# The use of the Sugeno integral as an aggregation operator

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## The discrete Sugeno integral as an aggregation function $M : [0,1]^n \to \mathbb{R}$

(Sugeno, 1974)

Let $N := \{1, \ldots, n\}$.

**Definition 1** *A fuzzy measure on* $N$ *is a set function* $\mu : 2^N \to [0,1]$ *such that*

$$\begin{array}{ll} i) & \mu(\emptyset) = 0, \;\; \mu(N) = 1, \\ ii) & S \subseteq T \quad \Rightarrow \quad \mu(S) \leq \mu(T) \end{array}$$

**Definition 2** *The Sugeno integral of* $x \in [0,1]^n$ *w.r.t. a fuzzy measure* $\mu$ *on* $N$ *is defined by*

$$\mathcal{S}_\mu(x) = \bigvee_{i=1}^{n} x_{(i)} \wedge \mu(A_{(i)})$$

*where* $(1), \ldots, (n)$ *is a permutation of indices such that* $x_{(1)} \leq \cdots \leq x_{(n)}$. *Also* $A_{(i)} := \{(i), \ldots, (n)\}$.

**Example:** If $x_3 \leq x_1 \leq x_2$ $(x_{(1)} \leq x_{(2)} \leq x_{(3)})$ then

$$\begin{aligned} &\mathcal{S}_\mu(x_1, x_2, x_3) \\ &= [x_3 \wedge \underbrace{\mu(3,1,2)}_{=1}] \vee [x_1 \wedge \mu(1,2)] \vee [x_2 \wedge \mu(2)] \end{aligned}$$

**Properties**

$i)$ $\min_i x_i \leq \mathcal{S}_\mu(x) \leq \max_i x_i$ (internality)

$ii)$ $\mathcal{S}_\mu(x, \ldots, x) = x$ for any $x \in [0,1]$ (idempotence)

$iii)$ $\mathcal{S}_\mu(x) \in \{x_1, \ldots, x_n\} \cup \{\mu(S) \mid S \subseteq N\}$

**Other forms:**

$$\boxed{\mathcal{S}_\mu(x) = \bigvee_{T \subseteq N} \Big[\mu(T) \wedge (\bigwedge_{i \in T} x_i)\Big]}$$

(Sugeno, 1974)

$$\begin{aligned}
&\mathcal{S}_\mu(x_1, x_2, x_3) = \\
&\mu(\emptyset) \vee [\mu(1) \wedge x_1] \vee [\mu(2) \wedge x_2] \vee [\mu(3) \wedge x_3] \\
&\vee[\mu(1,2) \wedge x_1 \wedge x_2] \vee [\mu(1,3) \wedge x_1 \wedge x_3] \\
&\vee[\mu(2,3) \wedge x_2 \wedge x_3] \vee [\mu(1,2,3) \wedge x_1 \wedge x_2 \wedge x_3]
\end{aligned}$$

If $x_3 \leq x_1 \leq x_2$ then...

$$\boxed{\mathcal{S}_\mu(x) = \bigwedge_{T \subseteq N} \Big[\mu(N \setminus T) \vee (\bigvee_{i \in T} x_i)\Big]}$$

(Greco, 1987)

$$\boxed{\mathcal{S}_\mu(x) = \bigwedge_{i=1}^{n} x_{(i)} \vee \mu(A_{(i+1)})}$$

(Marichal, 1997)

$$\boxed{\mathcal{S}_\mu(x) = \text{median}\Big[\underbrace{x_1,\ldots,x_n}_{n},\underbrace{\mu(A_{(2)}),\ldots,\mu(A_{(n)})}_{n-1}\Big]}$$

(Kandel and Byatt, 1978)

Example: If $x_3 \leq x_1 \leq x_2$ then

$$\mathcal{S}_\mu(x_1,x_2,x_3) = \text{median}[x_1,x_2,x_3,\mu(1,2),\mu(2)]$$

Particularly, for any $i \geq 2$,

$$x_{(i-1)} < \mathcal{S}_\mu(x) < x_{(i)} \quad \Rightarrow \quad \mathcal{S}_\mu(x) = \mu(A_{(i)})$$

