On Sugeno integral as an aggregation function

by Jean-Luc Marichal
University of Liège

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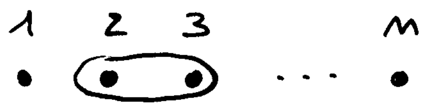
1) Definitions

Let $N = \{1, \dots, n\}$ be a set of criteria

Fuzzy measure on N :

$$\mu : 2^N \to [0,1] \quad \text{s.t.} \quad \begin{cases} \mu_\emptyset = 0, \quad \mu_N = 1 \\ S \subseteq T \Rightarrow \mu_S \leq \mu_T \end{cases}$$



$\mu_{\{2,3\}}$: weight related to $\{2,3\}$.

Aggregation function :

$$M : [0,1]^n \to \mathbb{R} \qquad (x_1, \dots, x_n) \mapsto M(x_1, \dots, x_n)$$

Sugeno integral : Sugeno (1974)

Let $x = (x_1, \dots, x_n) \in [0,1]^n$. The Sugeno integral of x ($x : N \to [0,1]$) w.r.t. μ is defined by

$$S_\mu(x) := \bigvee_{i=1}^{n} \left[x_{(i)} \wedge \mu_{\{(i), \dots, (n)\}} \right]$$

where $x_{(1)} \leq \dots \leq x_{(n)}$.

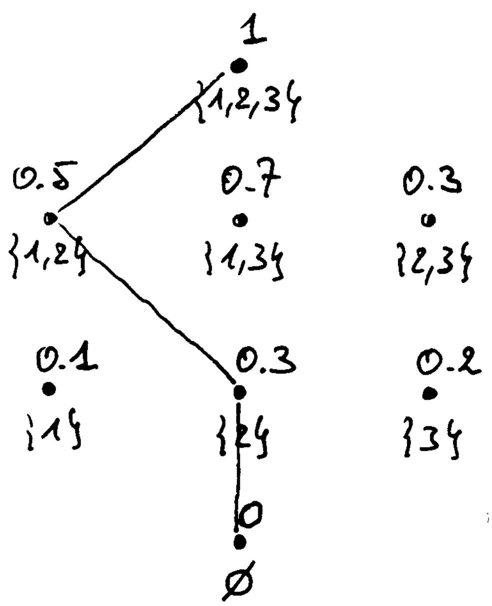
$$S_\mu(x) = \left[x_{(1)} \wedge \mu_{\{(1),\dots,(m)\}}\right] \vee \dots \vee \left[x_{(m)} \wedge \mu_{\{(m)\}}\right]$$

<u>Example:</u> $N = \{1,2,3\}$

$x \in [0,1]^3$ with $x_3 \leq x_1 \leq x_2$

$(1) = 3 \quad (2) = 1 \quad (3) = 2$

$$S_\mu(x_1, x_2, x_3) = (x_3 \wedge \mu_{\{3,1,2\}}) \vee (x_1 \wedge \mu_{\{1,2\}}) \vee (x_2 \wedge \mu_{\{2\}})$$



$S_\mu(0.4, 0.5, 0.2)$

$= (0.2 \wedge 1) \vee (0.4 \wedge 0.5) \vee (0.5 \wedge 0.3)$

$= 0.2 \vee 0.4 \vee 0.3$

$= 0.4$

Dubois and Prade (1986):

$$S_\mu(x) = \text{median}\,(x_1, \dots, x_m, \mu_{\{(2),\dots,(m)\}}, \dots, \mu_{\{(m)\}})$$

$$\text{median}\,(0.4, 0.5, 0.2, 0.5, 0.3) = 0.4$$

$$0.2 < 0.3 < \boxed{0.4} < 0.5 = 0.5$$

2) Alternative representations

Pseudo-Boolean function : $f : \{0,1\}^n \to [0,1]$

$$T \subseteq N \quad \longleftrightarrow \quad e_T \in \{0,1\}^n$$

$$\{1,2,4\} \qquad\qquad (1,1,0,1,0,\dots)$$

If f is increasing and fulfils $f(0,\dots,0)=0$, $f(1,\dots,1)=1$

Then

$$f \quad \longleftrightarrow \quad \mu \quad \text{(fuzzy measure)}$$

$$f(e_T) = \mu_T$$

Such an f can be put in the form :

$$\boxed{f(x) = \bigvee_{T \subseteq N} \Big[\mu_T \wedge \big(\bigwedge_{i \in T} x_i\big)\Big], \qquad x \in \{0,1\}^n}$$

Indeed,

$$f(e_S) = \bigvee_{T \subseteq N} \Big[\mu_T \wedge \underbrace{\big(\bigwedge_{i \in T} (e_S)_i\big)}_{=0 \text{ iff } T \not\subseteq S}\Big]$$

