Interaction between criteria

In classical MCDM procedures (ELECTRE & PROMETHEE ...) we consider additive weights : $p(i)$ related to criterion i such that

$$\sum_{i=1,\ldots,n} p(i) = 1$$

in order to introduce compensation via weighted sums in aggregation phase :

$$\begin{aligned} C(a,b) &= \sum_{i=1,\ldots,n} p(i) r_{ab}^{(i)} = p_{ab}^{+} + p_{ab}^{=} \\ r_{ab}^{(i)} &= 1 \text{ if } a \geq_i b \\ &= 0 \text{ otherwise.} \end{aligned}$$

Additive model : $\begin{cases} \mu(1,\ldots,n) = 1 \\ \mu(i,j) = \mu(i) + \mu(j) \\ \mu(i) = p(i). \end{cases}$

This model can be extended in a Choquet capacity (fuzzy measure, Sugeno measure) :

X : set of criteria $\quad \mathcal{P}(X)$: power set of X

$$\mu : \mathcal{P}(X) \to [0,1]$$

$$\begin{cases} \mu(P) \geq 0, & P \in \mathcal{P}(X) \\ \mu(\emptyset) = 0 & \\ \mu(X) = 1 \ \text{(not compulsary)} & \\ \mu(Q) \leq \mu(P) \text{ if } Q \subseteq P & \end{cases}$$

Pseudo-Boolean functions and Choquet capacity

μ can be rewritten under the form of a *unique* pseudo-Boolean function

$$f(x_1, \ldots, x_n) = \sum_{L \subset X} \left[a(L) \prod_{i \in L} x_i \right], \quad x_i \in \{0, 1\}$$

$$\begin{aligned}
\mu(P) &= f(\ldots \underset{i \in P}{\underset{\uparrow}{\overset{i}{1}}} \ldots \underset{j \notin P}{\underset{\uparrow}{\overset{j}{0}}}\ ) \\
&= f(x_1, \ldots, x_n \mid x_i = 1 \text{ if } i \in P, \\
&\qquad\qquad\qquad\qquad x_i = 0 \text{ otherwise}) \\
&= \sum_{L \subset P} a(L)
\end{aligned}$$

$$
\begin{cases}
\mu(\emptyset) = 0 \\
\mu(i) = a(i) \\
\mu(i,j) = a(i) + a(j) + a(i,j) \\
\vdots \\
\mu(X) = 1
\end{cases}
$$

$$\downarrow$$

$$
\begin{cases}
a(i,j) = 0 : \text{additivity} \\
a(i,j) > 0 : \text{synergy effect (superadditivity)} \\
a(i,j) < 0 : \text{redundancy effect (subadditivity)}
\end{cases}
$$

Möbius transform

$$\mu(P) = \sum_{L \subset P} a(L)$$

$$\begin{cases} a(\emptyset) = 0 \\ \mu(i) = a(i) \\ \mu(i,j) = a(i) + a(j) + a(ij) \\ \vdots \end{cases}$$

$$a(P) = \sum_{L \subset P} (-1)^{|P|-|L|} \mu(L)$$

$$\begin{cases} a(i) = \mu(i) \\ a(i,j) = \mu(\emptyset) - \mu(i) - \mu(j) + \mu(i,j) \\ \vdots \end{cases}$$

From $\mu \rightarrow a$: Möbius transform

$\underline{n=1}$ $$\begin{pmatrix} a(\emptyset) \\ a(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mu(\emptyset) \\ \mu(1) \end{pmatrix}$$

$\tilde{a}_{(1)} = M_{(1)} \tilde{\mu}_{(1)}$.

$\underline{n > 1}$: $\mu(P)$ and $a(P)$ are represented as vectors where the elements of $\mathcal{P}(\cancel{X})$ are represented in the following order :

$$\tilde{\mu}_{(n)} = \{\underbrace{\underline{\mu(\emptyset)}\ \underline{\mu(1)}}_{n=1}\ \underbrace{\mu(2)\ \mu(1,2)}_{n=2}$$

$$\underbrace{\underline{\mu(3)\mu(1,3)\mu(2,3)}\ \underline{\mu(1,2,3)}}_{n=3}\ldots\}$$

