1$ for all $k' \in [4] \setminus K$. Thus $\mathbf{y}_l = \mathbf{z}$. But this contradicts $\mathbf{y}_l \in \rho$ and $\mathbf{z} \notin \rho$.

Therefore $\xi_j \in \text{pPol } \rho$. \square

Lemma 4.5. *$\mathcal{I}_{\text{Str}}(C_{01})$ has the cardinality of the continuum.*

Proof. Let $X_J := [\{\xi_j \mid j \in J\} \cup \text{Str}(L)]$ for every $J \subseteq \mathbb{N} \setminus \{0\}$. Then the set $I := \{X_J \mid J \subseteq \mathbb{N} \setminus \{0\}\}$ has the cardinality of the continuum.

Let $T := \text{pPol } \rho_C$.

- $T \cap \text{Op}(\mathfrak{Z}) = \text{Pol } \rho_C = C_{01}$.
- Since $\text{Str}(C_{01}) \subseteq T$ we have $T \star \text{Str}(C_{01}) \subseteq T$, and $\text{Str}(C_{01}) \star T \subseteq T$.
- By Lemma 4.4 and the definition of I , we get $X \cap T \neq Y \cap T$ for all $X, Y \in I$ with $X \neq Y$.

Then we apply Lemma 3.13, and yield the result. \square

By setting $T = \text{pPol } \rho_1$ in the previous proof we obtain the proof for the following lemma.

Lemma 4.6. $\mathcal{I}_{\text{Str}}(\Omega_1)$ has the cardinality of the continuum.

Now we can conclude from Theorem 4.2, and Lemmas 4.3, 4.5, 4.6, that the following theorem holds.

Theorem 4.7. Let $D \subseteq L$. Then $\mathcal{I}_{\text{Str}}(D)$ has the cardinality of the continuum.

5. The clone $T_{0,2}$ and its subclones

In this section we first give an alternative proof for the fact that $\mathcal{I}_{\text{Str}}(T_{0,2})$ has the cardinality of the continuum. The relations used are similar to the ones given in [3], but the proof presented here only uses relations.

5.1. Alternative proof for $\mathcal{I}_{\text{Str}}(T_{0,2})$ is continuum. The proof given in this section uses some ideas from the proof in [3], but changes the basic building blocks of the relations used. Furthermore, while the former proof depends on working with functions, the following proof only deals with relations.

Let $\rho_{0,2} := \{(0,0), (0,1), (1,0)\}$. We remember that $T_{0,2} = \text{Pol } \rho_{0,2}$.

Let $R_{C,n}^{0,2}$ and $R_{K,n}^{0,2}$ be two n -ary relations defined by

$$R_{C,n}^{0,2}(x_1, \dots, x_n) := \bigwedge_{i \in [n]} \rho_{0,2}(x_i, x_{i+1 \bmod n}),$$

$$R_{K,n}^{0,2}(x_1, \dots, x_n) := \bigwedge_{\substack{i,j \in [n] \\ i \neq j}} \rho_{0,2}(x_i, x_j).$$

Furthermore, let

$$R_n^{0,2} := R_{C,n}^{0,2} \times R_{K,n}^{0,2}.$$

The names C and K in the indices of the relations are in correspondance with the circular graph C_n and the complete graph K_n on n vertices. The relations $R_{C,n}^{0,2}$ have the same definition as R_{\uparrow}^k in [3]. The idea behind replacing R_{\downarrow}^k with $R_{K,n}^{0,2}$ stems from the following fact about graphs:

- Let $n' > n \geq 3$ be two odd numbers. Then there is no graph homomorphism from C_n into $C_{n'}$.

- For $n' > n \geq 3$ there is no graph homomorphism from $K_{n'}$ into K_n .
- For $n', n \geq 3$ with $n \neq n'$ there is no graph homomorphism from K_n into $C_{n'}$.

The relation $R_n^{0,2}$ in this model represents the disjoint union $C_n \uplus K_n$ of C_n and K_n . Let $G \rightarrow H$ denote the fact, that there is some graph homomorphism from G to H . We consider the possible homomorphisms from $K_{n'} \uplus C_{n'}$ to $K_n \uplus C_n$. Then we see

- for $n' > n \geq 3$ that $C_{n'} \uplus K_{n'} \not\rightarrow C_n \uplus K_n$, since $K_{n'} \not\rightarrow K_n$ and $K_{n'} \not\rightarrow C_n$; and
- for $n > n' \geq 3$ that any homomorphism from $C_{n'} \uplus K_{n'}$ to $C_n \uplus K_n$ actually maps into K_n , since $C_{n'} \not\rightarrow C_n$ and $K_{n'} \not\rightarrow C_n$. But for the construction of $R_n^{0,2}$ this would mean that the first n coordinates are not essential, a contradiction.

For $n = 5$ the relations $R_{C,5}^{0,2}$ and $R_{K,5}^{0,2}$ look like this:

$$R_{C,5}^{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \quad R_{K,5}^{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Lemma 5.1. *Let $n \geq 2$. Then $\text{Pol } R_n^{0,2} = \text{Pol } R_{C,n}^{0,2} = \text{Pol } R_{K,n}^{0,2} = T_{0,2}$.*

Proof. By construction we have that $T_{0,2} \subseteq \text{Pol } R_{C,n}^{0,2}$, $T_{0,2} \subseteq \text{Pol } R_{K,n}^{0,2}$, and $\text{Pol } R_n^{0,2} = \text{Pol } R_{C,n}^{0,2} \cap \text{Pol } R_{K,n}^{0,2}$.

