Cumulative Distribution Functions and Moments of Weighted Lattice Polynomials

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Sketch of the Presentation

Part I : Weighted lattice polynomials

- Definitions
- Representation and characterization

Part II : Cumulative distribution functions of aggregation operators

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- Applications

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Part I : Weighted lattice polynomials

Lattice polynomials

Let L be a lattice with lattice operations $\wedge$ and $\vee$

We assume that L is

- bounded (with bottom 0 and top 1)
- distributive

Definition (Birkhoff 1967)

An n-ary *lattice polynomial* is a well-formed expression involving n variables $x_1, \ldots, x_n \in L$ linked by the lattice operations $\wedge$ and $\vee$ in an arbitrary combination of parentheses

Example.

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

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Lattice polynomial functions

Any lattice polynomial naturally defines a *lattice polynomial function* (l.p.f.) $p : L^n \to L$.

Example.

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

If p and q represent the same function, we say that p and q are equivalent and we write $p = q$

Example.

$$x_1 \vee (x_1 \wedge x_2) = x_1$$

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Disjunctive and conjunctive forms of l.p.f.'s

Notation. $[n] := \{1, \ldots, n\}$.

Proposition (Birkhoff 1967)

Let $p : L^n \to L$ be any l.p.f.
Then there are nonconstant set functions $v, w : 2^{[n]} \to \{0, 1\}$, with $v(\varnothing) = 0$ and $w(\varnothing) = 1$, such that

$$p(x) = \bigvee_{\substack{S \subseteq [n] \\ v(S)=1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ w(S)=0}} \bigvee_{i \in S} x_i.$$

Example.

$$(x_1 \wedge x_2) \vee x_3 = (x_1 \vee x_3) \wedge (x_2 \vee x_3)$$

$$v(\{3\}) = v(\{1,2\}) = 1$$
$$w(\{1,3\}) = w(\{2,3\}) = 0$$

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The set functions v and w, which generate p, are not unique :

$$x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2)$$

Notation. $\mathbf{1}_S :=$ characteristic vector of $S \subseteq [n]$ in $\{0,1\}^n$.

Proposition (Marichal 2002)

From among all the set functions v that disjunctively generate the l.p.f. p, only one is isotone :

$$v(S) = p(\mathbf{1}_S)$$

From among all the set functions w that conjunctively generate the l.p.f. p, only one is antitone :

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Particular cases : order statistics

Denote by $x_{(1)}, \ldots, x_{(n)}$ the *order statistics* resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order : $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$.

Proposition (Ovchinnikov 1996, Marichal 2002)

$$p \text{ is a symmetric l.p.f.} \quad \Longleftrightarrow \quad p \text{ is an order statistic}$$

Notation. Denote by $os_k : L^n \to L$ the kth order statistic function.

$$os_k(x) := x_{(k)}$$

Then we have

$$os_k(\mathbf{1}_S) = 1 \quad \Longleftrightarrow \quad |S| \geqslant n - k + 1$$

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Weighted lattice polynomials

We can generalize the concept of l.p.f. by regarding some variables as parameters.

Example. For $c \in L$, we consider

$$p(x_1, x_2) = (c \vee x_1) \wedge x_2$$

Definition

$p : L^n \to L$ is an n-ary *weighted lattice polynomial* function (w.l.p.f.) if there exist parameters $c_1, \ldots, c_m \in L$ and a l.p.f. $q : L^{n+m} \to L$ such that

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n, c_1, \ldots, c_m)$$

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Disjunctive and conjunctive forms of w.l.p.f.'s

Proposition (Lausch & Nöbauer 1973)

Let $p : L^n \to L$ be any w.l.p.f.

Then there are set functions $v, w : 2^{[n]} \to L$ such that

$$p(x) = \bigvee_{S \subseteq [n]} \Big[v(S) \wedge \bigwedge_{i \in S} x_i \Big] = \bigwedge_{S \subseteq [n]} \Big[w(S) \vee \bigvee_{i \in S} x_i \Big].$$

Remarks.

- p is a l.p.f. iff v and w range in $\{0, 1\}$, with $v(\varnothing) = 0$ and $w(\varnothing) = 1$
- Any w.l.p.f. is entirely determined by 2^n parameters, even if more parameters have been considered to construct it.

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Particular case : the Sugeno integral

Let us generalize the concept of discrete Sugeno integral in the framework of bounded distributive lattices.

Definition (Sugeno 1974)

An L-valued *fuzzy measure* on $[n]$ is an isotone set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = 0$ and $\mu([n]) = 1$.