For any $k \in N$,

$$\boxed{\mathcal{S}_\mu(x) = \text{median}\Big[\underbrace{\mathcal{S}_\mu(x \,|\, x_k = 1)}_{\text{ind. of } x_k},\underbrace{\mathcal{S}_\mu(x \,|\, x_k = 0)}_{\text{ind. of } x_k},x_k\Big]}$$

(Marichal, 1999)

Example:

$$\mathcal{S}_\mu(x_1,x_2) = \text{median}[\underbrace{\mathcal{S}_\mu(x_1,1)}_{x_1\vee\mu(2)},\underbrace{\mathcal{S}_\mu(x_1,0)}_{x_1\wedge\mu(1)},x_2]$$

Particularly, for any $S \not\ni k$,

$$\mu(S) < \mathcal{S}_\mu(x) < \mu(S \cup k) \quad \Rightarrow \quad \mathcal{S}_\mu(x) = x_k$$

**Interpretation of $\mu$:**

$\mu(S)$ = importance of the combination $S$ of criteria

$e_S$ := characteristic vector of $S$ in $\{0,1\}^n$

$$\boxed{\mu(S) = \mathcal{S}_\mu(e_S)}$$

Example: $(n = 4)$

$$\begin{aligned} \mu(2,4) &= \mathcal{S}_\mu(0,1,0,1) \\ \mu(1,2,4) &= \mathcal{S}_\mu(1,1,0,1) \end{aligned}$$

**The Sugeno integral is a very natural concept**
From

- $n$ variables $x_1,\ldots,x_n \in [0,1]$
- $m$ constants $r_1,\ldots,r_m \in [0,1]$,

we can form a lattice polynomial

$$P_{r_1,\ldots,r_m}(x_1,\ldots,x_n)$$

in a usual manner using $\wedge$, $\vee$, and parenthese.
If such a polynomial fulfills

$$\begin{aligned} P_{r_1,\ldots,r_m}(0,\ldots,0) = 0 \\ P_{r_1,\ldots,r_m}(1,\ldots,1) = 1 \end{aligned}$$

then there exists a fuzzy measure $\mu$ on $N$ such that $P_{r_1,\ldots,r_m} = \mathcal{S}_\mu$ (Marichal, 2000).

Example:

$$P_{r_1,r_2}(x_1,x_2,x_3) = ((x_1 \vee r_2) \wedge x_3) \vee (x_2 \wedge r_1)$$

## Some axiomatic characterizations of the discrete Sugeno integral

**Theorem 1** *(Marichal, 1998) Let* $M : [0,1]^n \to \mathbb{R}$*.*
*The following assertions are equivalent:*

- *There exists a fuzzy measure* $\mu$ *on* $N$ *such that* $M = \mathcal{S}_\mu$
- $M$ *is increasing in each argument and fulfills*
$$M(x_1 \wedge r, \ldots, x_n \wedge r) = M(x_1, \ldots, x_n) \wedge r$$
$$M(x_1 \vee r, \ldots, x_n \vee r) = M(x_1, \ldots, x_n) \vee r$$
*for any* $r, x_1, \ldots, x_n \in [0,1]$
- $M$ *is idempotent, increasing in each argument, and fulfills*
$$M(e_S \wedge r), M(e_S \vee r) \in \{M(e_S), r\}$$
*for any* $S \subseteq N$ *and any* $r \in [0,1]$
- $M$ *is idempotent, increasing in each argument, and fulfills*
$$M(x \wedge x') = M(x) \wedge M(x')$$
$$M(x \vee x') = M(x) \vee M(x')$$
*for any* $x, x' \in [0,1]^n$ *such that* $(x_i - x_j)(x'_i - x'_j) \geq 0$*,* $i, j \in N$

**The discrete Sugeno integral as a tool to aggregate ordinal values**

Let $X = \{r_1 < \cdots < r_k\}$ be a finite ordinal scale.

