AGGREGATION OPERATORS FOR MULTICRITERIA DECISION AID

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1. Aggregation in MCDM

Set of alternatives $A = \{a, b, c, \ldots\}$
Set of criteria $N = \{1, \ldots, n\}$.

For all $i \in N$, ω_i = weight associated to criterion i.

Profile: $a \in A \quad \rightarrow \quad (x_1^a, \ldots, x_n^a) \in E^n, \quad E$ = real interval.
x_i^a = partial score of a w.r.t. criterion i.

Aggregation operator $M : E^n \rightarrow F$
Example: $\mathrm{WAM}_\omega(x) = \sum_{i=1}^{n} \omega_i \, x_i$ with $\sum_{i=1}^{n} \omega_i = 1, \ \omega_i \geq 0$.

	criterion 1	$\cdots$	criterion n	global score
alternative a	x_1^a	$\cdots$	x_n^a	$M(x_1^a, \ldots, x_n^a)$
alternative b	x_1^b	$\cdots$	x_n^b	$M(x_1^b, \ldots, x_n^b)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Phases of multicriteria decision making procedures:

1. Modelling phase: How to construct x_i^a and ω_i?
2. Aggregation phase: How to build $M(x_1^a, \ldots, x_n^a)$?
3. Exploitation phase: Which are the best alternatives?

Hypotheses:

- The weights ω_i are defined according to a cardinal scale
- All the partial scores x_i^a are commensurable.

2. Some aggregation operators

Continuity (Co)
Increasing monotonicity (In):

$$x_i \leq x_i' \ \forall i \ \Rightarrow \ M(x_1, \ldots, x_n) \leq M(x_1', \ldots, x_n')$$

Idempotence (Id):

$$M(x, \ldots, x) = x$$

Associativity (A):

$$M(x_1, M(x_2, x_3)) = M(M(x_1, x_2), x_3)$$
$$M(x_1, \ldots, x_n, x_{n+1}) = M(M(x_1, \ldots, x_n), x_{n+1})$$

An *extended aggregation operator* is a sequence $M = (M^{(n)})_{n \in \mathbb{N}_0}$ of aggregation operators $M^{(n)} : E^n \to F$. The set of all those sequences is denoted by $A(E, F)$.

Theorem
$M \in A(E, \mathbb{R})$ fulfils (Co, In, Id, A) if and only if there exist $\alpha, \beta \in E$ such that

$$M^{(n)}(x) = (\alpha \wedge x_1) \vee (\bigvee_{i=2}^{n-1} (\alpha \wedge \beta \wedge x_i)) \vee (\beta \wedge x_n) \vee (\bigwedge_{i=1}^{n} x_i) \quad \forall n \in \mathbb{N}_0$$

+ Symmetry (Sy)

Theorem
$M \in A(E, \mathbb{R})$ fulfils (Sy, Co, In, Id, A) if and only if there exists $\alpha \in E$ such that

$$M^{(n)}(x) = \text{median}(\bigwedge_{i=1}^{n} x_i, \bigvee_{i=1}^{n} x_i, \alpha) \quad \forall n \in \mathbb{N}_0.$$

Decomposability (D):

for all $k \leq n$,

$$M^{(k)}(x_1, \ldots, x_k) = M^{(k)}(x'_1, \ldots, x'_k)$$
$$\Downarrow$$
$$M^{(n)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = M^{(n)}(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n)$$

Strict increasing monotonicity (SIn)

Theorem (Kolmogoroff-Nagumo, 1930)
$M \in A(E, \mathbb{R})$ fulfils (Sy, Co, SIn, Id, D) if and only if there exists a continuous strictly monotonic function $f : E \to \mathbb{R}$ such that

$$M^{(n)}(x) = f^{-1}[\frac{1}{n} \sum_{i=1}^{n} f(x_i)], \quad n \in \mathbb{N}_0.$$

Remarks

1. The family of $M \in A(E, \mathbb{R})$ that satisfy (Sy, Co, In, Id, D) has a rather intricate structure (see §3.2.2).
2. (Sy, Co, SIn, Id, D) $\Leftrightarrow$ (Co, SIn, Id, SD) (see §3.2.1).