Since $\rho_{0,2} = \text{pr}_{1,2} R_{C,n}^{0,2} = \text{pr}_{1,2} R_{K,n}^{0,2}$ we obtain $\text{Pol } R_{C,n}^{0,2} \subseteq T_{0,2}$, and $\text{Pol } R_{K,n}^{0,2} \subseteq T_{0,2}$. From this follows $\text{Pol } R_{C,n}^{0,2} = T_{0,2} = \text{Pol } R_{K,n}^{0,2}$, and consequently $T_{0,2} = \text{Pol } R_n^{0,2}$. \square

Let $\hat{\mathbb{N}} := \{n \in \mathbb{N} \mid n \text{ odd}, n \geq 3\}$. Let $n \in \hat{\mathbb{N}}$ and $M \subseteq \hat{\mathbb{N}} \setminus \{n\}$ for the remainder of this section. We want to show that

$$\text{pPol } R_n^{0,2} \not\supseteq \bigcap_{m \in M} \text{pPol } R_m^{0,2} \quad (5.1)$$

holds. We assume to the contrary, that (5.1) is false. This means that by Theorem 2.1 we can write

$$R_n^{0,2} := \{\mathbf{x} \in 2^{2n} \mid \mathbf{x}_i \in R_m^{0,2} \text{ for all } i \in \gamma_m \text{ and } m \in M\} \quad (5.2)$$

for some auxiliary relations γ_m for all $m \in M$. Furthermore, we can assume that no condition is superfluous.

Lemma 5.2. *Let $m \in M$ with $\gamma_m \neq \emptyset$, and $\mathbf{i} \in \gamma_m$.*

Then $\mathbf{i}_{[m]} \subseteq [n]$ or $\mathbf{i}_{[m]} \subseteq [n+1, 2n]$.

Similarly, $\mathbf{i}_{[m+1, 2m]} \subseteq [n]$ or $\mathbf{i}_{[m+1, 2m]} \subseteq [n+1, 2n]$.

Proof. We only consider the first statement; the second one follows similarly.

Assume the statement is not true. Then there is some $j \in [m]$ such that $i_j \in [n]$ and $i_{j+1 \bmod m} \in [n+1, 2n]$. By construction of $R_n^{0,2}$ (or, more specifically $R_{C,n}^{0,2}$) this means, that $\rho_{0,2}(x_{i_j}, x_{i_{j+1 \bmod m}})$ holds, i.e., x_{i_j} and $x_{i_{j+1 \bmod m}}$ can not both be 1 at the same time. But by construction of $R_n^{0,2}$ we have

$$(0, \dots, 0, \underset{i_j}{\uparrow} 1, 0, \dots, 0, \underset{i_{j+1 \bmod m}}{\uparrow} 1, 0, \dots, 0) \in R_n^{0,2}.$$

This is a contradiction, and thus $\mathbf{i}_{[m]} \subseteq [n]$ or $\mathbf{i}_{[m]} \subseteq [n+1, 2n]$. \square

Lemma 5.3. *Let $m \in M$ and $\gamma_m \neq \emptyset$.*

Then $m < n$.

Proof. Assume to the contrary, that $m \in M$, $\gamma_m \neq \emptyset$, and $m > n$.

Then there is some $\mathbf{i} \in \gamma_m$. By Lemma 5.2 we have $\mathbf{i}_{[m+1, 2m]} \subseteq [n]$ or $\mathbf{i}_{[m+1, 2m]} \subseteq [n+1, 2n]$. Thus $i_j = i_{j'}$ for some $j, j' \in [m+1, 2m]$ with $j \neq j'$. By construction of $R_n^{0,2}$ (or, more specifically $R_{K,n}^{0,2}$) this means, that $\rho_{0,2}(x_{i_j}, x_{i_{j'}})$ holds, i.e., $x_{i_j} = 0$. But by construction of $R_n^{0,2}$ we have

$$(0, \dots, 0, \underset{i_j}{\uparrow} 1, 0, \dots, 0) \in R_n^{0,2}.$$

This is a contradiction, and thus $m < n$. \square

Since $R_n^{0,2}$ is not a trivial relation, there is at least one $m < n$ with non-empty γ_m . Thus we can assume that $n \geq 5$.

Lemma 5.4. *Let $m \in M$, $m < n$, and $\mathbf{i} \in \gamma_m \neq \emptyset$.*

Then $\mathbf{i}_{[m]} \subseteq [n+1, 2n]$, and $\mathbf{i}_{[m+1, 2m]} \subseteq [n+1, 2n]$.

Proof. We only consider the first statement; the second one follows similarly.

Assume to the contrary that $\mathbf{i}_{[m]} \not\subseteq [n+1, 2n]$ holds. By Lemma 5.2 we have $\mathbf{i}_{[m]} \subseteq [n]$.