This scale can be viewed as an ordered $k$-uple of numbers in $[0,1]$:

$$0 = r_1 < \cdots < r_k = 1$$

These numbers are defined up to an increasing bijection $\varphi : [0,1] \to [0,1]$.

We want to aggregate $n$ numbers $x_1, \ldots, x_n \in X$ by a function $M : [0,1]^n \to \mathbb{R}$

**Definition 3** *(Orlov, 1981)*
$M : [0,1]^n \to \mathbb{R}$ *is comparison meaningful from an ordinal scale if, for any increasing bijection* $\varphi : [0,1] \to [0,1]$ *and any* $x, x' \in [0,1]^n$,

$$M(x) \leq M(x') \quad \Leftrightarrow \quad M(\varphi(x)) \leq M(\varphi(x'))$$

*where* $\varphi(x) := (\varphi(x_1), \ldots, \varphi(x_n))$.

The arithmetic mean violates this property.
Example:

$$\begin{aligned} 0.4 = (0.3 + 0.5)/2 &< (0.1 + 0.8)/2 = 0.45 \\ 0.55 = (0.4 + 0.7)/2 &> (0.1 + 0.8)/2 = 0.45 \end{aligned}$$

**Proposition 1** *(Ovchinnikov, 1996)*
*If* $M : [0,1]^n \to \mathbb{R}$ *is internal (⇒ idempotent) and comparison meaningful then*

$$M(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$$

(⇒ $M : X^n \to X$)

**Proposition 2** *(Marichal, 1999)*
$M : [0,1]^n \to \mathbb{R}$ *is idempotent, continuous, and comparison meaningful if and only if there exists a* $\{0,1\}$*-valued fuzzy measure* $\mu$ *on* $N$ *such that* $M = \mathcal{S}_\mu$.

Weakness of this model:

$$M(e_S) = \mathcal{S}_\mu(e_S) = \mu(S) \in \{0,1\} = \{r_1, r_k\} \quad !!$$

The importance of any subset of criteria is always an extreme value of $X$.

**Proposition 3** *(Marichal, 1999)*
$M : [0,1]^n \to \mathbb{R}$ *is non-constant, continuous, and comparison meaningful if and only if there exists a* $\{0,1\}$*-valued fuzzy measure* $\mu$ *on* $N$ *and a continuous and strictly monotonic function* $g : [0,1] \to \mathbb{R}$ *such that* $M = g \circ \mathcal{S}_\mu$.

**Let us enrich the aggregation model:**

For each set function $v : 2^N \to [0,1]$ such that $v(\emptyset) = 0$ and $v(N) = 1$, we define an aggregation function

$$M_v : [0,1]^n \to \mathbb{R}.$$

However,

$$\begin{cases} x_i \in X \\ v(S) \in X \qquad \text{(cf. } \mu(S) = \mathcal{S}_\mu(e_S)) \end{cases}$$

The mapping $(x, v) \mapsto M_v(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, is comparison meaningful.

**Theorem 2** *(Marichal, 1999)*
*The set of functions $M_v : [0,1]^n \to \mathbb{R}$ ($v$ as defined above) such that*

- $i)$ *$M_v$ is idempotent (for all $v$)*
- $ii)$ *$(x, v) \mapsto M_v(x)$ is comparison meaningful and continuous*

*identifies with the class of the Sugeno integrals on $[0,1]^n$.*

**Open problem:** Suppress continuity or replace it by increasing monotonicity

**The use of the Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework**

$A = \{a, b, c, \ldots\}$ set of actions (alternatives)

$N = \{1, \ldots, i, \ldots, n\}$ set of criteria

Each $i \in N$ is represented by

$$g_i : A \to X_i$$

$$X_i = \{r_1^{(i)} < \cdots < r_{k_i}^{(i)}\} \qquad \text{(finite ordinal scale)}$$