$$\tilde{a}_{(n)} = \{a(\emptyset)a(1)a(2)a(1,2)\ldots\}$$

$\underline{n = 2}$

$$\begin{pmatrix} a(\emptyset) \\ a(1) \\ a(2) \\ a(1,2) \end{pmatrix} = \left(\begin{array}{cc|cc} +1 & 0 & 0 & 0 \\ -1 & +1 & 0 & 0 \\ \hline -1 & 0 & +1 & 0 \\ +1 & -1 & -1 & +1 \end{array}\right) \begin{pmatrix} \mu(\emptyset) \\ \mu(1) \\ \mu(2) \\ \mu(1,2) \end{pmatrix}$$

$$\tilde{a}_{(a)} = M_{(2)}\tilde{\mu}_{(2)}$$

Two equivalent representations

$$\tilde{a}_{(1)} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \tilde{\mu}_{(1)} = M_{(1)}\tilde{\mu}_{(1)} \text{ dim. } 2$$

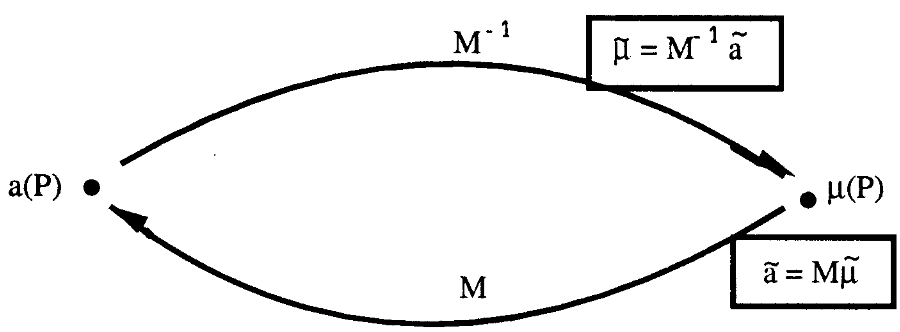
$$\tilde{a}_{(2)} = \left[\begin{array}{c|c} 1M_{(1)} & 0M_{(1)} \\ \hline -1M_{(1)} & 1M_{(1)} \end{array}\right] \tilde{\mu}_{(2)} = M_{(2)}\tilde{\mu}_{(2)} \text{ (dim. } 2^2$$

$$\vdots$$

$$\tilde{a}_{(n)} = \underbrace{\left[\begin{array}{c|c} 1M_{(n-1)} & 0M_{(n-1)} \\ \hline -1M_{(n-1)} & 1M_{(n-1)} \end{array}\right]}_{M_{(n)}} \tilde{\mu}_{(n)} \text{ (dim. } 2^n)$$

$M_{(n)}$ is a triangular fractal matrix : $M_{(1)} \otimes \cdots \otimes M_{(1)}$

(tensorial product of matrix $M_{(1)}$)



Möbius Transform

Weights and power indices

Back to cooperative game theory.

$\mu(P)$: weight of coalition

$$: \begin{cases} 1 \text{ if } P \text{ has the majority} \\ 0 \text{ otherwise} \end{cases}$$

swing related to $\{i\}$:

$$\mu(P) = 0 \nearrow \mu(P \cup \{i\}) = 1$$

Banzhaf power index : "probability" of a swing related to $\{i\}$.

$$b_i(X) = \frac{1}{2^{n-1}} \sum_{P \subset X - \{i\}} [\mu(P \cup \{i\}) - \mu(P)]$$

$2^{n-1} = |\mathcal{P}(X - \{i\})|$.

b_i has some nice properties :

1) If $\{i\}$ is a dummy :
$\mu(P \cup \{i\}) = \mu(P), \forall P \subset X - \{i\} \Rightarrow$
$b_i(X) = 0,$
the real weight of $\{i\}$ is zero (even if $\mu(i) > 0$).

2) If $\{i\}$ and $\{j\}$ are symmetric players :
$\mu(P \cup \{i\}) = \mu(P \cup \{j\}), \forall P \subset X - \{i, j\} \Rightarrow b_i(X) = b_j(X)$
they have the same real power (even if $\mu(i) \neq \mu(j)$).