If $|\mathbf{i}_{[m]}| \leq 2$ then there is some $j \in [m]$ with $i_j = i_{j+1 \bmod m}$, implying $x_{i_j} = 0$. But by construction of $R_n^{0,2}$ we have

$$(0, \dots, 0, \underset{i_j}{\uparrow} 1, 0, \dots, 0) \in R_n^{0,2}.$$

Thus $|\mathbf{i}_{[m]}| \geq 3$. Since $R_{C,n}^{0,2}$ has a circular structure, and $m \leq n-2$, we have some $j, j' \in [m]$ with $j' = j+1 \bmod m$ and $|i_j - i_{j'} \bmod n| \geq 2$. But $i_j, i_{j'} \in [n]$ and by construction of $R_n^{0,2}$ (or, more specifically $R_{C,n}^{0,2}$) this means, that $\rho_{0,2}(x_{i_j}, x_{i_{j'}})$ holds, i.e., x_{i_j} and $x_{i_{j'}}$ can not both be 1 at the same time. But by construction of $R_n^{0,2}$ we have

$$(0, \dots, 0, \underset{i_j}{\uparrow} 1, 0, \dots, 0, \underset{i_{j'}}{\uparrow} 1, 0, \dots, 0) \in R_n^{0,2}.$$

This is a contradiction, and thus $\mathbf{i}_{[m]} \not\subseteq [n]$. \square

This shows that on the right hand side of (5.2) the variables x_1, \dots, x_n are inessential. But this contradicts the fact, that these variables are essential in $R_n^{0,2}$. Thus follows:

Theorem 5.5. $\text{pPol } R_n^{0,2} \not\supseteq \bigcap_{m \in M} \text{pPol } R_m^{0,2}$.

Corollary 5.6. *Let $X, Y \subseteq \hat{\mathbb{N}}$ be non-empty sets. Then*

$$\bigcap_{n \in X} \text{pPol } R_n^{0,2} = \bigcap_{m \in Y} \text{pPol } R_m^{0,2} \iff X = Y.$$

5.2. The subclones of $T_{0,2}$. Now we look at the intervals $\mathcal{I}_{\text{Str}}(D)$ for all subclones $D \subseteq T_{0,2}$. We use the fact that $T_{0,2} \subseteq T_0 = \text{Pol}\{0\}$, and let $T := \text{cPol}(\{0\}, \emptyset)$. In this way the conditions (i) and (ii) of Lemma 3.13 are fulfilled due to Corollary 3.12. The only condition left to show is (iii) for the set I defined by

$$I := \left\{ \bigcap_{n \in X} \text{pPol } R_n^{0,2} \mid X \subseteq \hat{\mathbb{N}}, X \neq \emptyset \right\}.$$

Lemma 5.7. *Let $n \in \hat{\mathbb{N}}$ and $M \subseteq \hat{\mathbb{N}} \setminus \{n\}$.*

Then $\text{pPol } R_n^{0,2} \not\supseteq T \cap \bigcap_{m \in M} \text{pPol } R_m^{0,2}$.

Proof. We need to show that there is some

$$F \in \left(T \cap \bigcap_{m \in M} \text{pPol } R_m^{0,2} \right) \setminus \text{pPol } R_n^{0,2}.$$

By Theorem 5.5 we have that there is some l -ary partial function

$$f \in \left(\bigcap_{m \in M} \text{pPol } R_m^{0,2} \right) \setminus \text{pPol } R_n^{0,2}.$$

If $\mathbf{0} \notin \text{dom } f$, then $F := f \in T$ and thus we are done.

We now assume that $\mathbf{0} \in \text{dom } f$. Since $f \notin \text{pPol } R_n^{0,2}$ there is some matrix M with columns $\mathbf{x}_1, \dots, \mathbf{x}_l \in R_n^{0,2}$ and rows $\mathbf{y}_1, \dots, \mathbf{y}_{2n}$ such that

- $f(\mathbf{x}_1, \dots, \mathbf{x}_l) \notin R_n^{0,2}$, and
- $\mathbf{y}_1, \dots, \mathbf{y}_{2n} \in \text{dom } f$.

Let $f' \leq f$ be defined by $\text{dom } f' := \{\mathbf{y}_1, \dots, \mathbf{y}_{2n}\}$ and $f'(\mathbf{y}_i) := f(\mathbf{y}_i)$ for all $i \in [2n]$. Thus we see that

$$f' \in \left(\bigcap_{m \in M} \text{pPol } R_m^{0,2} \right) \setminus \text{pPol } R_n^{0,2}.$$

If $\mathbf{0} \notin \text{dom } f'$, then $F := f' \in T$ and thus we are done.

Thus there is some $j \in [2n]$, such that $\mathbf{y}_j = \mathbf{0}$. We define the $(l+1)$ -ary partial function g by

$$\begin{aligned} \text{dom } g &:= \{(1, \mathbf{y}_j)\} \cup \{(0, \mathbf{y}_i) \mid i \in [2n] \setminus \{j\}\}, \\ g(1, \mathbf{y}_j) &:= f'(\mathbf{y}_j), \\ g(0, \mathbf{y}_i) &:= f'(\mathbf{y}_i) \text{ for all } i \in [2n] \setminus \{j\}. \end{aligned}$$

As $g \leq \nabla f' \leq \nabla f$ we see that $g \in \bigcap_{m \in M} \text{pPol } R_m^{0,2}$.