**Example**:

Application for an academic position at University of Liège

| | | | | | |
|---|---|---|---|---|---|
| | Exc. | V.Good | Good | Sat. | Weak |
| Scientific value of CV: | ☐ | ☐ | ☐ | ☐ | ☐ |
| | Exc. | V.Good | Sat. | Weak | V.Weak |
| Teaching effectiveness: | ☐ | ☐ | ☐ | ☐ | ☐ |
| | Positive | Medium | Negative | | |
| Interview: | ☐ | ☐ | ☐ | | |

Profile related to action $a \in A$ :

$$a : \Big(\underbrace{g_1(a)}_{\in X_1}, \ldots, \underbrace{g_i(a)}_{\in X_i}, \ldots, \underbrace{g_n(a)}_{\in X_n}\Big) \in \prod_{i=1}^{n} X_i$$

$a$ : (V.Good, Sat., Medium)

How can we aggregate $(g_1(a), \ldots, g_n(a))$ ?
(Roubens, 1999)

Let $X = \{r_1 < \cdots < r_k\}$ be the finite ordinal scale of global evaluations.

| | A1 | A2 | B | C |
|---|---|---|---|---|
| Global evaluation: | □ | □ | □ | □ |

We assume the existence of mappings

$$U_i : X_i \to X \qquad (i \in N)$$

enabling to express all the partial evaluations in the same scale $X$:

$$a : \Big(\underbrace{U_1(g_1(a))}_{\in X}, \ldots, \underbrace{U_n(g_n(a))}_{\in X}\Big)$$

These “commensurateness” mappings must be non-decreasing and such that

$$U_i(r_1^{(i)}) = r_1 = 0 \quad \text{and} \quad U_i(r_{k_i}^{(i)}) = r_k = 1$$

Example (continued)

$$\begin{aligned} X_1 &: \text{A1} = U_1(\text{E}) \leq \cdots \leq U_1(\text{W}) = \text{C} \\ X_2 &: \text{A1} = U_1(\text{E}) \leq \cdots \leq U_1(\text{VW}) = \text{C} \\ X_3 &: \text{A1} = U_1(\text{P}) \leq \cdots \leq U_1(\text{N}) = \text{C} \end{aligned}$$

We define an aggregation function $M : X^n \to X$ that determines the global evaluation

$$g(a) = M[\underbrace{U_1(g_1(a))}_{\in X}, \ldots, \underbrace{U_n(g_n(a))}_{\in X}] \in X$$

As a consequence,

all actions are comparable in terms of a WEAK ORDER defined on $A$.

Example (continued)
What is the global evaluation of the profile

$$a : (\text{V.Good, Sat., Medium})?$$

$$g(a) = M[U_1(\text{VG}), U_2(\text{S}), U_3(\text{M})] = ?$$

| A1 | A2 | B | C |
|---|---|---|---|
| □ | □ | □ | □ |

?

Given an aggregation function $M : X^n \to X$, how can we identify the commensurateness mappings $U_i$ ?

## Identification of mappings $U_i$ when $M = \mathcal{S}_\mu$

(Marichal and Roubens, 1999)

1) $\mathcal{S}_\mu$ is uniquely determined by $\mu$

$$\mu(S) = \mathcal{S}_\mu(e_S)$$

$\longrightarrow$ provided by the decision maker ($2^n - 2$ questions)

However, we often have

$$\mathcal{S}_\mu(0,1,0,1,1) = 0, \ldots$$

$\longrightarrow \approx n$ questions

$$\mu(N \setminus i) = \mathcal{S}_\mu(e_{N \setminus i}) \qquad (i \in N)$$

Example (continued)

The decision maker gives

$$\mu(1,2,3) = \text{A1}$$
$$\mu(1,2) = \text{A2}$$
$$\mu(1,3) = \mu(1) = \text{B}$$
$$\mu(2,3) = \text{C}$$

2) Fix $i \in N$. We have to determine $U_i : X_i \to X$, that is,

$$U_i(r_j^{(i)}), \qquad j = 1, \ldots, k_i$$

(cf. $X_i = \{r_1^{(i)} < \cdots < r_{k_i}^{(i)}\}$)