The quasi-linear means (Aczél, 1948):

$$M(x) = f^{-1}[\sum_{i=1}^{n} \omega_i f(x_i)], \quad \text{with } \sum_{i=1}^{n} \omega_i = 1, \; \omega_i \geq 0.$$

The weighted arithmetic means:

$$\text{WAM}_\omega(x) = \sum_{i=1}^{n} \omega_i x_i, \quad \text{with } \sum_{i=1}^{n} \omega_i = 1, \; \omega_i \geq 0.$$

General bisymmetry (GB):

$M^{(1)}(x) = x$ for all $x \in E$, and

$$\begin{aligned} & M^{(p)}(M^{(n)}(x_{11},\ldots,x_{1n}),\ldots,M^{(n)}(x_{p1},\ldots,x_{pn})) \\ = \; & M^{(n)}(M^{(p)}(x_{11},\ldots,x_{p1}),\ldots,M^{(p)}(x_{1n},\ldots,x_{pn})) \end{aligned}$$

for all matrices

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pn} \end{pmatrix} \in E^{p\times n}.$$

Stability for the admissible positive linear transformations (SPL):

$$M(r\,x_1+s,\ldots,r\,x_n+s) = r\,M(x_1,\ldots,x_n)+s$$

for all $x \in E^n$ and all $r>0, s\in \mathbb{R}$ such that $r\,x_i+s\in E$ for all $i \in N$.

We can assume w.l.o.g. that $E = [0,1]$

Theorem

$M \in A([0,1],\mathbb{R})$ *fulfils (In, SPL, GB) if and only if*

- *either:* $\forall n \in \mathbb{N}_0$, $\exists S \subseteq \{1,\ldots,n\}$ *such that* $M^{(n)} = \min_S$,
- *or:* $\forall n \in \mathbb{N}_0$, $\exists S \subseteq \{1,\ldots,n\}$ *such that* $M^{(n)} = \max_S$,
- *or:* $\forall n \in \mathbb{N}_0$, $\exists \omega \in [0,1]^n$ *such that* $M^{(n)} = \mathrm{WAM}_\omega$.

$$\min_S(x) := \bigwedge_{i\in S} x_i \qquad \max_S(x) := \bigvee_{i\in S} x_i$$

Theorem

$M \in A([0,1],\mathbb{R})$ *fulfils (SIn, SPL, GB) if and only if for all* $n \in \mathbb{N}_0$*, there exists* $\omega \in]0,1[^n$ *such that* $M^{(n)} = \mathrm{WAM}_\omega$.

3. The weighted arithmetic means

$$\mathrm{WAM}_\omega(x) = \sum_{i\in N} \omega_i\, x_i, \quad \text{with} \ \sum_{i\in N} \omega_i = 1, \ \ \omega_i \geq 0.$$

Definition
For any $S \subseteq N$, we define $e_S \in \{0,1\}^n$ as the binary profile whose i-th component is 1 iff $i \in S$.

We observe that

$$\mathrm{WAM}_\omega(e_{\{i\}}) = \omega_i$$

The weight ω_i can be viewed as the global score obtained with the profile e_i

Additivity (Add):

$$M(x_1 + x_1', \ldots, x_n + x_n') = M(x_1, \ldots, x_n) + M(x_1', \ldots, x_n')$$

Theorem
$M : [0,1]^n \to \mathbb{R}$ *fulfils (In, SPL, Add) if and only if there exists* $\omega \in [0,1]^n$ *such that* $M = \mathrm{WAM}_\omega$.

Remark
Weighted arithmetic means can be used only when criteria are "independent" !!!

Example of correlated criteria:

Statistics	Probability	Algebra
0.3	0.3	0.4

Preferential independence

Let x, x' be two profiles in $[0,1]^n$.
The profile x is said to be preferred to the profile x' ($x \succeq x'$) if $M(x) \geq M(x')$.