Because $\mathbf{x}_0 := (0, \dots, 0, \underset{j}{1}, 0, \dots, 0) \in R_n^{0,2}$ we have

$$g(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_l) = f(\mathbf{x}_1, \dots, \mathbf{x}_l) \notin R_n^{0,2},$$

and all rows of the matrix formed by the columns $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_l$ belong to the domain of g . Therefore $g \notin \text{pPol } R_n^{0,2}$ holds, and this implies

$$g \in \left(\bigcap_{m \in M} \text{pPol } R_m^{0,2} \right) \setminus \text{pPol } R_n^{0,2}.$$

If $\mathbf{0} \notin \text{dom } g$, then $F := g \in T$ and thus we are done. Otherwise, repeating the steps from f' to g yields finally a desired F . \square

Corollary 5.8. *Let $X, Y \in I$ with $X \neq Y$. Then $X \cap T \neq Y \cap T$.*

Theorem 5.9. *Let $D \subseteq T_{0,2}$ be a clone on $\text{Op}(\mathbb{2})$.*

Then $\mathcal{I}_{\text{Str}}(D)$ has the cardinality of the continuum.

Proof. By Corollary 5.8 and the properties of T mentioned at the beginning of this subsection all conditions of Lemma 3.13 hold, and therefore $|\mathcal{I}_{\text{Str}}(D)| \geq |I|$. \square

6. Continuum on Λ

From the results of the previous sections we see that the clones Λ , $\Lambda \cap T_1$, V , and $V \cap T_0$ are the only clones for which we need to determine the size of $\mathcal{I}_{\text{Str}}(C)$. We will show in this section that $\mathcal{I}_{\text{Str}}(\Lambda)$ and $\mathcal{I}_{\text{Str}}(\Lambda \cap T_1)$ have both the cardinality of the continuum. By the symmetry of Post's lattice this implies the same statement for $\mathcal{I}_{\text{Str}}(V)$ and $\mathcal{I}_{\text{Str}}(V \cap T_0)$.

Creignou, Kolaitis and Zanuttini have given in [4] the set of relations defining the smallest element in the interval $\mathcal{I}_{\text{Str}}(C)$ for each Boolean clone C . They call this the plain basis. Since the least element in $\mathcal{I}_{\text{Str}}(C)$ is $\text{Str}(C)$ for each total clone C , we can conclude from [4] that

$$\text{Str}(\Lambda) = \text{pPol}\{\lambda_k \mid k \geq 1\}$$

where $\lambda_k(y, x_1, \dots, x_k) \equiv (y \vee \neg x_1 \vee \dots \vee \neg x_k)$. Equivalently, $\lambda_k = 2^{k+1} \setminus \{(0, 1, \dots, 1)\}$. The clone Λ is denoted by E in [4], and the plain basis can be found in the entry IE of Table 2 in [4].

Any n -ary relation ρ in the partial co-clone of $\text{Str}(\Lambda)$ can be constructed from a selection of λ_k , i.e., there are (possibly empty) $(k+1)$ -ary auxiliary relations γ_k on $[n]$ for each $k \geq 1$ such that

$$\rho(x_1, \dots, x_n) = \bigwedge_{k \geq 1} \bigwedge_{\mathbf{i} \in \gamma_k} \lambda_k(x_{i_1}, \dots, x_{i_{k+1}}). \quad (6.1)$$

Since λ_k is totally symmetric on the last k coordinates, and $\lambda_k(y, x_1, \dots, x_k) = \lambda_{k+1}(y, x_1, x_1, \dots, x_k)$, the tuples $\mathbf{i} \in \gamma_k$ can be represented by pairs of the form $(i_1, \{i_j \mid j \in [2, k+1]\})$. This notation makes the symmetry of the relation more obvious, and exposes the special element more visibly.

For such pairs (i, J) with $i \in [n]$ and $J \subseteq [n]$ we can define the n -ary relation $\lambda_{(i,J)}^n$ by

$$\lambda_{(i,J)}^n(x_1, \dots, x_n) \equiv (x_i \vee \bigvee_{j \in J} \neg x_j).$$

We note that $\lambda_{(i,J)}^n = 2^n$ whenever $i \in J$, due to the tautology $x_i \vee \neg x_i$ in the definition of $\lambda_{(i,J)}^n$.

Let $\Gamma \subseteq \{(i, J) \mid i \in [n], J \subseteq [n], i \notin J\}$. Then we define the relation $\lambda_\Gamma^n \in \text{Rel}^{(n)}(2)$ by

$$\lambda_\Gamma^n := \bigwedge_{(i,J) \in \Gamma} \lambda_{(i,J)}^n(x_1, \dots, x_n).$$

Then equation (6.1) holds if and only if there is some suitable Γ with

$$\rho(x_1, \dots, x_n) = \lambda_\Gamma^n.$$

Lemma 6.1. *Let $i \in [n]$, $J \subseteq J' \subseteq [n]$. Then $\lambda_{(i,J)}^n \subseteq \lambda_{(i,J')}^n$.*

Proof. Follows from the definition. \square

6.1. Monsters. In this subsection we define some relations R_m^Λ for $m \geq 3$, for which we show that they are independent. These are used to show that $\mathcal{I}_{\text{Str}}(\Lambda)$ has the cardinality of the continuum. The relations R_m^Λ will be called *monsters*, as they “kill” this problem.

Let $m \geq 3$. We define $\Gamma_m \subseteq \{(i, J) \mid i \in [m+1], J \subseteq [m+1], i \notin J\}$ and $R_m^\Lambda \in \text{Rel}^{(m+1)}(2)$ by

$$\begin{aligned} \Gamma_m &:= \{(1, [2, m+1])\} \cup \\ &\quad \{(i, \{1, j_1, j_2\}) \mid i, j_1, j_2 \in [2, m+1], |\{i, j_1, j_2\}| = 3\}, \\ R_m^\Lambda &:= \lambda_{\Gamma_m}^{m+1}. \end{aligned}$$

A more visual representation of Γ_4 and R_4^Λ is given in Table 2.