The *Sugeno integral* of a function $x : [n] \to L$ with respect to μ is defined by

$$\mathcal{S}_\mu(x) := \bigvee_{S \subseteq [n]} \Big[\mu(S) \wedge \bigwedge_{i \in S} x_i \Big]$$

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Notation. The median function is the function $os_2 : L^3 \to L$.

Proposition (Marichal 2006)

For any w.l.p.f. $p : L^n \to L$, there is a fuzzy measure $\mu : 2^{[n]} \to L$ such that

$$p(x) = \text{median}\big[p(\mathbf{1}_\varnothing), \mathcal{S}_\mu(x), p(\mathbf{1}_{[n]})\big]$$

Corollary (Marichal 2006)

Consider a function $f : L^n \to L$.
The following assertions are equivalent :

- f is a Sugeno integral
- f is an idempotent w.l.p.f., that is such that $f(x, \ldots, x) = x$
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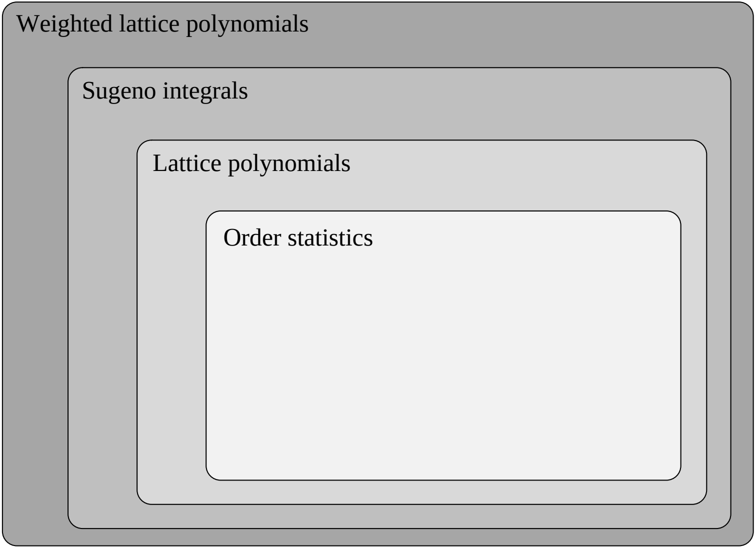
Inclusion properties



The median based decomposition formula

Let $f : L^n \to L$ and $k \in [n]$ and define $f_k^0, f_k^1 : L^n \to L$ as

$$f_k^0(x) := f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n)$$
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Example. For $n = 2$ we have

$$f(x_1, x_2) = \text{median}\big[f(x_1, 0), x_2, f(x_1, 1)\big]$$

with

$$\begin{aligned} f(x_1, 0) &= \text{median}\big[f(0,0), x_1, f(1,0)\big] \quad \text{(w.l.p.f.)} \\ f(x_1, 1) &= \text{median}\big[f(0,1), x_1, f(1,1)\big] \quad \text{(w.l.p.f.)} \end{aligned}$$

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The median based decomposition formula characterizes the w.l.p.f.'s

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Part II : Cumulative distribution functions of aggregation operators

Cumulative distribution functions of aggregation operators

Consider

- an aggregation operator $A : \mathbb{R}^n \to \mathbb{R}$
- n independent random variables $X_1, \ldots, X_n$, with cumulative distribution functions $F_1(x), \ldots, F_n(x)$

$$\left.\begin{array}{c} X_1 \\ \vdots \\ X_n \end{array}\right\} \longrightarrow \quad Y_A = A(X_1, \ldots, X_n)$$

Problem. We are searching for the cumulative distribution function (c.d.f.) of Y_A :

$$\boxed{F_A(y) := \Pr[Y_A \leqslant y]}$$

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From the c.d.f. of Y_A, we can calculate the expectation

$$\mathbf{E}\big[g(Y_A)\big] = \int_{-\infty}^{\infty} g(y)\,\mathrm{d}F_A(y)$$

for any measurable function g.

Some useful examples :

$g(x)$	$\mathbf{E}\big[g(Y_A)\big]$
x	expected value of Y_A
x^r	raw moments of Y_A
$\big[x - \mathbf{E}(Y_A)\big]^r$	central moments of Y_A
e^{tx}	moment-generating function of Y_A

Cumulative distribution functions of aggregation operators

From the c.d.f. of Y_A, we can calculate the expectation

$$\mathbf{E}\big[g(Y_A)\big] = \int_{-\infty}^{\infty} g(y)\,\mathrm{d}F_A(y)$$

for any measurable function g.