Definition
The subset S of criteria is said to be *preferentially independent* of $N \setminus S$ if, for all $x, x' \in [0,1]_S$ and all $y, z \in [0,1]_{N \setminus S}$, we have

$$(x, y) \succeq (x', y) \quad \Leftrightarrow \quad (x, z) \succeq (x', z).$$

Theorem (Scott and Suppes, 1958)
If a weighted arithmetic mean is used as an aggregation operator then every subset S of criteria is preferentially independent of $N \setminus S$.

Example:

	price	consumption	comfort
car 1	10.000 Euro	10 ℓ/100 km	very good
car 2	10.000 Euro	9 ℓ/100 km	good
car 3	30.000 Euro	10 ℓ/100 km	very good
car 4	30.000 Euro	9 ℓ/100 km	good

No weighted arithmetic mean can model the following preferences:

$$\text{car 2} \succeq \text{car 1} \quad \text{and} \quad \text{car 3} \succeq \text{car 4}.$$

4. The Choquet integral

Definition (Choquet, 1953; Sugeno, 1974)
A (discrete) fuzzy measure on N is a set function $\mu : 2^N \to [0,1]$ satisfying

i) $\mu_\emptyset = 0$, $\mu_N = 1$,
ii) $S \subseteq T \Rightarrow \mu_S \leq \mu_T$.

μ_S is regarded as the weight of importance of the combination S of criteria.

A fuzzy measure is additive if $\mu_{S\cup T} = \mu_S + \mu_T$ whenever $S \cap T = \emptyset$.

When the fuzzy measure is not additive then some criteria interact. For example, we should have

$$\mu_{\{\text{St},\text{Pr}\}} < \mu_{\{\text{St}\}} + \mu_{\{\text{Pr}\}}.$$

We search for a suitable aggregation operator $M_\mu : [0,1]^n \to \mathbb{R}$, which generalizes the weighted arithmetic mean.

As for the weighted arithmetic means, we assume that the weight μ_S is defined as the global score of the profile e_S:

$$\mu_S = M_\mu(e_S) \quad (S \subseteq N).$$

We observe that μ can be expressed in a unique way as:

$$\mu_S = \sum_{T \subseteq S} a_T \qquad (S \subseteq N)$$

where $a_T \in \mathbb{R}$.
a viewed as a set function on N is called the Möbius transform of μ, which is given by:

$$a_S = \sum_{T \subseteq S} (-1)^{|T|-|S|} \mu_T \qquad (S \subseteq N).$$

For example,

$$\begin{aligned} a_\emptyset &= 0, \\ a_{\{i\}} &= \mu_{\{i\}}, \\ a_{\{i,j\}} &= \mu_{\{i,j\}} - [\mu_{\{i\}} + \mu_{\{j\}}] \\ &\leq 0 \quad \text{(overlap effect)} \\ &\geq 0 \quad \text{(positive synergy)} \\ &= 0 \quad \text{(no interaction)} \end{aligned}$$

If μ is additive then we have $a_S = 0$ for all $S \subseteq N$, $|S| \geq 2$.

$$M_\mu(x) = \sum_{i \in N} a_{\{i\}} \, x_i \qquad \text{(weighted arithmetic mean).}$$

When μ is not additive, we can introduce

$$\begin{aligned} M_\mu(x) &= \sum_{i \in N} a_{\{i\}} \, x_i + \sum_{\{i,j\} \subseteq N} a_{\{i,j\}} \, [x_i \wedge x_j] + \ldots \\ &= \sum_{T \subseteq N} a_T \bigwedge_{i \in T} x_i. \end{aligned}$$

(Choquet integral)

Such a function satisfies (Co), (In), (Id), and (SPL).
It violates (Add).