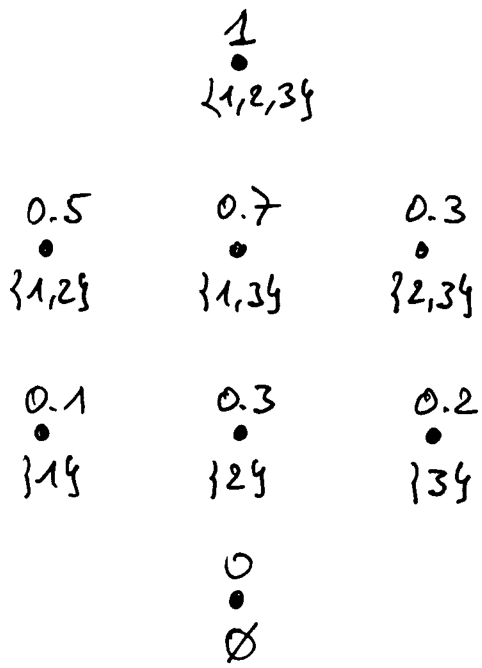
$$= \bigvee_{T \subseteq S} \mu_T$$

$$= \mu_S$$

<u>Theorem</u> For all $x \in [0,1]^n$, we have

$$S_\mu(x) = \bigvee_{i=1}^{n} \left[x_{(i)} \wedge \mu_{\{(i),\ldots,(n)\}} \right]$$

$$= \bigvee_{T \subseteq N} \left[\mu_T \wedge \left(\bigwedge_{i \in T} x_i \right) \right]$$



$$S_\mu(0.4, 0.5, 0.2)$$

$$= (0.1 \wedge 0.4) \vee (0.3 \wedge 0.5) \vee (0.2 \wedge 0.2)$$

$$\vee\, (0.5 \wedge 0.4) \vee (0.7 \wedge 0.2) \vee (0.3 \wedge 0.2)$$

$$\vee\, (1 \wedge 0.2)$$

$$= 0.1 \vee 0.3 \vee 0.2 \vee 0.4 \vee 0.2 \vee 0.2 \vee 0.2$$

$$= 0.4$$

We also have

$$S_\mu(x) = \bigwedge_{i=1}^{n} \left[x_{(i)} \overset{\vee}{\cancel{\wedge}} \mu_{\{(i+1),\ldots,(n)\}} \right]$$

$$= \bigwedge_{T \subseteq N} \left[\mu_{N \setminus T} \vee \left(\bigvee_{i \in T} x_i \right) \right]$$

3) Axiomatic characterizations

Theorem Let $M : [0,1]^n \to [0,1]$ be an agg. function.

$$\boxed{\exists\, \mu : 2^N \to [0,1] \text{ s.t. } M = S_\mu}$$

$$\Updownarrow$$

- M is increasing
- $M(x_1 \vee r, \dots, x_n \vee r) = M(x_1, \dots, x_n) \vee r, \quad x \in [0,1]^n, r \in [0,1$
- $\qquad \wedge \qquad\qquad \wedge \qquad\qquad\qquad \wedge$

$$\Updownarrow$$

- M is increasing
- $M(x, \dots, x) = x, \qquad x \in [0,1]$
- $M(x_1 \vee r, \dots, x_n \vee r) = M(x_1, \dots, x_n)$ or r, $x \in \{0,1\}^n, r \in [0,$
- $\qquad \wedge \qquad\qquad \wedge$