Some useful examples :

$g(x)$	$\mathbf{E}\big[g(Y_A)\big]$
x	expected value of Y_A
x^r	raw moments of Y_A
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Cumulative distribution functions of aggregation operators

If $F_A(y)$ is absolutely continuous, then Y_A has a probability density function (p.d.f.)

$$f_A(y) := \frac{\mathrm{d}}{\mathrm{d}y} F_A(y)$$

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Example : the arithmetic mean

$$AM(x_1,\ldots,x_n)=\frac{1}{n}\sum_{i=1}^{n}x_i$$

$F_{AM}(y)$ is given by the convolution product of $F_1,\ldots,F_n$

$$F_{AM}(y)=(F_1*\cdots*F_n)(ny)$$

For uniform random variables $X_1,\ldots,X_n$ on $[0,1]$, we have

$$F_{AM}(y)=\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(ny-k)_+^n\qquad(y\in[0,1])$$

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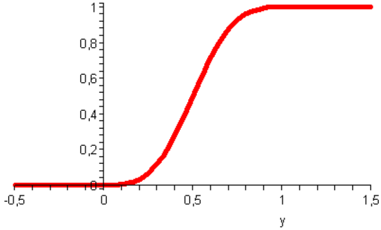
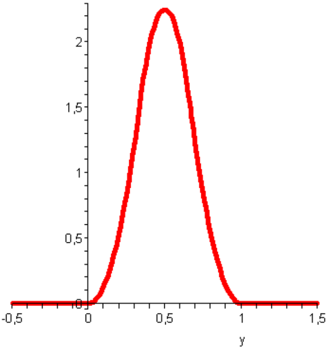


Graph of $F_{AM}(y)$



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Example : Łukasiewicz t-norm

$$T_L(x_1,\ldots,x_n) = \max\left[0, \sum_{i=1}^{n} x_i - (n-1)\right]$$

$$\begin{aligned} F_{T_L}(y) &= \Pr\Big[\max\left[0, \textstyle\sum_i X_i - (n-1)\right] \leqslant y\Big] \\ &= \Pr\Big[0 \leqslant y \quad \text{and} \quad \textstyle\sum_i X_i - (n-1) \leqslant y\Big] \\ &= \Pr[0 \leqslant y]\,\Pr\Big[\textstyle\sum_i X_i \leqslant y+n-1\Big] \\ &= H_0(y)\,F_{AM}\left(\tfrac{y+n-1}{n}\right) \end{aligned}$$

where $H_c(y)$ is the *Heaviside* function

$$H_c(y) = \mathbf{1}_{[c,+\infty[}(y)$$

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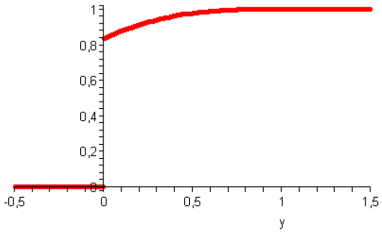

Graph of $F_{T_L}(y)$

Remark.
$F_{T_L}(y)$ is discontinuous
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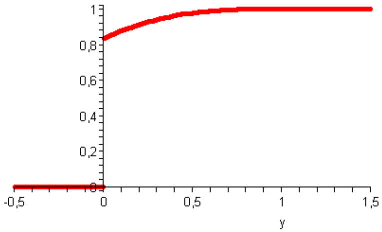

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Example : order statistics on $\mathbb{R}$

$$os_k(x_1, \ldots, x_n) = x_{(k)}$$

$$F_{os_k}(y) = \sum_{\substack{S \subseteq [n] \\ |S| \geqslant k}} \prod_{i \in S} F_i(y) \prod_{i \in [n] \setminus S} [1 - F_i(y)]$$

(see e.g. David & Nagaraja 2003)

Examples.

$$F_{os_1}(y) = 1 - \prod_{i=1}^{n} [1 - F_i(y)]$$

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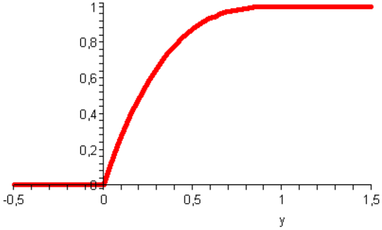