Definition (Choquet, 1953)
Let μ be a fuzzy measure on N. The (discrete) Choquet integral of the profile $x : N \to [0,1]$ w.r.t. μ is defined by

$$\mathcal{C}_\mu(x) = \sum_{i=1}^{n} x_{(i)} \, [\mu_{\{(i),\ldots,(n)\}} - \mu_{\{(i+1),\ldots,(n)\}}]$$

with the convention that $x_{(1)} \leq \cdots \leq x_{(n)}$.

Particular cases:

- When μ is additive, $\mathcal{C}_\mu$ identifies with the weighted arithmetic mean (Lebesgue integral):

$$\mathcal{C}_\mu(x) = \sum_{i=1}^{n} x_i \, \mu_{\{i\}} = \sum_{i=1}^{n} \omega_i \, x_i$$

- $\mathcal{C}_\mu$ is symmetric (Sy) iff μ depends only on the cardinality of subsets (Grabisch, 1995). Setting

$$\omega_i := \mu_{\{(i),\ldots,(n)\}} - \mu_{\{(i+1),\ldots,(n)\}},$$

  we see that $\mathcal{C}_\mu$ identifies with an *ordered weighted averaging* operator (OWA):

$$\mathrm{OWA}_\omega(x) = \sum_{i=1}^{n} \omega_i \, x_{(i)} \quad \text{with} \quad \sum_{i=1}^{n} \omega_i = 1, \;\; \omega_i \geq 0$$

  (Yager, 1988)

Two profiles $x, x' \in [0,1]^n$ are said to be *comonotonic* if there exists a permutation π of N such that

$$x_{\pi(1)} \leq \cdots \leq x_{\pi(n)} \quad \text{and} \quad x'_{\pi(1)} \leq \cdots \leq x'_{\pi(n)}.$$

Comonotonic additivity (CoAdd):

$$M(x_1 + x'_1, \ldots, x_n + x'_n) = M(x_1, \ldots, x_n) + M(x'_1, \ldots, x'_n)$$

for any two comonotonic profiles $x, x' \in [0,1]^n$.

Theorem (Schmeidler, 1986)
$M : [0,1]^n \to \mathbb{R}$ fulfils (In, SPL, CoAdd) if and only if there exists a fuzzy measure μ on N such that $M = \mathcal{C}_\mu$.

Theorem
The aggregation operator $M_\mu : [0,1]^n \to \mathbb{R}$

- *is* linear *w.r.t. the fuzzy measure μ:*
  there exist 2^n functions $f_T : [0,1]^n \to \mathbb{R}$, $T \subseteq N$, such that

$$M_\mu = \sum_{T \subseteq N} a_T \, f_T \quad \forall \mu,$$

- *satisfies (In),*
- *satisfies (SPL),*
- *and is such that*

$$M_\mu(e_S) = \mu_S, \quad (S \subseteq N),$$

if and only if $M_\mu = \mathcal{C}_\mu$.

5. Behavioral analysis of aggregation

5.1 Shapley power index

Given $i \in N$, it may happen that

- $\mu_{\{i\}} = 0$,
- $\mu_{T\cup\{i\}} \gg \mu_T$ for many $T \not\ni i$

The overall importance of $i \in N$ should not be solely determined by $\mu_{\{i\}}$, but also by all $\mu_{T\cup\{i\}}$ such that $T \not\ni i$.

The marginal contribution of i in combination $T \subseteq N$ is defined by

$$\mu_{T\cup\{i\}} - \mu_T$$

The *Shapley power index* for i is defined as an average value of the marginal contributions of i alone in all combinations:

$$\begin{aligned} \phi_\mu(i) &:= \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\not\ni i\\ |T|=t}}[\mu_{T\cup\{i\}} - \mu_T] \\ &= \sum_{T\not\ni i}\frac{(n-t-1)!\,t!}{n!}[\mu_{T\cup\{i\}} - \mu_T] \\ &= \sum_{T\ni i}\frac{1}{t}\,a_T \end{aligned}$$

This index has been introduced axiomatically by Shapley (1953) in game theory.