We ask the decision maker to appraise

$$\mathcal{S}_\mu(U_i(r_j^{(i)})e_i + e_{N\setminus i}), \qquad j = 1, \ldots, k_i$$

Example:

$$\begin{aligned}
\mathcal{S}_\mu(U_1(\text{E}), 1, 1) &= \text{A1} = \mu(1, 2, 3) \\
\mathcal{S}_\mu(U_1(\text{VG}), 1, 1) &= \text{A2} \\
\mathcal{S}_\mu(U_1(\text{G}), 1, 1) &= \text{A2} \\
\mathcal{S}_\mu(U_1(\text{S}), 1, 1) &= \text{B} \\
\mathcal{S}_\mu(U_1(\text{W}), 1, 1) &= \text{C} = \mu(2, 3)
\end{aligned}$$

By increasing monotonicity, we have

$$\mu(N \setminus i) \leq \mathcal{S}_\mu(U_i(r_j^{(i)})e_i + e_{N\setminus i}) \leq \mu(N)$$

More precisely, we have

$$\mathcal{S}_\mu(U_i(r_j^{(i)})e_i + e_{N\setminus i}) = \text{median}[\mu(N), \mu(N\setminus i), U_i(r_j^{(i)})]$$

$$\begin{aligned}
\mathcal{S}_\mu(U_1, 1, 1) &= \text{median}[\text{A1}, \text{C}, U_1] \\
&= U_1
\end{aligned}$$

$\Rightarrow U_1$ is completely determined !

Identification of $U_2$:

$$\begin{aligned}
\mathcal{S}_\mu(1, U_2(\mathsf{E}), 1) &= \mathsf{A1} = \mu(1,2,3) \\
\mathcal{S}_\mu(1, U_2(\mathsf{VG}), 1) &= \mathsf{A1} \\
\mathcal{S}_\mu(1, U_2(\mathsf{S}), 1) &= \mathsf{A1} \\
\mathcal{S}_\mu(1, U_2(\mathsf{W}), 1) &= \mathsf{A2} \\
\mathcal{S}_\mu(1, U_2(\mathsf{VW}), 1) &= \mathsf{B} = \mu(1,3)
\end{aligned}$$

$$\mathcal{S}_\mu(1, U_2, 1) = \mathsf{median}[\mathsf{A1}, \mathsf{B}, U_2] = U_2 \vee \mathsf{B}$$

$\Rightarrow U_2$ is completely determined !

Identification of $U_3$:

$$\begin{aligned}
\mathcal{S}_\mu(1, 1, U_3(\mathsf{P})) &= \mathsf{A1} = \mu(1,2,3) \\
\mathcal{S}_\mu(1, 1, U_3(\mathsf{M})) &= \mathsf{A2} \\
\mathcal{S}_\mu(1, 1, U_3(\mathsf{N})) &= \mathsf{A2} = \mu(1,2)
\end{aligned}$$

$$\mathcal{S}_\mu(1, 1, U_3) = \mathsf{median}[\mathsf{A1}, \mathsf{A2}, U_3] = U_3 \vee \mathsf{A2}$$

$\Rightarrow U_3(\mathsf{M}) \in \{\mathsf{A2}, \mathsf{B}, \mathsf{C}\}$ not completely determined

However,

$$\mathcal{S}_\mu(U_1, U_2, U_3(\mathsf{M})) = (\mathsf{B} \wedge U_1) \vee (\mathsf{A2} \wedge U_1 \wedge U_2)$$

# The Sugeno integral and preferential independence

Example: Evaluation of students w.r.t. 3 subjects

$$X = \{\text{E} > \text{VG} > \text{G} > \text{S} > \text{W} > \text{VW}\}$$

| student | St | Pr | Al |
|---|---|---|---|
| $a$ | E | G | VG |
| $b$ | E | VG | G |
| $c$ | S | G | VG |
| $d$ | S | VG | G |

Profile of student $a$ : $x^a = (\text{E}, \text{G}, \text{VG})$, etc.