3) Hammer and Holzman proved in 1992

$$\min_{p_i} \sum_{R \subset X} [\mu(R) - \hat{\mu}(R)]^2$$

with $\hat{\mu}(R) = \sum_{i \subset R} p_i$ (related additive measure)
is such that $p_i = b_i(X)$ but

$$\hat{\mu}(X) = \sum_{i \in X} b_i(X) \neq 1 \quad !!!$$

Example related to Owen (1968)

4 stock holders in a company :

(1) : 10 shares of stock
(2) : 30 shares
(3) : 30 shares
(4) : 40 shares

weight is : $\frac{10}{110}$; $\frac{30}{110}$; $\frac{30}{110}$; $\frac{40}{110}$ (9%; 27%; 27%; 36%)

Consider now :

$$
\begin{array}{lll}
\mu(1)=0 & \mu(1,2)=0 & \mu(i,j,k)=1,\ \forall i,j,k \\
\mu(2)=0 & \mu(1,3)=0 & \\
\mu(3)=0 & \mu(1,4)=0 & \\
\mu(4)=0 & \mu(2,3)=1 & \\
 & \mu(2,4)=1 & \\
 & \mu(3,4)=1 &
\end{array}
$$

$$
\begin{cases}
b_1(X) & = 0 \\
b_2(X) & = \frac{1}{8}\{\overbrace{\mu(23) - \mu(3)}^{\text{swing}} + \mu(24) - \mu(4) \\
& \quad + \mu(124) - \mu(14) + \mu(123) - \mu(13)\} \\
& = \frac{4}{8} \\
b_3(X) & = \frac{4}{8} \\
b_4(X) & = \frac{1}{8}\{\mu(24) - \mu(2) + \mu(34) - \mu(4) \\
& \quad + \mu(124) - \mu(12) + \mu(134) - \mu(13)\} \\
& = \frac{4}{8}
\end{cases}
$$

We switch from the weights (9%; 27%; 27%; 36%) to the power indices $(0, \underbrace{.50, .50, .50}_{\text{same weight}})$

Back to MCDM

• Practically, it is useless to think on a general Choquet capacity to determine the weights of criteria including interaction between criteria.

For n criteria, $2^{n\cancel{-1}}$ weights should be determined !!

$\mu(1), \ldots, \mu(n), \mu(1,2), \mu(1,3), \ldots, \mu(1,2,3), \ldots$

We might restrict ourselves to a second order model :

$$a(P) = 0, \ \forall P : |P| > 2$$

$(\Leftrightarrow J(P) = 0, \ \forall P : |P| > 2)$

$$2^{n\cancel{-1}} \searrow \begin{pmatrix} n+1 \\ 2 \end{pmatrix} = \frac{n(n+1)}{2} \text{ parameters.}$$

• Extension of ELECTRE II using interaction.

We have seen that

$$C(a,b) = \sum_{i=1,\dots,n} p(i) r_{ab}^{(i)} = \sum_{\substack{i \\ a \geq_i b}} p_i$$

The weighted sum can be extended to a Choquet integral.

Def. : If $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$, the Choquet integral for $(x_1, \dots, x_n)$:

$$\begin{aligned} C[x_1, \dots, x_n] \quad = \quad & x_{(1)}[1 - \mu(2, \dots, n)] \\ & + x_{(2)}[\mu(2, \dots, n) - \mu(3, \dots, n)] \\ & + x_{(3)}[\mu(3, \dots, n) - \mu(4, \dots, n)] \\ & \quad \vdots \\ & + x_{(n)}[\mu(n)]. \end{aligned}$$