Furthermore, we define the ternary relation R_Λ^Λ by

$$R_\Lambda^\Lambda := \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 1, 1)\}.$$

As shown by Blochina in [2] (see also Section 10.2 [12]) the relation R_Λ^Λ characterizes the clone Λ , i.e.,

$$\Lambda = \text{Pol } R_\Lambda^\Lambda.$$

i	J	1	2	3	4	5
1	2, 3, 4, 5	0	1	1	1	1
2	1, 3, 4	1	0	1	1	
2	1, 3, 5	1	0	1		1
2	1, 4, 5	1	0		1	1
3	1, 2, 4	1	1	0	1	
3	1, 2, 5	1	1	0		1
3	1, 4, 5	1		0	1	1
4	1, 2, 3	1	1	1	0	
4	1, 2, 5	1	1		0	1
4	1, 3, 5	1		1	0	1
5	1, 2, 3	1	1	1		0
5	1, 2, 4	1	1		1	0
5	1, 2, 5	1		1	1	0

TABLE 2. Visual representation of Γ_4 and of the forbidden tuples in R_4^Λ . For example, the condition $(2, \{1, 4, 5\})$ forbids the tuples $(1, 0, x_3, 1, 1)$ for all $x_3 \in \mathbb{2}$. This means that $(1, 0, 0, 1, 1), (1, 0, 1, 1, 1) \notin R_4^\Lambda$.

Now we give some properties of the relations R_m^Λ .

Lemma 6.2. *Let $m \geq 3$. Then $\text{pPol } R_m^\Lambda \subseteq \text{pPol } R_\Lambda^\Lambda$.*

Proof. We have the following connections:

$$\begin{aligned}
\lambda_{\{(1, \{2, 3\}), (2, \{1, 3\})\}}^3(x_1, x_2, x_3) &= R_m^\Lambda(x_1, x_2, x_3, \dots, x_3) \\
\lambda_{(1, \{2\})}^2(x_1, x_2) &= R_m^\Lambda(x_1, x_2, \dots, x_2) \\
\lambda_1(x_1, x_2) &= \lambda_{(1, \{2\})}^2(x_1, x_2) \\
R_\Lambda^\Lambda(x_1, x_2, x_3) &= \lambda_{\{(1, \{2, 3\}), (2, \{1, 3\})\}}^3(x_1, x_2, x_3) \wedge \\
&\quad \lambda_1(x_2, x_1) \wedge \lambda_1(x_3, x_1)
\end{aligned}$$

Therefore R_Λ^Λ is constructible from R_m^Λ as

$$R_\Lambda^\Lambda = \{\mathbf{x} \in \mathbb{2}^3 \mid \mathbf{x}_{(1, 2, 3, \dots, 3)}, \mathbf{x}_{(2, 1, \dots, 1)}, \mathbf{x}_{(3, 1, \dots, 1)} \in R_m^\Lambda\}.$$

Thus $\text{pPol } R_m^\Lambda \subseteq \text{pPol } R_\Lambda^\Lambda$ by Theorem 2.1. \square

Corollary 6.3. *Let $m \geq 2$. Then $\text{Pol } R_m^\Lambda = \Lambda$, i.e., $\text{pPol } R_m^\Lambda \in \mathcal{I}_{\text{Str}}^\subseteq(\Lambda)$.*

Proof. Since R_m^Λ can be constructed from $\{\lambda_k \mid k \geq 1\}$ and $\text{pPol}\{\lambda_k \mid k \geq 1\} = \text{Str}(\Lambda)$ we have $\Lambda \subseteq \text{pPol } R_m^\Lambda$, and thus $\Lambda \subseteq \text{pPol } R_m^\Lambda \cap \text{Op}(\mathbb{2})$.

On the other hand we have $\text{pPol } R_m^\Lambda \subseteq \text{pPol } R_\Lambda^\Lambda$ and thus $\text{pPol } R_m^\Lambda \cap \text{Op}(\mathbb{2}) \subseteq \text{pPol } R_\Lambda^\Lambda \cap \text{Op}(\mathbb{2}) = \text{Pol } R_\Lambda^\Lambda = \Lambda$. \square

Lemma 6.4. *Let $m \geq 3$. Then the following properties hold.*

$$(i) \ (1, \dots, 1) \in R_m^\Lambda.$$

- (ii) $(1, \dots, 1, \underset{\uparrow i}{0}, 1, \dots, 1) \notin R_m^\Lambda$ for all $i \in [m+1]$.
- (iii) $\{0\} \times (2^m \setminus \{(1, \dots, 1)\}) \subseteq R_m^\Lambda$.
- (iv) $(0, \dots, 0, \underset{\uparrow i}{1}, 0, \dots, 0, \underset{\uparrow j}{1}, 0, \dots, 0) \in R_m^\Lambda$ for all $i, j \in [m+1]$ with $i < j$.

Proof.

