Behavioral analysis of aggregation in multicriteria decision aid

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SKETCH OF THE PRESENTATION

Assumptions : cardinal setting, commensurable evaluations

aggregation of decision criteria

Weighted arithmetic mean
Additive measure
Problem: interaction phenomena ?

Choquet integral
Fuzzy measure
Problem: how to interpret it ?

Behavioral indices :
- global importance of criteria
- influence of criteria
- interaction among criteria
- tolerance of the decision maker
- dispersion of the importance of criteria

Aggregation in multicriteria decision making

- Alternatives $A = \{a, b, c, \ldots,\}$

- Criteria $N = \{1, 2, \ldots, n\}$

- Profile $a \in A \longrightarrow \qquad (x_1^a, \ldots, x_n^a) \in \mathbb{R}^n$

  commensurable partial scores
  (defined on the same interval scale)

- Aggregation operator $M : \mathbb{R}^n \to \mathbb{R}$
  $M : [0,1]^n \to [0,1]$

Alternative	crit. 1	$\cdots$	crit. n	global score
a	x_1^a	$\cdots$	x_n^a	$M(x_1^a, \ldots, x_n^a)$
b	x_1^b	$\cdots$	x_n^b	$M(x_1^b, \ldots, x_n^b)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Example : Evaluation of students w.r.t. three subjects: statistics, probability, algebra.

Student	St	Pr	Al
a	19	15	18
b	19	18	15
c	11	15	18
d	11	18	15

$\longrightarrow$

St	Pr	Al
0.95	0.75	0.90
0.95	0.90	0.75
0.55	0.75	0.90
0.55	0.90	0.75

(marks are expressed on a scale from 0 to 20)

An often used operator: the weighted arithmetic mean

$$\mathrm{WAM}_\omega(x) := \sum_{i=1}^{n} \omega_i x_i$$

with $\sum_i \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in N$

$$\left.\begin{array}{r} \omega_{\mathrm{St}} = 35\% \\ \omega_{\mathrm{Pr}} = 35\% \\ \omega_{\mathrm{Al}} = 30\% \end{array}\right\} \Rightarrow$$

Student	global evaluation
a	0.750
b	0.872
c	0.725
d	0.732

$$\boxed{b \succ a \succ d \succ c}$$

$$\begin{aligned} \mathsf{WAM}_\omega(1,0,0) &= \omega_{\mathsf{St}} = 0.35 \\ \mathsf{WAM}_\omega(0,1,0) &= \omega_{\mathsf{Pr}} = 0.35 \\ \mathsf{WAM}_\omega(1,1,0) &= 0.70\ !!! \end{aligned}$$

What is the importance of $\{\mathsf{St},\mathsf{Pr}\}$?

Definition (Choquet, 1953; Sugeno, 1974)
A fuzzy measure on N is a set function $v : 2^N \to [0,1]$ such that
 $i)$ $v(\emptyset) = 0, v(N) = 1$
 $ii)$ $S \subseteq T \Rightarrow v(S) \leq v(T)$

$$\begin{aligned} v(S) &= \text{weight of } S \\ &= \text{degree of importance of } S \\ &= \text{power of } S \text{ to make the decision alone} \\ &\ \text{(without the remaining criteria)} \end{aligned}$$

A fuzzy measure is additive if

$$v(S \cup T) = v(S) + v(T) \quad \text{if } S \cap T = \emptyset$$

$\to$ independent criteria

$$v(\mathsf{St},\mathsf{Pr}) = v(\mathsf{St}) + v(\mathsf{Pr})\ (= 0.70)$$

The discrete Choquet integral

Definition

Let $v \in \mathcal{F}_N$. The (discrete) Choquet integral of $x \in \mathbb{R}^n$ w.r.t. v is defined by

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)}[v(A_{(i)}) - v(A_{(i+1)})]$$

with the convention that $x_{(1)} \leq \cdots \leq x_{(n)}$.
Also, $A_{(i)} = \{(i), \ldots, (n)\}$.