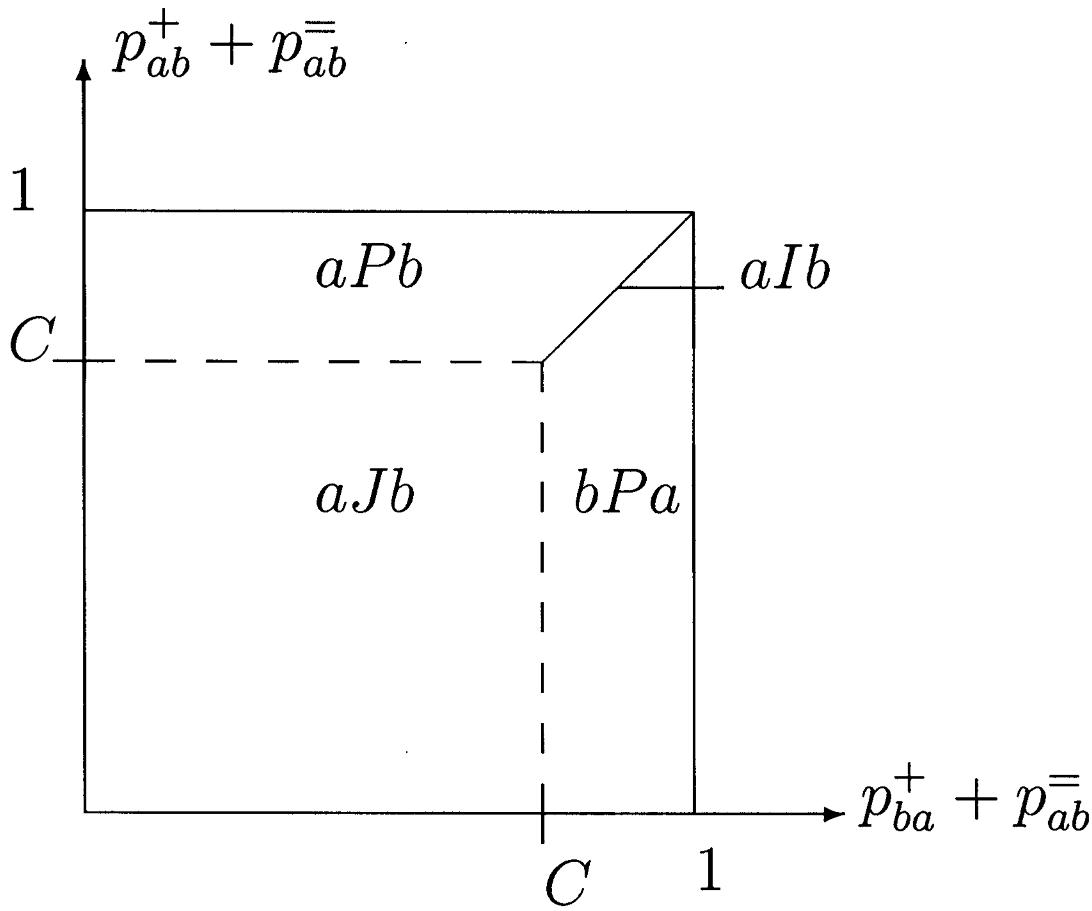
$$(x_1, \dots, x_n) \to (r_{ab}^{(1)} \dots r_{ab}^{(n)}) \in \{0,1\}^n$$

$$C[r_{ab}^{(1)} \dots r_{ab}^{(n)}] = \mu(i_1, \dots, i_k \mid a \geq_{i_1} b, \dots, a \geq_{i_k} b)$$

