The use of the discrete Sugeno integral in multicriteria decision making

by

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WE ARE CONCERNED WITH A BASIC QUESTION IN MCDM

How do we aggregate ordinal information ?

$A = \{a, b, c, \ldots\}$ set of actions (alternatives)
$N = \{1, \ldots, i, \ldots, n\}$ set of criteria

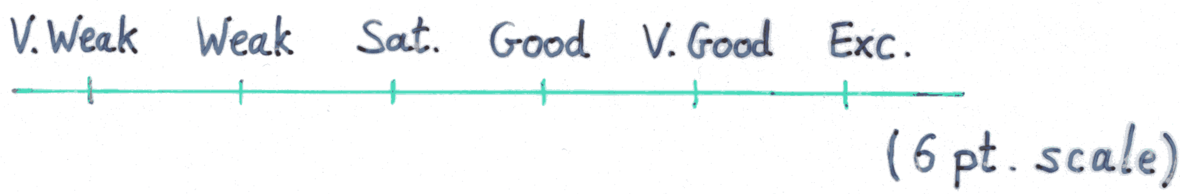
Each $i \in N$ is represented by

$$g_i : A \to X_i$$

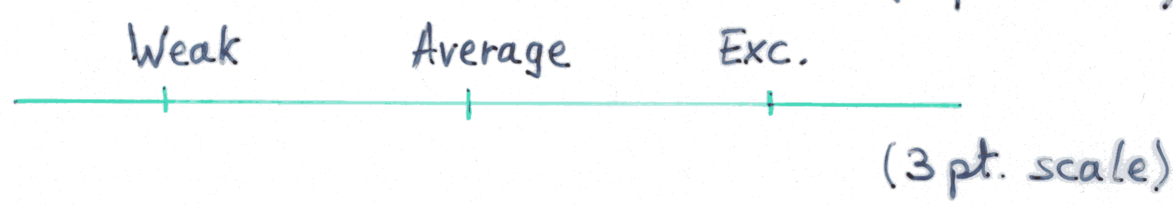
$$X_i = \{r_1^{(i)} < \cdots < r_{k_i}^{(i)}\} \qquad \text{(ordinal scale)}$$

Examples:

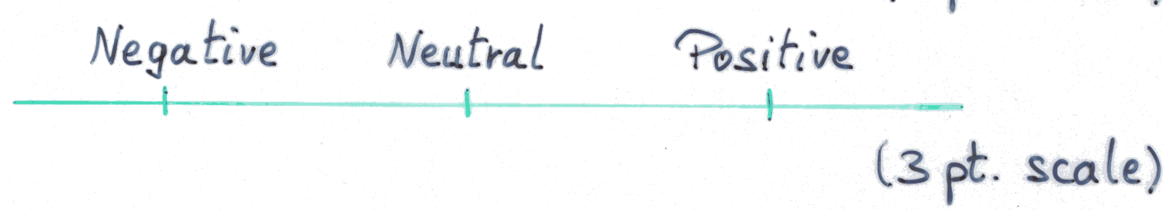
X_1 :



X_2 :



X_3 :



Profile related to action $a \in A$:

$$\big(\underbrace{g_1(a)}_{\in X_1}, \ldots, \underbrace{g_i(a)}_{\in X_i}, \ldots, \underbrace{g_n(a)}_{\in X_n}\big) \in \prod_{i=1}^{n} X_i$$

We will assume the commensurability among the scales, i.e., we assume the existence of

ordinal utilities $\qquad U_i : X_i \rightarrow X$

$X = \{r_1 < \cdots < r_k\}$ (common ordinal scale)

(Roubens, 1999)

and we define an aggregation function $M : X^n \rightarrow X$ that determines the

global evaluation

$$g(a) = M[\underbrace{U_1(g_1(a))}_{\in X}, \ldots, \underbrace{U_n(g_n(a))}_{\in X}] \in X$$

As a consequence,

all actions are comparable in terms of a WEAK ORDER defined on A

APPLICATION FOR AN ACADEMIC POSITION AT ULg (1998)