Example: If $x_3 \leq x_1 \leq x_2$, we have

$$\begin{aligned} \mathcal{C}_v(x_1, x_2, x_3) &= x_3\,[v(3,1,2) - v(1,2)] \\ &\quad + x_1\,[v(1,2) - v(2)] \\ &\quad + x_2\,v(2) \end{aligned}$$

Particular case:

$$\boxed{v \text{ additive} \quad \Rightarrow \quad \mathcal{C}_v = \mathrm{WAM}_\omega}$$

Indeed,

$$\mathcal{C}_v(x) = \sum_{i=1}^{n} x_{(i)} v((i)) = \sum_{i=1}^{n} x_i \underbrace{v(i)}_{\omega_i}$$

Properties of the Choquet integral

Linearity w.r.t. the fuzzy measure :
There exist 2^n functions $f_T : \mathbb{R}^n \to \mathbb{R}$ ($T \subseteq N$) such that

$$\mathcal{C}_v = \sum_{T \subseteq N} v(T)\, f_T \qquad (v \in \mathcal{F}_N)$$

Indeed, on can show that

$$\mathcal{C}_v(x) = \sum_{T \subseteq N} v(T) \underbrace{\sum_{K \supseteq T} (-1)^{|K|-|T|} \min_{i \in K} x_i}_{f_T(x)}$$

Stability w.r.t. positive linear transformations :
For any $x \in \mathbb{R}^n, r > 0, s \in \mathbb{R}$,

$$\mathcal{C}_v(r\, x_1 + s, \ldots, r\, x_n + s) = r\, \mathcal{C}_v(x_1, \ldots, x_n) + s$$

Example : marks obtained by students
- on a $[0, 20]$ scale : 16, 11, 7, 14
- on a $[0, 1]$ scale : 0.80, 0.55, 0.35, 0.70
- on a $[-1, 1]$ scale : 0.60, 0.10, -0.30, 0.40

Remark : The partial scores may be embedded in $[0, 1]$

Monotonicity

For any $x, x' \in \mathbb{R}^n$, one has

$$x_i \leq x'_i \quad \forall i \in N \qquad \Rightarrow \qquad \mathcal{C}_v(x) \leq \mathcal{C}_v(x')$$

$\mathcal{C}_v$ is properly weighted by v

$$\mathcal{C}_v(e_S) = v(S) \qquad\qquad (S \subseteq N)$$

$e_S =$ characteristic vector of S in $\{0,1\}^n$

Example : $e_{\{1,3\}} = (1, 0, 1, 0, \ldots)$

Independent criteria	Dependent criteria
$\mathrm{WAM}_\omega(e_{\{i\}}) = \omega_i$	$\mathcal{C}_v(e_{\{i\}}) = v(i)$
$\mathrm{WAM}_\omega(e_{\{i,j\}}) = \omega_i + \omega_j$	$\mathcal{C}_v(e_{\{i,j\}}) = v(i,j)$

Example :

$$\begin{array}{ccccc} v(\mathrm{St},\mathrm{Pr}) & < & v(\mathrm{St}) & + & v(\mathrm{Pr}) \\ \| & & \| & & \| \\ \mathcal{C}_v(1,1,0) & & \mathcal{C}_v(1,0,0) & & \mathcal{C}_v(0,1,0) \end{array}$$

Axiomatic characterization of the class of Choquet integrals with n arguments

Theorem

The operators $M_v : \mathbb{R}^n \to \mathbb{R}$ $(v \in \mathcal{F}_N)$ are

- **linear w.r.t. the underlying fuzzy measure v :**

  M_v is of the form

  $$M_v = \sum_{T \subseteq N} v(T)\, f_T \qquad (v \in \mathcal{F}_N)$$

  where f_T's are independent of v

- **stable for the positive linear transformations :**

  $$M_v(r\,x_1 + s, \ldots, r\,x_n + s) = r\,M_v(x_1, \ldots, x_n) + s$$

  for all $x \in \mathbb{R}^n, r > 0, s \in \mathbb{R}$

- **non-decreasing in each argument (monotonic)**

- **properly weighted by** v **:**

$$M_v(e_S) = v(S) \qquad (S \subseteq N, v \in \mathcal{F}_N)$$

if and only if $M_v = \mathcal{C}_v$ for all $v \in \mathcal{F}_N$.