- (i) Since $\mathbf{1} \in \lambda_{(i,J)}$ for any i and J , we have $\mathbf{1} \in R_m^\Lambda$.
- (ii) If $i = 1$, then $(0, 1, \dots, 1) \notin \lambda_{(1,[2,m+1])}^{m+1} \supseteq R_m^\Lambda$. Otherwise, if $\mathbf{x} = (1, \dots, 1, 0, 1, \dots, 1)$ then $\mathbf{x}_{(i,1,j_1,j_2)} = (0, 1, 1, 1) \notin \lambda_{(i,\{1,j_1,j_2\})}^4$. Thus $\mathbf{x} \notin \lambda_{(i,\{1,j_1,j_2\})}^{m+1} \supseteq R_m^\Lambda$.
- (iii) By the definition of $\lambda_{(i,J)}^{m+1}$ we see that $(0, x_2, \dots, x_{m+1}) \in \lambda_{(i,J)}^{m+1}$ if $\mathbf{1} \in J$. Thus

$$\begin{aligned} (\{0\} \times 2^m) \cap R_m^\Lambda &= (\{0\} \times 2^m) \cap \lambda_{(1,[2,m+1])}^{m+1} \\ &= \{0\} \times (2^m \setminus \{(1, \dots, 1)\}). \end{aligned}$$

- (iv) Let $\mathbf{x} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$. Since the set J contains at least three elements for every condition $\lambda_{(i,J)}^{m+1}$ in the construction of R_m^Λ , there is some $j \in J$ with $\mathbf{x}_j = 0$. Thus $\mathbf{x} \in \lambda_{(i,J)}^{m+1}$, and consequently $\mathbf{x} \in R_m^\Lambda$. \square

6.2. Monsters are good. Similar to the case of $T_{0,2}$ we want to show that there are continuum many strong partial clones with total part equal to Λ .

Let $\hat{\mathbb{N}} := \{n \in \mathbb{N} \mid n \geq 3\}$. Let $n \in \hat{\mathbb{N}}$ and $M \subseteq \hat{\mathbb{N}} \setminus \{n\}$ for the rest of this section. We want to show that

$$\text{pPol } R_n^\Lambda \not\supseteq \bigcap_{m \in M} \text{pPol } R_m^\Lambda \quad (6.2)$$

holds. We assume to the contrary, that (6.2) is false. This means that by Theorem 2.1 we can write

$$R_n^\Lambda := \{\mathbf{x} \in 2^{n+1} \mid \mathbf{x}_i \in R_m^\Lambda \text{ for all } i \in \gamma_m \text{ and } m \in M\} \quad (6.3)$$

with some auxiliary relations γ_m for all $m \in M$. Furthermore, we can assume that no condition is superfluous.

Lemma 6.5. *Let $m \geq 3$, $i \in \gamma_m$, and distinct $j, j' \in [m+1]$. Then $i_j \neq i_{j'}$.*

Proof. Assume to the contrary that there are distinct $j, j' \in [m+1]$ with $i_j = i_{j'}$.

There are a few cases distinguished by the size of the set $[i]$. For each $x \in [m+1]$ let $t_x := \{y \in [m+1] \mid i_y = i_x\}$.

- $|i| = 1$. Since $(0, \dots, 0), (1, \dots, 1) \in R_m^\Lambda$ the condition $\mathbf{x}_i \in R_m^\Lambda$ is superfluous in contradiction to the assumption for (6.3).
- $|i| = 2$. We have three subcases.

- $|t_1| = 1$. Let $\mathbf{x} := (0, \dots, 0, \underset{\uparrow i_2}{1}, 0, \dots, 0) \in R_n^\Lambda$. But we have $\mathbf{x}_i = (0, 1, \dots, 1) \notin R_m^\Lambda$, i.e., this case can not appear in the construction of R_n^Λ .
- $|t_1| = 2$. For each constraint $\lambda_{x,Y}$ in the construction of R_m^Λ we have some $y \in Y \setminus t_1$. Thus $\{x, y\} \subseteq [m+1] \setminus t_1$, i.e., these coordinates get identified. Therefore this constraint is superfluous. Since this holds for every such constraint the complete condition $\mathbf{x}_i \in R_m^\Lambda$ is superfluous.
- $|t_1| \geq 3$. Let $\{1, y_2, y_3\} := Y \subseteq t_1$ with $|Y| = 3$. Let $z := \min([m+1] \setminus t_1)$, and define $\mathbf{x} := (0, \dots, 0, \underset{\uparrow i_1}{1}, 0, \dots, 0) \in R_n^\Lambda$.
From $R_m^\Lambda \subseteq \lambda_{z,Y}^{m+1}$ and $(\mathbf{x}_i)_{(z,1,y_2,y_3)} = (0, 1, 1, 1) \notin \lambda_{1,\{2,3,4\}}^4$ follows that $\mathbf{x}_i \notin R_m^\Lambda$. This contradicts $\mathbf{x} \in R_n^\Lambda$ and therefore this case can not happen.

- $|\mathbf{i}| \geq 3$.

Since there are distinct $j, j' \in [m+1]$ with $i_j \neq i_{j'}$ there is some $x \in [m+1]$ with $|t_1 \cup t_x| \geq 3$.

Let $t' := t_1 \cup t_x$. Since $|\mathbf{i}| \geq 3$ we have $t' \neq [\mathbf{i}]$, and thus the proof for $|\mathbf{i}| = 2$ and $|t_1| \geq 3$ works if we replace t_1 by t' .