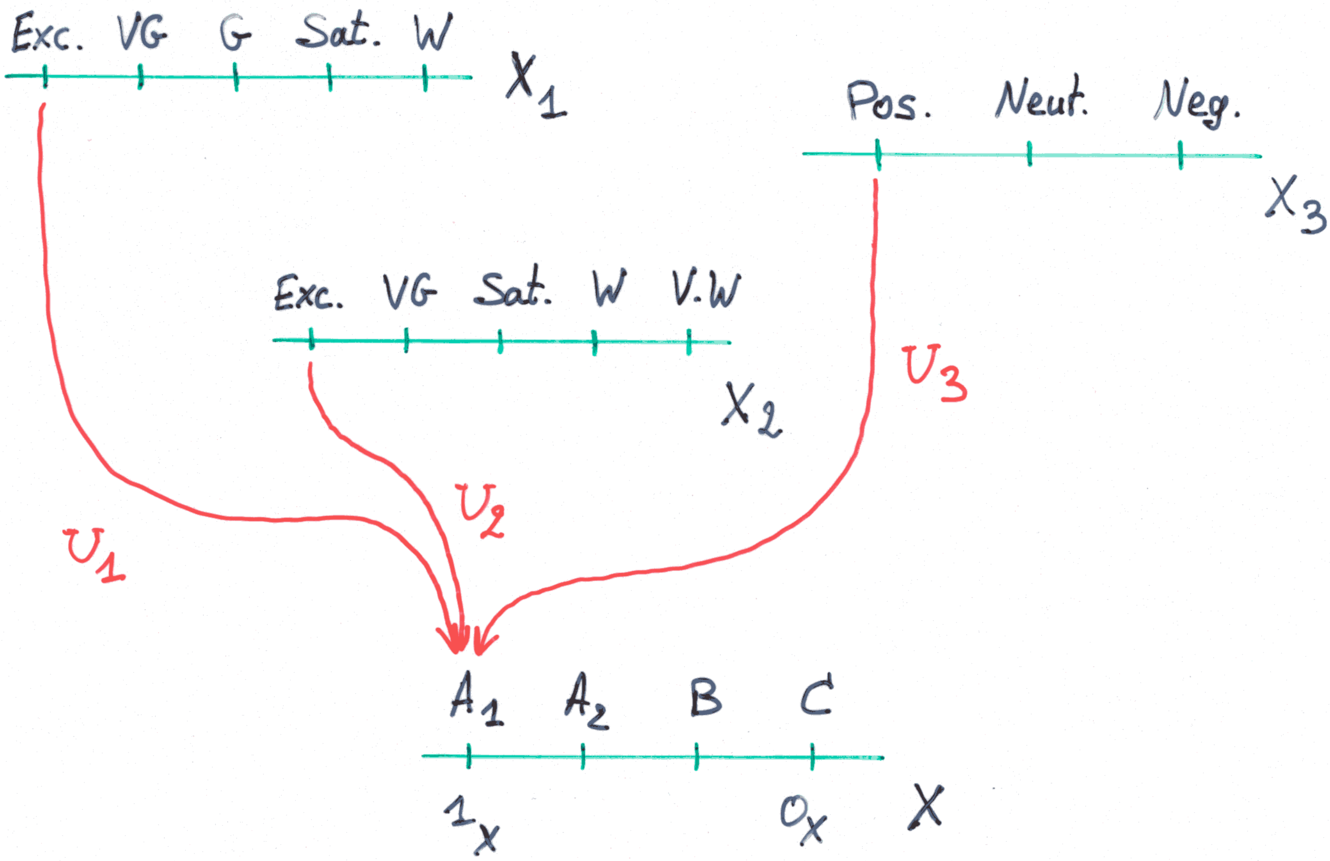
	Exc	V.G	G	Sat	Weak
Scientific value of CV	☐	☑	☐	☐	☐
	Exc	V.G	Sat	Weak	V.W
Teaching effectiveness	☐	☐	☑	☐	☐
	Positive		Neutral		Neg.
Interview	☐		☑		☐

One has to deliver a global evaluation

A1	A2	B	C
☐	☐	☐	☐

?

We assume the commensurability among the ordinal scales



$$a : (\text{VG}, \text{Sat.}, \text{Neutral})$$

$$g(a) = M[U_1(\text{VG}), U_2(\text{Sat.}), U_3(\text{Neutral})] =?$$

We have to determine

- $M \longrightarrow$ axiomatic approach
- $U_i \ (i \in N) \longrightarrow$ by asking questions

The discrete Sugeno integral as a function $M : [0,1]^n \to [0,1]$

(Sugeno, 1974)

Definition 1 *A fuzzy measure on N is a set function $\mu : 2^N \to [0,1]$ such that*

$$\begin{array}{ll} i) & \mu(\emptyset) = 0, \;\; \mu(N) = 1, \\ ii) & S \subseteq T \quad \Rightarrow \quad \mu(S) \leq \mu(T) \end{array}$$

Definition 2 *The Sugeno integral of $x \in [0,1]^n$ w.r.t. a fuzzy measure μ on N is defined by*

$$\mathcal{S}_\mu(x) = \bigvee_{i=1}^{n} [x_{(i)} \wedge \mu(\{(i), \ldots, (n)\})]$$

where $(\cdot)$ is a permutation on N such that $x_{(1)} \leq \cdots \leq x_{(n)}$.