Back to the example of evaluation of students

Student	St	Pr	Al
a	19	15	18
b	19	18	15
c	11	15	18
d	11	18	15

Assumptions :

- St and Pr are more important than Al
- St and Pr are somewhat substitutive

Behavior of the decision maker :

When a student is good at statistics (19), it is preferable that he/she is better at algebra than probability, so

$$\boxed{a \succ b}$$

When a student is not good at statistics (11), it is preferable that he/she is better at probability than algebra, so

$$\boxed{d \succ c}$$

Additive model : WAM_ω

$$\left.\begin{array}{lcl} a \succ b & \Leftrightarrow & \omega_{\mathsf{Al}} > \omega_{\mathsf{Pr}} \\ d \succ c & \Leftrightarrow & \omega_{\mathsf{Al}} < \omega_{\mathsf{Pr}} \end{array}\right\} \quad \text{No solution !}$$

Non-additive model : $\mathcal{C}_v$

$$v(\mathsf{St}) = 0.35$$
$$v(\mathsf{Pr}) = 0.35$$
$$v(\mathsf{Al}) = 0.30$$

$$v(\mathsf{St}, \mathsf{Pr}) = 0.50 \qquad \text{(redundancy)}$$
$$v(\mathsf{St}, \mathsf{Al}) = 0.80 \qquad \text{(complementarity)}$$
$$v(\mathsf{Pr}, \mathsf{Al}) = 0.80 \qquad \text{(complementarity)}$$

$$v(\emptyset) = 0$$
$$v(\mathsf{St}, \mathsf{Pr}, \mathsf{Al}) = 1$$

Student	St	Pr	Al	Global evaluation
a	19	15	18	17.75
b	19	18	15	16.85
c	11	15	18	15.10
d	11	18	15	15.25

$$\boxed{a \succ b \succ d \succ c}$$

Particular cases of Choquet integrals

1) Weighted arithmetic mean

$$\mathsf{WAM}_\omega(x) = \sum_{i=1}^{n} \omega_i x_i\,, \quad \sum_{i=1}^{n} \omega_i = 1, \quad \omega_i \geq 0$$

Proposition
Let $v \in \mathcal{F}_N$. The following assertions are equivalents :

$i)$ v is additive
$ii)$ $\exists$ a weight vector ω such that $\mathcal{C}_v = \mathsf{WAM}_\omega$
$iii)$ $\mathcal{C}_v$ is additive, i.e. $\mathcal{C}_v(x+x') = \mathcal{C}_v(x) + \mathcal{C}_v(x')$

$$\begin{aligned} v(S) &= \sum_{i\in S} \omega_i \qquad (S \subseteq N) \\ \omega_i &= v(i) \qquad (i \in N) \end{aligned}$$

- arithmetic mean $\quad (\omega = (1/n, \ldots, 1/n))$

$$\mathsf{AM}(x) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

- k-th projection $\quad (\omega = e_{\{k\}})$

$$\mathsf{P}_k(x) = x_k$$

2) Ordered weighted averaging (Yager, 1988)

$$\mathrm{OWA}_\omega(x) = \sum_{i=1}^{n} \omega_i x_{(i)}\,, \quad \sum_{i=1}^{n} \omega_i = 1, \quad \omega_i \geq 0$$

with the convention that $x_{(1)} \leq \cdots \leq x_{(n)}$.