We define a weak order on $A = \prod_i X_i$ by

$$a \succeq b \quad \Leftrightarrow \quad M(x^a) \geq M(x^b)$$

We assume that $\succeq$ is given by the decision maker

- by monotonicity:

$$a \succeq c \quad \text{and} \quad b \succeq d$$

- **St** and **Pr** are somewhat substitutive but each of them is more important than **Al**

$$a \succeq b \quad \text{and} \quad d \succeq c$$

For any $S \subseteq N$ and any profile $x, y \in \prod_i X_i$, we define $xSy$ by

$$(xSy)_i := \begin{cases} x_i & \text{if } i \in S \\ y_i & \text{if } i \notin S \end{cases}$$

For instance: $(\mathsf{E}, \mathsf{VG}, \mathsf{G})\{2,3\}(\mathsf{S}, \mathsf{S}, \mathsf{S}) = (\mathsf{S}, \mathsf{VG}, \mathsf{G})$

## Independence conditions of the weak order $\succeq$

- Mutual independence (MI)

$$xSy \succeq x'Sy \quad \Leftrightarrow \quad xSz \succeq x'Sz \qquad (*)$$

for all $x, x', y, z \in \prod_i X_i$ and all $S \subseteq N$

- Coordinate independence (CI) (equiv. to MI)

$\equiv$ restriction of $(*)$ to $S = N \setminus \{k\}$ for all $k \in N$

- Weak separability (WS)

$\equiv$ restriction of $(*)$ to $S = \{k\}$ for all $k \in N$

**Theorem 3** *(Marichal, 1999)*
*Assume that there exists a fuzzy measure $\mu$ on $N$ such that $M = \mathcal{S}_\mu$. The following conditions are equivalent.*

$i)$ $\succeq$ *fulfills (MI)*
$ii)$ $\succeq$ *fulfills (WS)*
$iii)$ $\exists k \in N$ s.t. $\mathcal{S}_\mu(x) = x_k \; \forall x$

**Definition 4** $M : [0,1]^n \to \mathbb{R}$ *is comparison meaningful from independent ordinal scales if, for any increasing bijections* $\varphi_1, \ldots, \varphi_n : [0,1] \to [0,1]$ *and any* $x, x' \in [0,1]^n$,

$$M(x) \leq M(x') \quad \Leftrightarrow \quad M(\varphi(x)) \leq M(\varphi(x'))$$

*where* $\varphi(x) := (\varphi_1(x_1), \ldots, \varphi_n(x_n))$.

**Proposition 4** *(Marichal, 1999)*
$M : [0,1]^n \to \mathbb{R}$ *non-constant, continuous, and comparison meaningful from independent ordinal scales if and only if there exists* $k \in N$ *and a continuous and strictly monotonic function* $g : [0,1] \to \mathbb{R}$ *such that*

$$M(x) = g(x_k) \qquad (x \in [0,1]^n)$$

+ Idempotence $\Rightarrow M(x) = x_k$

## Weaker forms of independence for $\succeq$

- Directional mutual independence (DMI)

$$xSy \succ x'Sy \quad \Rightarrow \quad xSz \succeq x'Sz \qquad (**)$$

for all $x, x', y, z \in \prod_i X_i$ and all $S \subseteq N$

- Directional coordinate independence (DCI) (DMI $\Rightarrow$ DCI)

$\equiv$ restriction of $(**)$ to $S = N \setminus \{k\}$ for all $k \in N$

- Directional weak separability (DWS) (DMI $\Rightarrow$ DWS)

$\equiv$ restriction of $(**)$ to $S = \{k\}$ for all $k \in N$

**Proposition 5** *(Roubens, 1999)*
- *If a Sugeno integral represents $\succeq$ then the weak order fulfills* **DWS** *but violates* **DCI***.*
- *If a symmetric Sugeno integral (owmax or owmin) represents $\succeq$ then the weak order fulfills* **DCI** *but violates* **DMI***.*
- *If a maxitive (or minitive) Sugeno integral (wmax or wmin) represents $\succeq$ then the weak order fulfills* **DMI** *but violates* **MI***.*