Example: If $x_3 \leq x_1 \leq x_2$ $(x_{(1)} \leq x_{(2)} \leq x_{(3)})$ then

$$\mathcal{S}_\mu(x_1, x_2, x_3) = [x_3 \wedge \mu(3,1,2)] \vee [x_1 \wedge \mu(1,2)] \vee [x_2 \wedge \mu(2)]$$

Proposition 1 (Kandel and Byatt, 1978)

$$\mathcal{S}_\mu(x) = \text{median}[\underbrace{x_1, \ldots, x_n}_{n}, \underbrace{\mu(\{(2), \ldots, (n)\}), \ldots, \mu(\{(n)\})}_{n-1}]$$

$$\mathcal{S}_\mu(x_1, x_2, x_3) = \text{median}[x_1, x_2, x_3, \mu(1,2), \mu(2)]$$

Proposition 2 (Marichal, 1998)

$$\mathcal{S}_\mu(x) = \bigvee_{T \subseteq N} [\mu(T) \wedge (\bigwedge_{i \in T} x_i)]$$

Interpretation of μ:

$\mu(S)$ = importance of the combination S of criteria

$e_S :=$ characteristic vector of S in $\{0,1\}^n$

$$\boxed{\mu(S) = \mathcal{S}_\mu(e_S)}$$

Example: $(n = 4)$

$$\begin{aligned} \mu(\{2\}) &= \mathcal{S}_\mu(0,1,0,0) \\ \mu(\{2,4\}) &= \mathcal{S}_\mu(0,1,0,1) \\ \mu(\{1,2,4\}) &= \mathcal{S}_\mu(1,1,0,1) \end{aligned}$$

Characterization of the Sugeno integral

$$x_i := U_i(g_i) \in X \subseteq [0,1]$$
$$0 = \underbrace{r_1 < \cdots < r_k}_{X} = 1$$

We want to aggregate $x_1, \ldots, x_n$ by a function $M : [0,1]^n \to \mathbb{R}$

Remark: The numbers that are assigned to an ordinal scale $X \subseteq [0,1]$ are defined up to an automorphism $\varphi : [0,1] \to [0,1]$

Definition 3 (Orlov, 1981)
$M : [0,1]^n \to \mathbb{R}$ is comparison meaningful from an ordinal scale if, for any automorphism $\varphi : [0,1] \to [0,1]$ and any $x, x' \in [0,1]^n$,

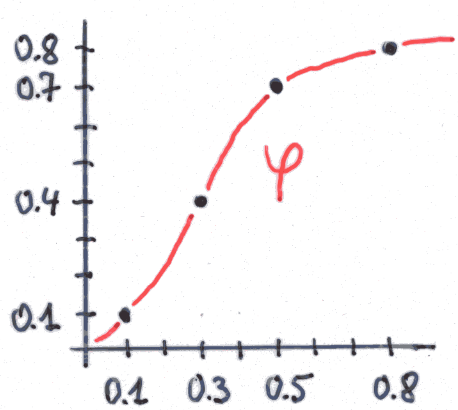
$$M(x) \leq M(x') \quad \iff \quad M(\varphi(x)) \leq M(\varphi(x'))$$

where $\varphi(x) := (\varphi(x_1), \ldots, \varphi(x_n))$.

The arithmetic mean violates this property

$$0.4 = \frac{0.3+0.5}{2} < \frac{0.1+0.8}{2} = 0.45$$

$$0.55 = \frac{0.4+0.7}{2} > \frac{0.1+0.8}{2} = 0.45$$



Proposition 3 (Ovchinnikov, 1996)
If $M : [0,1]^n \to \mathbb{R}$ *is comparison meaningful and idempotent* $(M(x,\ldots,x) = x)$ *then*

$$M(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$$

(cf. $M : X^n \to X$*)*

Proposition 4 (Marichal, 1999)
$M : [0,1]^n \to \mathbb{R}$ *is comparison meaningful, idempotent, and continuous, if and only if there exists a* $\{0,1\}$*-valued fuzzy measure* μ *on* N *such that*

$$M(x) = \mathcal{S}_\mu(x)$$

Weakness of this model:

$$M(e_S) = \mathcal{S}_\mu(e_S) = \mu(S) \in \{0,1\} \quad !!$$

The importance of any subset of criteria is always an extreme value of X.