Proposition (Grabisch, 1995)
Let $v \in \mathcal{F}_N$. The following assertions are equivalents :
$i)$ v is cardinality-based : $|S| = |S'| \Rightarrow v(S) = v(S')$
$ii)$ $\exists$ a weight vector ω such that $\mathcal{C}_v = \mathrm{OWA}_\omega$
$iii)$ $\mathcal{C}_v$ is a symmetric function.

$$\begin{aligned} v(S) &= \sum_{i=n-s+1}^{n} \omega_i \qquad (S \subseteq N, S \neq \emptyset) \\ \omega_{n-s} &= v(S \cup i) - v(S) \qquad (i \in N, S \subseteq N \setminus i) \end{aligned}$$

- arithmetic mean $\quad (\omega = (1/n, \ldots, 1/n))$

- k-th order statistic $\quad (\omega = e_{\{k\}})$

$$\mathrm{OS}_k(x) = x_{(k)}$$

Note. If $n = 2k - 1$ then $\mathrm{OS}_k = \text{median}$

3) Partial minima and maxima

Let $T \subseteq N$, with $T \neq \emptyset$.

$$\min\nolimits_T(x) = \min_{i \in T} x_i$$

$$v(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{else} \end{cases}$$

$$\max\nolimits_T(x) = \max_{i \in T} x_i$$

$$v(S) = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{else} \end{cases}$$

- minimum ($T = N$)

$$v(S) = \begin{cases} 1 & \text{if } S = N \\ 0 & \text{else} \end{cases}$$

- maximum ($T = N$)

$$v(S) = \begin{cases} 1 & \text{if } S \neq \emptyset \\ 0 & \text{else} \end{cases}$$

Behavioral analysis of aggregation

Given a fuzzy measure $v \in \mathcal{F}_N$,

how can we interpret it ?

$\downarrow$

Behavioral indices

global importance of criteria
influence of criteria
interaction among criteria
tolerance / intolerance of the decision maker
dispersion of the importance of criteria

Global importance of criteria

Given $i \in N$, it may happen that

- $v(i) = 0$
- $v(T \cup i) \gg v(T)$ for many $T \subseteq N \setminus i$

The overall importance of $i \in N$ should not be solely determined by $v(i)$, but by all $v(T \cup i)$ such that $T \subseteq N \setminus i$.

Marginal contribution of i in combination $T \subseteq N \setminus i$:

$$v(T \cup i) - v(T)$$

Shapley power index (Shapley, 1953)
= Average value of the marginal contribution of i alone in all combinations :

$$\phi(v,i) := \frac{1}{n} \sum_{t=0}^{n-1} \underbrace{\frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus i \\ |T|=t}} [v(T \cup i) - v(T)]}_{\text{average over all the subsets of the same size } t}$$

$$\phi(v,i) = \sum_{T \subseteq N \setminus i} \frac{(n-t-1)!\,t!}{n!}[v(T \cup i) - v(T)]$$

(proposed in MCDM by Murofushi in 1992)

Properties of the Shapley power index

$i)$ $\phi(v,i) \in [0,1]$ for all $i \in N$

$ii)$ $\sum_i \phi(v,i) = 1$

$iii)$ v additive $\Rightarrow \phi(v,i) = v(i)$ for all $i \in N$

Axiomatic characterization

Theorem (Shapley, 1953)
The numbers $\psi(v,i)$ $(i \in N, v \in \mathcal{F}_N)$

- **are linear w.r.t. the fuzzy measure v :**
  $\psi(v,i)$ is of the form

$$\psi(v,i) = \sum_{T \subseteq N} v(T)\, p_T^i \qquad (i \in N, v \in \mathcal{F}_N)$$

  where p_T^i's are independent of v

- **are symmetric, i.e., independent of the labels :**

$$\psi(v,i) = \psi(\pi v, \pi(i)) \qquad (i \in N, v \in \mathcal{F}_N)$$

  for any permutation π on N

- **fulfill the "null criterion" axiom :**

$$v(T \cup i) = v(T) \quad \forall T \subseteq N \setminus i \quad \Rightarrow \quad \psi(v,i) = 0$$

- **fulfill the "efficiency" axiom :**

$$\sum_{i=1}^{n} \psi(v,i) = 1 \qquad (v \in \mathcal{F}_N)$$

if and only if $\psi = \phi$ (Shapley power index).