5.2 Interaction index

Consider a pair $\{i,j\}$ of criteria. If

$$\underbrace{\mu_{T\cup\{i,j\}} - \mu_{T\cup\{i\}}}_{\substack{\text{contribution of } j \text{ in} \\ \text{the presence of } i}} \;<\; \underbrace{\mu_{T\cup\{j\}} - \mu_T}_{\substack{\text{contribution of } j \text{ in} \\ \text{the absence of } i}} \qquad \forall T \not\ni i,j$$

then there is an overlap effect between i and j.
Criteria i and j interfere in a positive way in case of $>$ and are independent of each other in case of $=$.

An *interaction index* for the pair $\{i,j\} \subseteq N$ is given by an average value of the marginal interaction between i and j, conditioned to the presence of elements of the subset $T \not\ni i,j$:

$$\begin{aligned} I_\mu(ij) &= \sum_{T \not\ni i,j} \frac{(n-t-2)!\,t!}{(n-1)!}\,[\mu_{T\cup\{i,j\}} - \mu_{T\cup\{i\}} - \mu_{T\cup\{j\}} + \mu_T] \\ &= \sum_{T \ni i,j} \frac{1}{t-1}\,a_T \end{aligned}$$

This interaction index has been proposed by Murofushi and Soneda (1993).

Notes

1. Interaction indices among a combination S of criteria have been introduced and characterized by Grabisch and Roubens (1998).
2. Another definition has also been introduced and investigated by Marichal and Roubens (1998) (see §5.4)

5.3 Degree of disjunction (cf. Dujmovic, 1974)

We observe that

$$\min x_i \le \mathcal{C}_\mu(x) \le \max x_i \quad \forall x \in [0,1]^n.$$

Define the *average value* of $\mathcal{C}_\mu$ as

$$m(\mathcal{C}_\mu) := \int_{[0,1]^n} \mathcal{C}_\mu(x)\, dx$$

We then have

$$\frac{1}{n+1} = m(\min) \le m(\mathcal{C}_\mu) \le m(\max) = \frac{n}{n+1}$$

A *degree of disjunction* of $\mathcal{C}_\mu$ corresponds to

$$\text{orness}(\mathcal{C}_\mu) := \frac{m(\mathcal{C}_\mu) - m(\min)}{m(\max) - m(\min)} \in [0,1].$$

Theorem

For any Choquet integral $\mathcal{C}_\mu$, we have

$$\text{orness}(\mathcal{C}_\mu) = \frac{1}{n-1} \sum_{T \subseteq N} \frac{n-t}{t+1}\, a_T$$

Moreover, we have

$$\begin{aligned}\text{orness}(\mathcal{C}_\mu) = 1 \quad &\Leftrightarrow \quad \mathcal{C}_\mu = \max\\ \text{orness}(\mathcal{C}_\mu) = 0 \quad &\Leftrightarrow \quad \mathcal{C}_\mu = \min\end{aligned}$$

$\mathcal{C}_\mu$	$\text{orness}(\mathcal{C}_\mu)$
WAM_ω	$1/2$
OWA_ω	$\frac{1}{n-1} \sum_{i=1}^{n} (i-1)\, \omega_i$

5.4 Veto and favor effects

Let $M : [0,1]^n \to [0,1]$ be an aggregation operator. A criterion $i \in N$ is a

- *veto* for M if

$$M(x_1, \ldots, x_n) \leq x_i \quad \forall x \in [0,1]^n$$

- *favor* for M if

$$M(x_1, \ldots, x_n) \geq x_i \quad \forall x \in [0,1]^n$$

(Dubois and Koning, 1991; Grabisch, 1997)

Given a criterion $i \in N$ and a fuzzy measure μ on N, how can we define a degree of veto of i for $\mathcal{C}_\mu$?

First attempt: Let $x \in [0,1]^n$ be a random variable uniformly distributed. A degree of veto of i is given by

$$\Pr[\mathcal{C}_\mu(x) \leq x_i].$$

However,

$$\Pr[\mathrm{WAM}_\omega(x) \leq x_i] = \begin{cases} 1, & \text{if } \omega_i = 1 \\ 1/2, & \text{otherwise} \end{cases}$$

is non-continuous w.r.t. the fuzzy measure !!!