Let us enrich the aggregation model:

For each set function $v : 2^N \to [0,1]$ s.t. $v(\emptyset) = 0$ and $v(N) = 1$, we define an aggregation function

$$M_v : [0,1]^n \to \mathbb{R}.$$

However,

$$\begin{cases} x_i \in X \\ v(S) \in X \quad (\text{cf. } \mu(S) = \mathcal{S}_\mu(e_S)) \end{cases}$$

$\Longrightarrow$ The mapping $(x, v) \mapsto M_v(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, is comparison meaningful.

Theorem 1 (Marichal, 1999)
The set of functions $M_v : [0,1]^n \to \mathbb{R}$ (v as defined above) such that

i) M_v is idempotent (for all v)
ii) $(x, v) \mapsto M_v(x)$ is comparison meaningful and continuou

identifies with the class of the Sugeno integrals on $[0,1]^n$.

Open problem: Suppress continuity or replace it by increasing monotonicity

Construction of the utilities U_i

(Marichal and Roubens, 1999)

1) $\mathcal{S}_\mu$ is uniquely determined by μ

$$\mu(S) = \mathcal{S}_\mu(e_S)$$

$\longrightarrow$ provided by the decision maker ($2^n - 2$ questions).
However, we often have

$$\mathcal{S}_\mu(0,1,0,1,1) = 0, \ldots$$

2) $$X_i = \{r_1^{(i)} < \ldots < r_{k_i}^{(i)}\}$$

We want to determine $U_i : X_i \to X$, that is,

$$U_i(r_j^{(i)}), \qquad j = 1, \ldots, k_i$$

a) Choose $S \subseteq N \setminus \{i\}$ s.t. the gap between $\mu(S)$ and $\mu(S \cup i)$ is maximum (often $S = N \setminus \{i\}$)

b) Ask the decision maker to appraise

$$\mathcal{S}_\mu(U_i(r_j^{(i)})e_i + e_S), \qquad j = 1, \ldots, k_i$$

We then have

$$\begin{aligned}
\mu(S) < \mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) < \mu(S \cup \{i\}) \;&\Rightarrow\; U_i(r_j^{(i)}) = \mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) \\
\mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) = \mu(S) \;&\Rightarrow\; U_i(r_j^{(i)}) \leq \mu(S) \\
\mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) = \mu(S \cup \{i\}) \;&\Rightarrow\; U_i(r_j^{(i)}) \geq \mu(S \cup \{i\}).
\end{aligned}$$

Example: Application for an academic position

Scientific value

$X_1 = \{\text{Weak} < \text{Sat.} < \text{Good} < \text{Very Good} < \text{Exc.}\}$

Teaching effectiveness:

$X_2 = \{\text{Very Weak} < \text{Weak} < \text{Sat.} < \text{Very Good} < \text{Exc.}\}$

Interview:

$X_3 = \{\text{Neg.} < \text{Neutral} < \text{Pos.}\}$

Global evaluation:

$X = \{C < B < A_2 < A_1\}$

1) The decision maker gives

$$\mu(1,2,3) = A_1$$
$$\mu(1,2) = A_2$$
$$\mu(1,3) = \mu(1) = B$$
$$\mu(2,3) = C$$

2) To determine U_1 he gives the following evaluations

$$\mathcal{S}_\mu(U_1(\text{VG}), 1, 1) = A_1 \Rightarrow U_1(\text{VG}) = A_1$$
$$\mathcal{S}_\mu(U_1(\text{G}), 1, 1) = A_2 \Rightarrow U_1(\text{G}) = A_2$$
$$\mathcal{S}_\mu(U_1(\text{S}), 1, 1) = B \Rightarrow U_1(\text{S}) = B$$

The same for U_2, U_3.

The Sugeno integral is a very natural concept

Consider

- n variables $x_1, \ldots, x_n \in [0, 1]$
- m constants $r_1, \ldots, r_m \in [0, 1]$

Construct a polynomial

$$P_{r_1,\ldots,r_m}(x_1, \ldots, x_n)$$

using $\wedge$, $\vee$, and parentheses.

Then if such a polynomial fulfills

$$P_{r_1,\ldots,r_m}(0, \ldots, 0) = 0 \quad \text{and} \quad P_{r_1,\ldots,r_m}(1, \ldots, 1) = 1$$

then it is a Sugeno integral on $[0, 1]^n$.

Example:

$$P_{r_1,r_2}(x_1, x_2, x_3) = ((x_1 \vee r_2) \wedge x_3) \vee (x_2 \wedge r_1)$$