v	$\phi(v,i)$
v_{WAM_ω}	ω_i
v_{OWA_ω}	$1/n$

Probabilistic interpretation

Define

$$\Delta_i\, \mathcal{C}_v(x) := \mathcal{C}_v(x \mid x_i = 1) - \mathcal{C}_v(x \mid x_i = 0)$$

(marginal contribution of criterion i on the aggregation at x)

We have

$$\phi(v,i) = \int_{[0,1]^n} \Delta_i\, \mathcal{C}_v(x)\, dx$$

that is,

$$\phi(v,i) = E[\Delta_i\, \mathcal{C}_v(x)]$$

where the expectation is defined from the uniform distribution over $[0,1]^n$.

$\phi(v,i)$ = expected value of the amplitude of the range of $\mathcal{C}_v$ that criterion i may control when assigning partial evaluations to the other criteria at random

Influence of criteria on the aggregation

Marginal contribution of $S \subseteq N$ in combination $T \subseteq N \setminus S$:

$$v(T \cup S) - v(T)$$

The influence of S on the aggregation operator $\mathcal{C}_v$ is defined as the average value of the marginal contribution of S in all outer combinations :

$$I(\mathcal{C}_v, i) := \frac{1}{n-s+1} \sum_{t=0}^{n-s} \underbrace{\frac{1}{\binom{n-s}{t}} \sum_{\substack{T \subseteq N \setminus S \\ |T|=t}} [v(T \cup S) - v(T)]}_{\substack{\text{average over all the subsets} \\ \text{of the same size } t}}$$

Properties of the influence function

i) $I(\mathcal{C}_v, S) \in [0, 1]$ for all $S \subseteq N$

ii) $I(\mathcal{C}_v, i) = \phi(v, i)$ for all $i \in N$

iii) v additive $\Rightarrow I(\mathcal{C}_v, S) = v(S)$ for all $S \subseteq N$

$\mathcal{C}_v$	$I(\mathcal{C}_v, S)$
WAM_ω	$\displaystyle\sum_{i\in S}\omega_i$
OWA_ω	$\displaystyle\frac{1}{n-s+1}\sum_{i=1}^{n}\omega_i\,\min(i,s,n-i+1,n-s+1)$

Probabilistic interpretation

We have

$$I(\mathcal{C}_v, S) = \int_{[0,1]^n}[\mathcal{C}_v(x \mid x_S = 1) - \mathcal{C}_v(x \mid x_S = 0)]\, dx$$

that is,

$$I(\mathcal{C}_v, S) = E[\mathcal{C}_v(x \mid x_S = 1) - \mathcal{C}_v(x \mid x_S = 0)]$$

$I(\mathcal{C}_v, S)$ = expected value of the amplitude of the range of $\mathcal{C}_v$ that criteria S may control when assigning partial evaluations to the other criteria at random

Interaction among criteria

Consider a pair $\{i, j\}$ of criteria. If

$$\underbrace{v(T \cup ij) - v(T \cup i)}_{\substack{\text{contribution of } j \text{ in} \\ \text{the presence of } i}} \quad < \quad \underbrace{v(T \cup j) - v(T)}_{\substack{\text{contribution of } j \text{ in} \\ \text{the absence of } i}} \qquad\qquad (T \subseteq N \backslash ij)$$

then there is an overlap effect between i and j.