$$\sum_{i=1,\ldots,n} p(i)r_{ab}^{(i)} = p_{ab}^{+} + p_{ab}^{=}$$

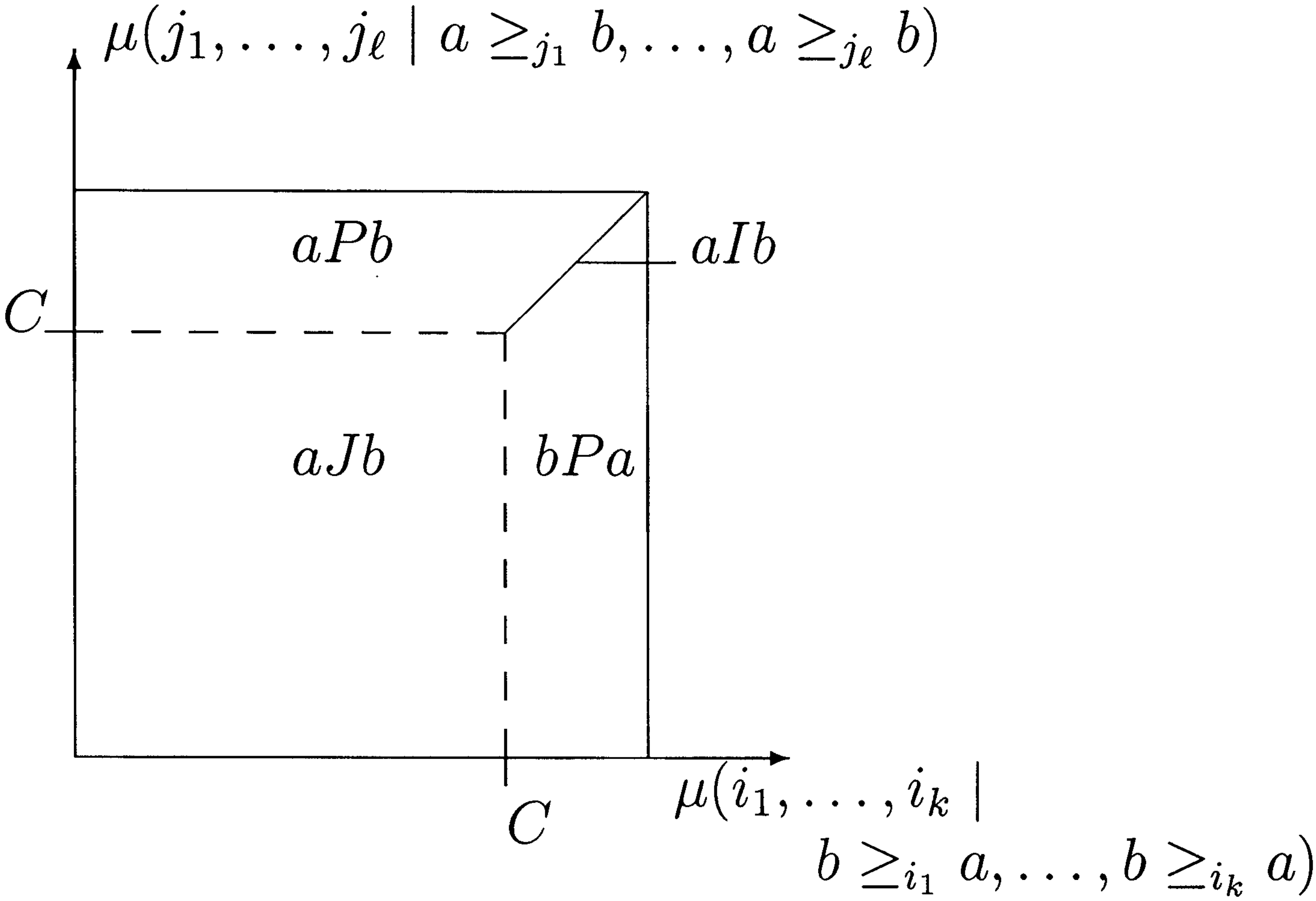
$$\Rightarrow \quad \mu(i_1, \ldots, i_k \mid a \geq_{i_1} b, \ldots, a \geq_{i_k} b)$$

Classical ELECTRE II (Concordance test)



(concordance level)

ELECTRE II with interaction between criteria



Interaction in a formal way

$$f(\underline{x}) = f(x_1, \ldots, x_n) = \sum_{L \subset X} \left[a(L) \prod_{i \in L} x_i \right];$$

$$\mu(P) = \sum_{L \subset P} a(L)$$

$$\begin{aligned}
\delta_i f(\underline{x}) &= f(x_1, \ldots, x_i = 1, \ldots, x_n) \\
&\quad - f(x_1, \ldots, x_i = 0, \ldots, x_n) \\
\delta_{ij} f(\underline{x}) &= \delta_i \left[\delta_j f(\underline{x}) \right] \\
&\vdots \\
\delta_P f(\underline{x}) &= \delta_{i_1} \left[\delta_{i_2 \ldots i_p} f(\underline{x}) \right] \text{ if } P = \{i_1, \ldots, i_p\}
\end{aligned}$$

$$\begin{aligned}
f(\underline{0}) &= f(0, \ldots, 0) = 0 \\
f(\underline{1}) &= f(1, \ldots, 1) = \sum_{L \subset X} a(L) \\
&= \mu(1, \ldots, n) = 1
\end{aligned}$$

$$\boxed{J(P) = \delta_P f\left(\underline{\frac{1}{2}}\right)}$$

with two particular cases :

$$
\begin{aligned}
J(i) &= \delta_i f\left(\frac{1}{\underline{2}}\right) = b_i(X) \\
&= \frac{1}{2^{n-1}} \sum_{L \subset X - \{i\}} [\mu(L \cup \{i\}) - \mu(L)] \\
J(i,j) &= \delta_{ij} f\left(\frac{1}{\underline{2}}\right) \\
&= \frac{1}{2^{n-\cancel{1}2}} \sum_{L \subset X - \{i,j\}} [\mu(L \cup \{i,j\} \\
&\qquad -\mu(L \cup \{i\}) - \mu(L \cup \{j\}) + \mu(L)]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{J}_{(2)} &= \begin{pmatrix} J(\emptyset) \\ J(1) \\ J(2) \\ J(1,2) \end{pmatrix} \\
&= \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} a(\emptyset) \\ a(1) \\ a(2) \\ a(1,2) \end{pmatrix} \\
&= R_{(2)} \tilde{a}_{(2)}
\end{aligned}
$$