□

Corollary 6.6. *Let $m > n$. Then $\gamma_m = \emptyset$.*

Lemma 6.7. *Let $m < n$ and $\mathbf{i} \in \gamma_m$. Then $1 \notin [\mathbf{i}]$.*

Proof. Assume to the contrary, that $1 \in [\mathbf{i}]$. By Lemma 6.5 there are no identifications, i.e., $i_j \neq i_{j'}$ for all distinct $j, j' \in [m+1]$.

There are two cases

- $i_1 = 1$. We may assume w.l.o.g. that $i_x = x$ for all $x \in [m+1]$.
We define $\mathbf{x} := (0, \underbrace{1, \dots, 1}_m, 0, \dots, 0)$. Then $\mathbf{x}_i = (0, 1, \dots, 1) \notin R_m^\Lambda$ but $\mathbf{x} \in R_n^\Lambda$. Thus this contradicts (6.3).
- $i_j = 1$ for some $j \in [2, m+1]$. W.l.o.g. let $j = 2$.
Let $\{u_1, u_2, u_3\} := \{i_1, i_3, i_4\}$ with $u_1 < u_2 < u_3$ and define

$$\mathbf{x} := (0, 0, \dots, 0, \underset{\uparrow u_1}{1}, 0, \dots, 0, \underset{\uparrow u_2}{1}, 0, \dots, 0, \underset{\uparrow u_3}{1}, 0, \dots, 0).$$

Then $\mathbf{x}_i = (1, 0, 1, 1, 0, \dots, 0) \notin R_m^\Lambda$ since $R_m^\Lambda \subseteq \lambda_{2,\{1,3,4\}}^{m+1}$. But $\mathbf{x} \in R_n^\Lambda$. Thus this contradicts (6.3).

□

Theorem 6.8. *Let $n \in \hat{\mathbb{N}}$ and $M \subseteq \hat{\mathbb{N}} \setminus \{n\}$.*

Then $\text{pPol } R_n^\Lambda \not\supseteq \bigcap_{m \in M} \text{pPol } R_m^\Lambda$.

Proof. From Corollary 6.6 and Lemma 6.7 follows that $1 \notin [i]$ for all $i \in \gamma_m$ and $m \in M$. Thus in the right hand side of (6.3) the variable x_1 is inessential. But this contradicts the fact, that this variables is essential in R_n^Λ . Therefore (6.3) is not true, and by Theorem 2.1 follows the statement of this theorem. \square

Corollary 6.9. *Let $X, Y \subseteq \hat{\mathbb{N}}$ be non-empty sets. Then*

$$\bigcap_{n \in X} \text{pPol } R_n^\Lambda = \bigcap_{m \in Y} \text{pPol } R_m^\Lambda \iff X = Y.$$

From this follows that I has the cardinality of the continuum and with $I \subseteq \mathcal{I}_{\text{Str}}(\Lambda)$ we obtain the following statement.

Theorem 6.10. *The interval $\mathcal{I}_{\text{Str}}(\Lambda)$ has the cardinality of the continuum.*

Theorem 6.11. *The interval $\mathcal{I}_{\text{Str}}(\Lambda \cap T_1)$ has the cardinality of the continuum.*

Proof. We have $c_0 \in \Lambda \setminus (\Lambda \cap T_1)$. Thus Lemma 3.16 is applicable, and by 6.10 follows that $\mathcal{I}_{\text{Str}}(\Lambda \cap T_1)$ has the cardinality of the continuum. \square

7. Conclusion

Combining Theorems 5.9, 6.10, 6.11, 4.7, and 1.1 we obtain the Dichotomy result for intervals of strong partial clones.

Theorem 7.1. *Let C be a total Boolean clone.*

Then $\mathcal{I}_{\text{Str}}(C)$ is either finite or has the cardinality of the continuum. Furthermore, $\mathcal{I}_{\text{Str}}(C)$ is finite if and only if $M \cap T_0 \cap T_1 \subseteq C$ or $S \cap T_0 \cap T_1 \subseteq C$.

7.1. Open questions. Does the dichotomy between finite intervals and intervals of continuum cardinality also hold if we consider the clones on some set A with $|A| \geq 3$? Or, does there exist some finite set A and some total clone C in $\text{Op}(A)$ such that the interval $\mathcal{I}_{\text{Str}}(C)$ is countably infinite? Another question in this direction is concerning the two different intervals $\mathcal{I}(C)$ and $\mathcal{I}_{\text{Str}}(C)$ for some total clone C in $\text{Op}(A)$. Clearly, $|\mathcal{I}_{\text{Str}}(C)| \leq |\mathcal{I}(C)|$ holds. In the Boolean case for each total clone C either both intervals are finite, or both intervals have the cardinality of the continuum. But is this also the case for every A with $|A| \geq 3$?

For some subclones of L , and (in principle) all subclones of Λ and V , respectively, we have shown a strong relation between the intervals. Let $C \in \{L, \Lambda, V\}$ and D a total Boolean clone with $D \subseteq C$ and $D \notin \{C_{01}, \Omega_1\}$. Then there is some partial class T , such that

$$(X \cap T) \cup \text{Str}(D) \subseteq (Y \cap T) \cup \text{Str}(D) \iff X \subseteq Y$$

and

$$(X \cap T) \cup \text{Str}(D) \in \mathcal{I}_{\text{Str}}(D)$$

hold for all $X, Y \in \mathcal{I}_{\text{Str}}(C)$. This means that there is some order-preserving embedding of the interval $\mathcal{I}_{\text{Str}}(C)$ into $\mathcal{I}_{\text{Str}}(D)$. The author would be interested, if such an embedding is possible for all pairs of Boolean clones C and

D with $D \subseteq C$? Since in this paper the structure of the lattice was used explicitly, for example for the subclones of L , a more difficult question arises: If the embedding is possible, can this be proven in general without directly using the description of all clones? What about this statement for some A with $|A| \geq 3$?