Marginal interaction between i and j, conditioned to the presence of $T \subseteq N \setminus ij$:

$$v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T)$$

$$\begin{cases} < 0 & \rightarrow i \text{ and } j \text{ are competitive} \\ > 0 & \rightarrow i \text{ and } j \text{ are complementary} \\ = 0 & \rightarrow i \text{ and } j \text{ do not interact} \end{cases}$$

Interaction index (Owen, 1972)
= Average value of the marginal interaction between i and j :

$$I(v, ij) := \frac{1}{n-1} \sum_{t=0}^{n-2} \underbrace{\frac{1}{\binom{n-2}{t}} \sum_{\substack{T \subseteq N \backslash ij \\ |T| = t}} [v(T \cup ij) - \ldots]}_{\substack{\text{average over all the subsets} \\ \text{of the same size } t}}$$

(proposed in MCDM by Murofushi and Soneda in 1993)

Probabilistic interpretation

Define

$$\begin{aligned}\Delta_{ij}\,\mathcal{C}_v(x) &= \Delta_i\Delta_j\,\mathcal{C}_v(x)\\ &= \mathcal{C}_v(x \mid x_i = x_j = 1) - \mathcal{C}_v(x \mid x_i = 1, x_j = 0)\\ &\quad -\mathcal{C}_v(x \mid x_i = 0, x_j = 1) + \mathcal{C}_v(x \mid x_i = x_j = 0)\end{aligned}$$

(marginal interaction between i and j at x)

We have

$$\begin{aligned} I(v,ij) &= \int_{[0,1]^n} \Delta_{ij}\,\mathcal{C}_v(x)\,dx \\ &= E[\Delta_{ij}\,\mathcal{C}_v(x)]\end{aligned}$$

Generalization to any combination S
(Grabisch and Roubens, 1998)

$$I(v,S) := E[\Delta_S\,\mathcal{C}_v(x)]$$

$$I(v,S) = \sum_{T\subseteq N\setminus S} \frac{(n-t-s)!\,t!}{(n-s+1)!} \sum_{K\subseteq S} (-1)^{s-k} v(K\cup T)$$

Properties of the interaction

$i)$ $I(v, ij) \in [-1, 1]$ for all $ij \in N$

$ii)$ $I(v, i) = \phi(v, i)$ for all $i \in N$

$iii)$ v additive $\Rightarrow I(v, S) = 0$ for all $S \subseteq N, |S| \geq 2$

v	$I(v, S), \|S\| \geq 2$
v_{WAM_ω}	0
v_{OWA_ω}	$\dfrac{1}{n-s+1} \sum_{i=1}^{s-2} \binom{s-2}{i} (-1)^{s-i} (\omega_{s-i-1} - \omega_{n-i})$

Conjunction and disjunction degrees

Average value of $\mathcal{C}_v$ over $[0,1]^n$:

$$E[\mathcal{C}_v(x)] = \int_{[0,1]^n} \mathcal{C}_v(x)\, dx$$

$\longrightarrow$ gives the average position of $\mathcal{C}_v$ within the interval $[0,1]$.

Since

$$\min x_i \leq \mathcal{C}_v(x) \leq \max x_i$$

we have

$$E(\min) \leq E(\mathcal{C}_v) \leq E(\max)$$

Conjunction degree :

$$\text{andness}(\mathcal{C}_v) := \frac{E(\max) - E(\mathcal{C}_v)}{E(\max) - E(\min)}$$

Disjunction degree :

$$\text{orness}(\mathcal{C}_v) := \frac{E(\mathcal{C}_v) - E(\min)}{E(\max) - E(\min)}$$

(Dujmović, 1974)

Properties

$i)$ andness($\mathcal{C}_v$), orness($\mathcal{C}_v$) $\in [0,1]$

$ii)$ andness($\mathcal{C}_v$) + orness($\mathcal{C}_v$) = 1

$iii)$ orness($\mathcal{C}_v$) = 0 (resp. 1) $\Leftrightarrow \mathcal{C}_v = \min$ (resp. max)

We have

$$\text{orness}(\mathcal{C}_v) = \frac{1}{n-1} \sum_{t=1}^{n-1} \underbrace{\frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} v(T)}_{\substack{\text{average over all} \\ \text{the subsets of} \\ \text{the same size } t}}$$

$\mathcal{C}_v$	orness($\mathcal{C}_v$)
WAM_ω	$1/2$
OWA_ω	$\underbrace{\frac{1}{n-1} \sum_{i=1}^{n} (i-1)\omega_i}_{\substack{\text{as proposed} \\ \text{by Yager in 1988}}}$