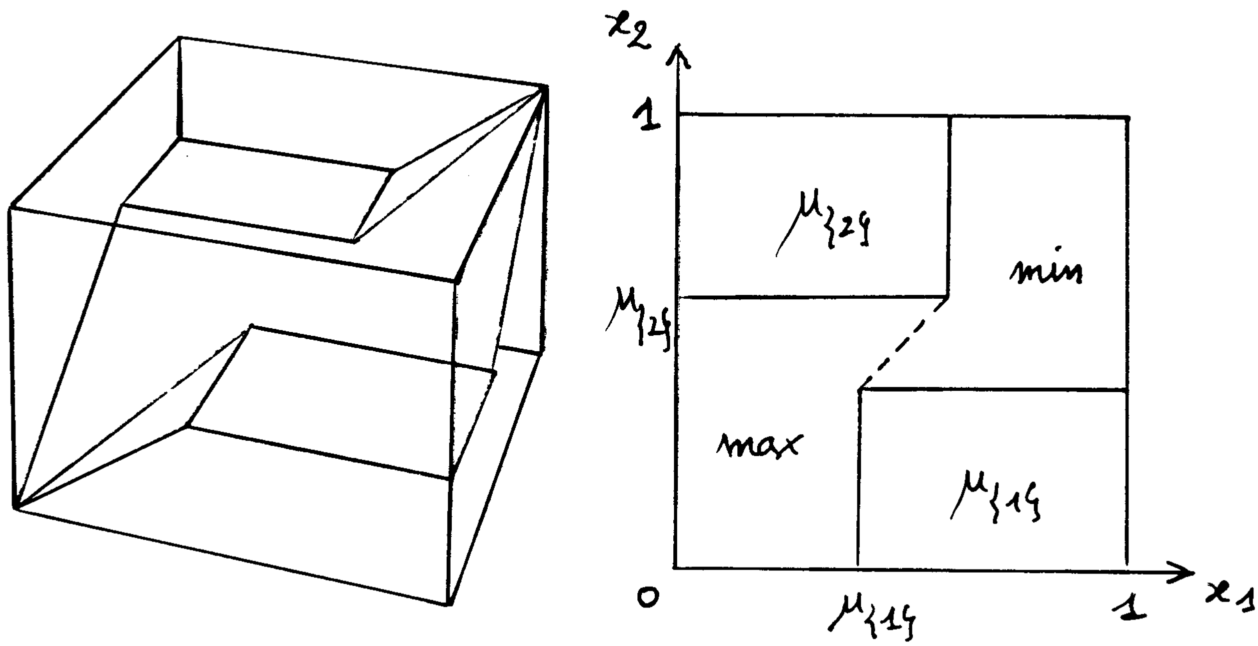
$$\Updownarrow$$

- M is increasing
- $M(x, \dots, x) = x, \qquad x \in [0,1]$
- $M(x \vee x') = M(x) \vee M(x'), \qquad x, x' \in [0,1]^n$ comonotonic
- $\qquad \wedge \qquad\qquad\qquad \wedge$

where x, x' are comonotonic if $\exists$ permutation π of N:

$$x_{\pi(1)} \leq \dots \leq x_{\pi(n)} \text{ and } x'_{\pi(1)} \leq \dots \leq x'_{\pi(n)}$$

On $[0,1]^2$:



$$S_\mu(x) = (\mu_{\{1\}} \wedge x_1) \vee (\mu_{\{2\}} \wedge x_2) \vee (x_1 \wedge x_2)$$

$\rightarrow$ Characterized by Fodor (1996) :

$M : [0,1]^2 \to [0,1]$ fulfils

- increasing monotonicity
- continuity
- $M(x,x) = x$
- $M(M(x,y),z) = M(x, M(y,z))$

if and only if $\exists\, \mu : M = S_\mu$.

4) Boolean Sugeno integrals

$B_\mu := S_\mu$ with a 0-1 fuzzy measure

$(\mu_T \in \{0,1\}, \quad T \subseteq N)$

$$\boxed{B_\mu(x) = \bigvee_{\substack{T \subseteq N \\ \mu_T = 1}} \bigwedge_{i \in T} x_i, \qquad x \in [0,1]^n}$$

$$B_\mu(x) \in \{x_1, \dots, x_n\}$$

Proposition: B_μ is a comparison meaningful function, that is a function $M : [0,1]^n \to [0,1]$ s.t.

$$M(x_1, \dots, x_n) \left\{ \begin{matrix} \leq \\ = \end{matrix} \right\} M(x'_1, \dots, x'_n)$$

$$\Downarrow$$

$$M(\phi(x_1), \dots, \phi(x_n)) \left\{ \begin{matrix} \leq \\ = \end{matrix} \right\} M(\phi(x'_1), \dots, \phi(x'_n))$$

for all $x, x' \in [0,1]^n$ and all strictly increasing $\phi : [0,1] \to [0,1]$

<u>Theorem</u> Let $M : [0,1]^n \to [0,1]$ be an agg. function

$$\boxed{\exists\, \mu : 2^N \to \{0,1\} \ \text{ s.t. } M = B_\mu}$$

$$\Updownarrow$$

- $\exists\, \mu : 2^N \to [0,1]$ s.t. $M = S_\mu$
- M is increasing
- $x'_i > x_i \quad \forall i \;\Rightarrow\; M(x'_1, \dots, x'_n) > M(x_1, \dots, x_n)$

$$\Updownarrow$$

$\exists\, \mu : 2^N \to [0,1]$ s.t. $M = S_\mu = C_\mu$

where $C_\mu(x) = \sum_{i=1}^{n} x_{(i)} \left[\mu_{\{(i), \dots, (n)\}} - \mu_{\{(i+1), \dots, (n)\}}\right]$

$$\Updownarrow$$

- M is increasing
- $M(rx_1 + s, \dots, rx_n + s) = r\,M(x_1, \dots, x_n) + s$, $\quad r > 0$, $s \in \mathbb{R}$
- $x \in \{0,1\}^n \;\Rightarrow\; M(x) \in \{0,1\}$

$$\Updownarrow$$

- M is increasing
- $M(x, \dots, x) = x$, $\quad x \in [0,1]$
- M is comparison meaningful