The partial classes introduced in Section 3 are an equivalent of the classes considered by Harnau in [9–11]. In there Harnau presents the Galois connection and also describes the closure operator for the relation pairs. The difference on the relational side between clones and strong partial clones is the omission of the projection operator. Does this also work when switching from classes to strong partial classes?

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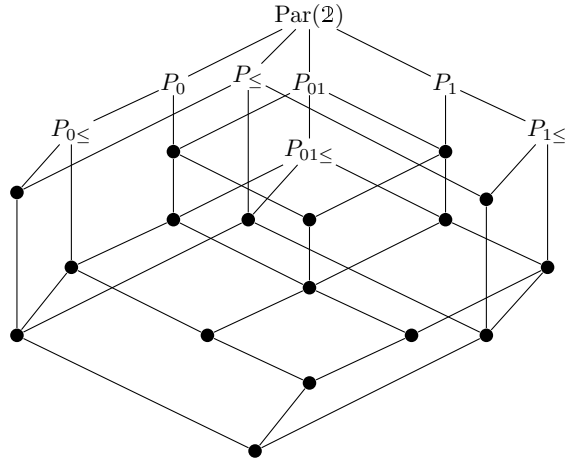
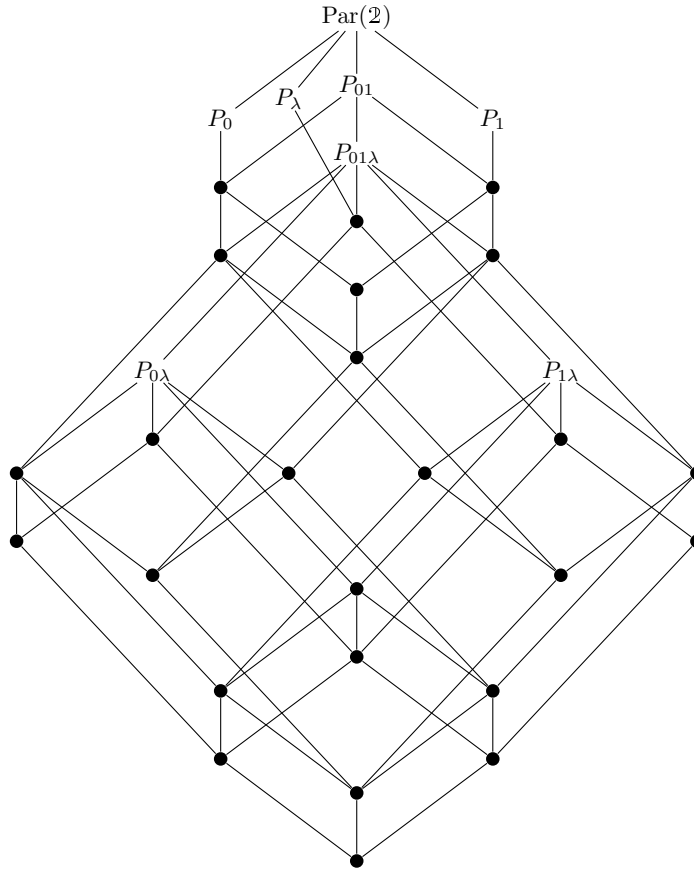
Appendix A. Finite intervals of strong clones

In the Figures 2 and 3 we present the two finite intervals $\mathcal{I}_{\text{Str}}^{\subseteq}(M \cap T_0 \cap T_1)$ and $\mathcal{I}_{\text{Str}}^{\subseteq}(S \cap T_0 \cap T_1)$, respectively. These were given in [8], but the drawings have been improved to show the structure in a better way. The following shorthand notation is used for some of these partial Boolean clones. All unlabeled points can be written as the intersection of some of these.

$$\begin{aligned} P_a &:= \text{pPol}\{a\} \text{ for } a \in \{0, 1\} \\ P_{01} &:= \text{pPol}\{(0, 1)\} \\ P_{\leq} &:= \text{pPol}\{(0, 0), (0, 1), (1, 1)\} \\ P_{a\leq} &:= \text{pPol}\{(a, 0, 0), (a, 0, 1), (a, 1, 1)\} \text{ for } a \in \{0, 1\} \\ P_{01\leq} &:= \text{pPol}\{(0, 1, 0, 0), (0, 1, 0, 1), (0, 1, 1, 1)\} \\ P_{\lambda} &:= \text{pPol}\{(0, 1), (1, 0)\} \\ P_{a\lambda} &:= \text{pPol}\{(a, 0, 1), (a, 1, 0)\} \text{ for } a \in \{0, 1\} \\ P_{01\lambda} &:= \text{pPol}\{(0, 1, 0, 1), (0, 1, 1, 0)\} \end{aligned}$$

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FIGURE 2. The interval $\mathcal{I}_{\text{Str}}^{\subseteq}(M \cap T_0 \cap T_1)$ FIGURE 3. The interval $\mathcal{I}_{\text{Str}}^{\subseteq}(S \cap T_0 \cap T_1)$