Veto and favor effects

A criterion $i \in N$ is

- a *veto* for $\mathcal{C}_v$ if

$$\mathcal{C}_v(x) \leq x_i \qquad (x \in [0,1]^n)$$

- a *favor* for $\mathcal{C}_v$ if

$$\mathcal{C}_v(x) \geq x_i \qquad (x \in [0,1]^n)$$

(Dubois and Koning, 1991; Grabisch, 1997)

Proposition
1) i is a veto for $\mathcal{C}_v$ iff $\exists\ \lambda \in [0,1[$ s.t.

$$x_i \leq \lambda \quad \Rightarrow \quad \mathcal{C}_v(x) \leq \lambda$$

2) i is a favor for $\mathcal{C}_v$ iff $\exists\ \lambda \in]0,1]$ s.t.

$$x_i \geq \lambda \quad \Rightarrow \quad \mathcal{C}_v(x) \geq \lambda$$

Problem :
Given $i \in N$ and $v \in \mathcal{F}_N$, how can we define a degree of veto (resp. favor) of i for $\mathcal{C}_v$?

First attempt :
Consider $[0,1]^n$ as a probability space with uniform distribution

$$\text{veto}(\mathcal{C}_v, i) := \Pr[\mathcal{C}_v(x) \leq x_i]$$

However,

$$\Pr[\text{WAM}_\omega(x) \leq x_i] = \begin{cases} 1 & \text{if } \omega_i = 1 \\ 1/2 & \text{else} \end{cases}$$

is non-continuous w.r.t. the fuzzy measure !!!

Second attempt : axiomatic characterization

$$\text{veto}(\mathcal{C}_v, i) := 1 - \frac{1}{n-1} \sum_{T \subseteq N \setminus i} \frac{(n-t-1)!\, t!}{(n-1)!}\, v(T)$$

$$\text{favor}(\mathcal{C}_v, i) := \frac{1}{n-1} \sum_{T \subseteq N \setminus i} \frac{(n-t-1)!\, t!}{(n-1)!}\, v(T \cup i) - \frac{1}{n-1}$$

Theorem
The numbers $\psi(\mathcal{C}_v, i)$ $(i \in N, v \in \mathcal{F}_N)$

- **are linear w.r.t. the fuzzy measure** v **:** $\psi(\mathcal{C}_v, i)$ is of the form
$$\psi(\mathcal{C}_v, i) = \sum_{T \subseteq N} v(T)\, p_T^i \qquad (i \in N, v \in \mathcal{F}_N)$$
where p_T^i's are independent of v

- **are symmetric, i.e., independent of the labels :**
$$\psi(\mathcal{C}_v, i) = \psi(\mathcal{C}_{\pi v}, \pi(i)) \qquad (i \in N, v \in \mathcal{F}_N)$$
for any permutation π on N

- **fulfill the "boundary" axiom :** $\forall T \subseteq N, \forall i \in T$
$$\psi(\mathrm{min}_T, i) = 1$$
(cf. $\mathrm{min}_T(x) \leq x_i$ whenever $i \in T$)

- **fulfill the "normalization" axiom :**
$$\psi(\mathcal{C}_v, i) = \psi(\mathcal{C}_v, j) \quad \forall i, j \in N$$
$$\Downarrow$$
$$\psi(\mathcal{C}_v, i) = \mathrm{andness}(\mathcal{C}_v) \quad \forall i \in N$$

if and only if $\psi = \text{veto}$.