Second attempt: Axiomatic characterization.

$$\mathrm{veto}(\mathcal{C}_\mu; i) := 1 - \frac{n}{n-1} \sum_{T \not\ni i} \frac{1}{t+1}\, a_T$$

(Similar definition for $\mathrm{favor}(\mathcal{C}_\mu; i)$)

Theorem

The real-valued function $\psi(\mathcal{C}_\mu; i)$ satisfies the

- *linearity axiom*:
  there exist real numbers p^i_T, $T \subseteq N$, $i \in N$, such that

$$\psi(\mathcal{C}_\mu; i) = \sum_{T \subseteq N} \mu_T \, p^i_T \quad \forall i \, \forall \mu,$$

- *symmetry axiom*:
  for any permutation π of N,

$$\psi(\mathcal{C}_\mu; i) = \psi(\mathcal{C}_{\pi\mu}; \pi(i)) \quad \forall i \, \forall \mu,$$

  where $\pi\mu$ is defined by $\pi\mu_{\{\pi(i)\}} = \mu_{\{i\}}$ for all i.

- *boundary axiom*:
  for all $S \subseteq N$ and all $i \in S$,

$$\psi(\min_S; i) = 1, \qquad (\text{cf. } \min_S(x) \leq x_i \; \forall i \in S)$$

- *normalization axiom*:

$$\psi(\mathcal{C}_\mu; i) = \psi(\mathcal{C}_\mu; j) \;\; \forall i, j \in N$$
$$\Downarrow$$
$$\psi(\mathcal{C}_\mu; i) = \text{andness}(\mathcal{C}_\mu) \;\; \forall i \in N.$$

if and only if $\psi(\mathcal{C}_\mu; i) = \text{veto}(\mathcal{C}_\mu; i)$.

5.5 Measure of dispersion

Consider a symmetric Choquet integral (OWA):

$$\mathrm{OWA}_\omega(x) = \sum_{i=1}^{n} \omega_i \, x_{(i)}.$$

Yager (1988) proposed to use the *entropy* of ω as degree of the use of the partial scores x:

$$\mathrm{disp}(\omega) = -\frac{1}{\ln n} \sum_{i=1}^{n} \omega_i \ln \omega_i \ \in [0,1]$$

Examples:

OWA_ω	ω	$\mathrm{orness}(\mathrm{OWA}_\omega)$	$\mathrm{disp}(\omega)$
AM	$(1/n, \ldots, 1/n)$	$1/2$	1
median	$(0, \ldots, 1, \ldots, 0)$	$1/2$	0

Measure of dispersion of a fuzzy measure:

$$\mathrm{disp}(\mu) := -\frac{1}{\ln n} \sum_{i=1}^{n} \sum_{T \not\ni i} \frac{(n-t-1)!\, t!}{n!} \, [\mu_{T\cup\{i\}} - \mu_T] \ln[\mu_{T\cup\{i\}} - \mu_T]$$

Theorem The following properties hold:

$$\begin{array}{rl}
i) & \mathrm{disp}(\mu_{\mathrm{WAM}_\omega}) = \mathrm{disp}(\mu_{\mathrm{OWA}_\omega}) = -\frac{1}{\ln n} \sum_{i=1}^{n} \omega_i \ln \omega_i \\
ii) & 0 \leq \mathrm{disp}(\mu) \leq 1 \\
iii) & \mathrm{disp}(\mu) = 1 \ \Leftrightarrow \ \mu = \mu_{\mathrm{AM}} \\
iv) & \mathrm{disp}(\mu) = 0 \ \Leftrightarrow \ \mu_S \in \{0,1\} \ \forall S \subseteq N \\
 & \qquad\qquad\qquad \Rightarrow \ \mathcal{C}_\mu(x) \in \{x_1, \ldots, x_n\}.
\end{array}$$