5) Weighted max and min functions

Definitions

$$\mathrm{wmax}_\omega(x) := \bigvee_{i=1}^{m} (\omega_i \wedge x_i), \quad \omega_i \in [0,1], \quad \bigvee_{i=1}^{m} \omega_i = 1$$

$$\mathrm{wmin}_\omega(x) := \bigwedge_{i=1}^{m} (\omega_i \vee x_i), \quad \omega_i \in [0,1], \quad \bigwedge_{i=1}^{m} \omega_i = 0$$

Theorem Let μ be a fuzzy measure on N.

$\exists\, \omega : S_\mu = \mathrm{wmax}_\omega$	$\exists\, \omega : S_\mu = \mathrm{wmin}_\omega$
$\Updownarrow$	$\Updownarrow$
μ is a possibility measure	μ is a necessity measure
$\mu(S \cup T) = \mu(S) \vee \mu(T)$	$\mu(S \cap T) = \mu(S) \wedge \mu(T)$
$\Updownarrow$	$\Updownarrow$
$S_\mu(x \vee x') = S_\mu(x) \vee S_\mu(x')$	$S_\mu(x \wedge x') = S_\mu(x) \wedge S_\mu(x')$
$x, x' \in [0,1]^m$	$x, x' \in [0,1]^m$

6) Ordered weighted max. and min. functions

Definitions

$$\text{owmax}_{\omega}(x) := \bigvee_{i=1}^{n} (\omega_i \wedge x_{(i)}), \quad \omega_i \in [0,1], \quad \bigvee_{i=1}^{n} \omega_i = 1$$

$$\text{owmin}_{\omega}(x) := \bigwedge_{i=1}^{n} (\omega_i \vee x_{(i)}), \quad \omega_i \in [0,1], \quad \bigwedge_{i=1}^{n} \omega_i = 0$$

Theorem Let μ be a fuzzy measure on N

$$\boxed{\exists\ \omega : S_\mu = \text{owmax}_\omega}$$

$$\Updownarrow$$

$$\boxed{\exists\ \omega' : S_\mu = \text{owmin}_{\omega'}}$$

$$\Updownarrow$$

$$\boxed{S_\mu \text{ is symmetric}}$$

$$\Updownarrow$$

$$\boxed{\left.\begin{array}{l} S, T \subseteq N \\ |S| = |T| \end{array}\right\} \Rightarrow \mu_S = \mu_T}$$

7) Ordered statistics

$$OS_k(x) := x_{(k)}, \quad k \in N, \quad x \in [0,1]^n$$

$$x_{(1)} \leq \dots \leq x_{(k)} \leq \dots \leq x_{(n)}$$

Max-min representation : Ovchinnikov (1996)

$$x_{(k)} = \bigvee_{\substack{T \subseteq N \\ |T| = n-k+1}} \bigwedge_{i \in T} x_i = \bigwedge_{\substack{T \subseteq N \\ |T| = k}} \bigvee_{i \in T} x_i$$

$$= \operatorname{median}(x_1, \dots, x_n, \underbrace{1, \dots, 1}_{k-1}, \underbrace{0, \dots, 0}_{n-k})$$

Theorem Let $M : [0,1]^n \to [0,1]$ be an agg. function

$$\boxed{\exists\, k \in N : M = OS_k}$$

$$\Updownarrow$$

- $\exists\, \mu : 2^N \to \{0,1\} : M = B_\mu$
- M is symmetric

$$\Updownarrow$$

- M is increasing
- M is symmetric
- M is idempotent
- M is comparison meaningful

8) Medians

$$N = \{1, \ldots, 2k-1\} \qquad (n = 2k-1)$$

$$\boxed{\text{median}(x_1, \ldots, x_{2k-1}) := x_{(k)}}$$

Max-min representation :

$$\text{median}(x_1, \ldots, x_{2k-1}) = \bigvee_{1 \le i_1 < \cdots < i_k \le 2k-1} (x_{i_1} \wedge \cdots \wedge x_{i_k})$$

Example :

$$\text{median}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$$

Theorem Let $M : [0,1]^{2k-1} \to [0,1]$ be an agg. functi[on]

$$\boxed{M = \text{median}}$$

$$\Updownarrow$$

- $\exists\, \ell \in \{1, \ldots, 2k-1\}$ s.t. $M = OS_\ell$
- $M(1-x_1, \ldots, 1-x_n) = 1 - M(x_1, \ldots, x_n)$, $x \in [0,1]^n$