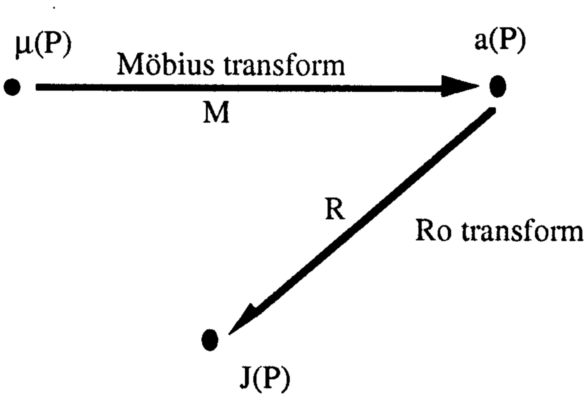
but

$$R_{(2)} = \left[\begin{array}{cc} 1R_{(1)} & \frac{1}{2}\,R_{(1)} \\ 0R_{(1)} & 1R_{(1)} \end{array} \right]$$

and we obtain a fractal representation :

$$\begin{aligned} \tilde{J}_{(1)} &= R_{(1)}\tilde{a}_{(1)} \\ \tilde{J}_{(2)} &= R_{(2)}\tilde{a}_{(2)} \\ &\vdots \\ \tilde{J}_{(n)} &= \left[\begin{array}{cc} 1R_{(n-1)} & \frac{1}{2}\,R_{(n-1)} \\ 0 & 1R_{(n-1)} \end{array} \right] \tilde{a}_{(n)} \end{aligned}$$

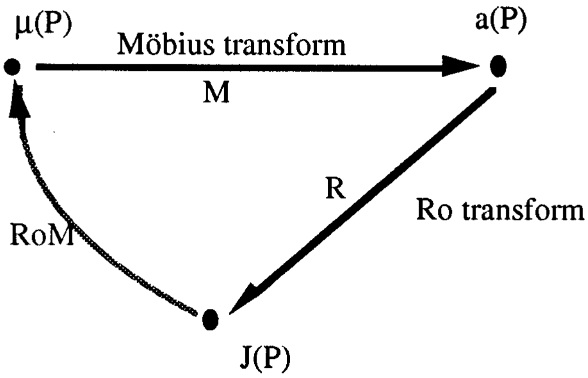
A third equivalent representation



$$J(P) = \delta_P f\left(\frac{1}{2}\right) = \sum_{L \subset X-P} \frac{1}{2^{|L|}}\, a(L \cup P)$$

$$\begin{aligned}
J(\emptyset) &= \sum_{L \subset X} \frac{1}{2^{|L|}}\, a(L) \\
J(i) &= \sum_{L \subset X-\{i\}} \frac{1}{2^{|L|}}\, a(L \cup \{i\}) \\
J(i,j) &= \sum_{L \subset X-\{i,j\}} \frac{1}{2^{|L|}}\, a(L \cup \{i,j\}) \\
&\vdots \\
\tilde{J}_{(1)} &= \begin{pmatrix} J(\emptyset) \\ J(1) \end{pmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a(\emptyset) \\ a(1) \end{pmatrix} \\
&= R_{(1)} \tilde{a}_{(1)}
\end{aligned}$$

Three equivalent presentations of Choquet capacities of fractal style



$$M_{(1)} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \; R_{(1)} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$$R_{(1)} \circ M_{(1)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{bmatrix}$$

$$\tilde{J}_{(n)} = \begin{bmatrix} \frac{1}{2} R_{(n-1)} \circ M_{(n-1)} & \frac{1}{2} R_{(n-1)} \circ M_{(n-1)} \\ -1 R_{(n-1)} \circ M_{(n-1)} & 1 R_{(n-1)} \circ M_{(n-1)} \end{bmatrix}$$

of course inversion is easy and is also fractal

$$M_{(1)}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \; R_{(1)}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$$M_{(1)}^{-1} \circ R_{(1)}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$$