Properties

$i)$ $\text{veto}(\mathcal{C}_v, i), \text{favor}(\mathcal{C}_v, i) \in [0,1]$

$ii)$ $$\frac{1}{n}\sum_{i=1}^{n} \text{veto}(\mathcal{C}_v, i) = \text{andness}(\mathcal{C}_v)$$

$iii)$ $$\frac{1}{n}\sum_{i=1}^{n} \text{favor}(\mathcal{C}_v, i) = \text{orness}(\mathcal{C}_v)$$

$\mathcal{C}_v$	$\text{veto}(\mathcal{C}_v, i)$	$\text{favor}(\mathcal{C}_v, i)$
WAM_ω	$\frac{1}{2} + \frac{n(\omega_i - 1/n)}{2(n-1)}$	$\frac{1}{2} + \frac{n(\omega_i - 1/n)}{2(n-1)}$
OWA_ω	$\frac{1}{n-1} \sum_{j=1}^{n} (n-j)\omega_j$	$\frac{1}{n-1} \sum_{j=1}^{n} (j-1)\omega_j$

Measure of dispersion

$$H(v) := \sum_{i=1}^{n} \sum_{T \subseteq N \setminus i} \frac{(n-t-1)!\, t!}{n!}\, h[v(T \cup i) - v(T)]$$

where

$$h(x) = \begin{cases} -x \log_n x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$H(v)$ measures the degree to which the aggregation function $\mathcal{C}_v$ uses its arguments

Properties

$i)$ $\quad H(v) \in [0,1]$

$ii)$ $\quad H(v_{\mathsf{WAM}_\omega}) = H(v_{\mathsf{OWA}_\omega}) = -\sum_{i=1}^{n} \omega_i \log_n \omega_i$

$iii)$ $\quad H(v) = 1 \quad \Leftrightarrow \quad v = v_{\mathsf{AM}}$

$iv)$ $\quad H(v) = 0 \quad \Leftrightarrow \quad v(S) \in \{0,1\}$

$\qquad\qquad\qquad\quad \Leftrightarrow \quad \mathcal{C}_v(x) \in \{x_1, \ldots, x_n\}$

Back to the example :

Global importance of criteria

$\phi(v, \text{St}) = 0.292$
$\phi(v, \text{Pr}) = 0.292$
$\phi(v, \text{Al}) = 0.417$

Influence of criteria

$I(\mathcal{C}_v, \text{St} \cup \text{Pr}) = 0.600$
$I(\mathcal{C}_v, \text{St} \cup \text{Al}) = 0.725$
$I(\mathcal{C}_v, \text{Pr} \cup \text{Al}) = 0.725$

Interaction among criteria

$I(v, \text{St} \cup \text{Pr}) = -0.25$
$I(v, \text{St} \cup \text{Al}) = 0.10$
$I(v, \text{Pr} \cup \text{Al}) = 0.10$

Conjunction degree

$\text{orness}(\mathcal{C}_v) = 0.517$

Veto and favor degrees

$\text{veto}(\mathcal{C}_v, \text{St}) = 0.437$ $\quad$ $\text{favor}(\mathcal{C}_v, \text{St}) = 0.500$
$\text{veto}(\mathcal{C}_v, \text{Pr}) = 0.437$ $\quad$ $\text{favor}(\mathcal{C}_v, \text{Pr}) = 0.500$
$\text{veto}(\mathcal{C}_v, \text{Al}) = 0.575$ $\quad$ $\text{favor}(\mathcal{C}_v, \text{Al}) = 0.550$

Dispersion of the importance of criteria

$H(v) = 0.820$

Inverse problem :

How to assess v from the behavior of the decision maker ?

$\downarrow$

maximize $\quad H(v)$

subject to

$a \succ b \qquad$ (i.e. $\mathcal{C}_v(19, 15, 18) > \mathcal{C}_v(19, 18, 15)$)

$d \succ c$

$\left.\begin{array}{l} v(\mathsf{St}) \\ v(\mathsf{Pr}) \end{array}\right\} > v(\mathsf{Al}) \qquad$ (local importances)

$I(v, \mathsf{St} \cup \mathsf{Pr}) < 0 \qquad$ (substitutiveness)

$0.45 < \text{orness}(\mathcal{C}_v) < 0.55 \qquad$ (tolerance)

$v(\emptyset) = 0, v(N) = 1$

Monotonicity of v

etc.

Objective function : strictly concave
Constraints : linear w.r.t. v