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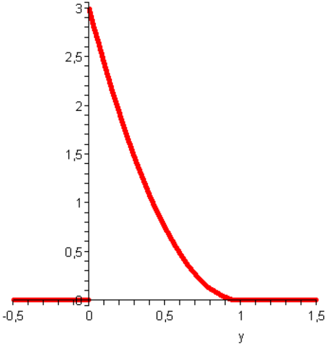

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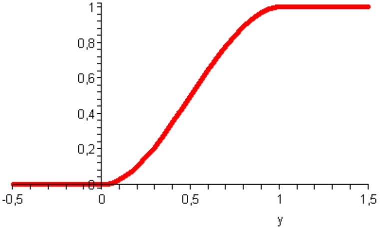

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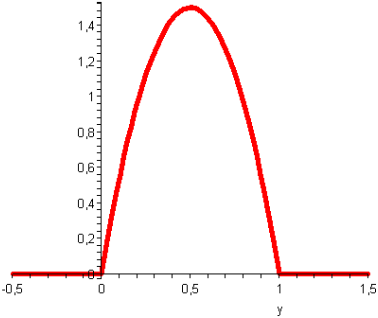

Graph of $f_{os_2}(y)$

New results : lattice polynomial functions on $\mathbb{R}$

Let $p : L^n \to L$ be a l.p.f. on $L = [0,1]$
It can be extended to an aggregation function from $\mathbb{R}^n$ to $\mathbb{R}$.

$$p(x_1,\ldots,x_n) = \bigvee_{\substack{S\subseteq[n]\\ p(\mathbf{1}_S)=1}} \bigwedge_{i\in S} x_i = \bigwedge_{\substack{S\subseteq[n]\\ p(\mathbf{1}_{[n]\setminus S})=0}} \bigvee_{i\in S} x_i$$

Note. $\wedge = \min$, $\vee = \max$

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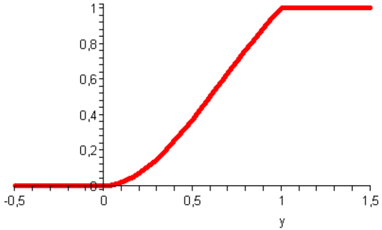
Example. $p(x) = (x_1 \wedge x_2) \vee x_3$

Uniform random variables X_1, X_2, X_3 on $[0,1]$



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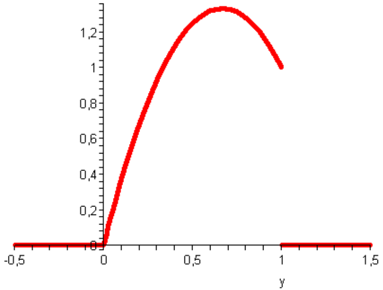

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Alternate expressions of $F_p(y)$

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$$m_v(S) := \sum_{T \subseteq S} (-1)^{|S|-|T|} \, v(T)$$

Alternate expressions of $F_p(y)$

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New results : lattice polynomial functions on $\mathbb{R}$

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Let $p : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ be a w.l.p.f. on $\overline{\mathbb{R}} = [-\infty, +\infty]$

Notation. $\mathbf{e}_S :=$ characteristic vector of S in $\{-\infty, +\infty\}^n$

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Example. $p(x) = (c \wedge x_1) \vee x_2$

Uniform random variables X_1, X_2 on $[0, 1]$
$F(y) = \text{median}[0, y, 1]$

S	$p(\mathbf{e}_S)$
$\varnothing$	$-\infty$
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$\{2\}$	$+\infty$
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$F_p(y) = F(y)\Big(F(y)+H_c(y)[1-F(y)]\Big)$

Graph of $F_p(y)$ for $c = 1/2$

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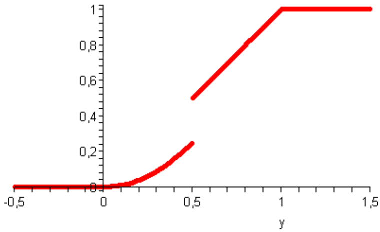

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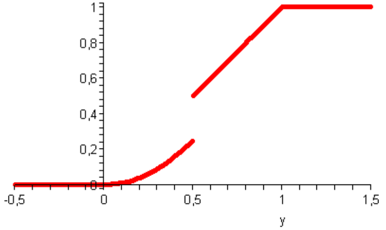

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Application : computation of certain integrals

Example. Given a w.l.p.f. $p : [0,1]^n \to [0,1]$ and a measurable function $g : [0,1] \to \overline{\mathbb{R}}$, compute

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$$\int_{[0,1]^n} \mathcal{S}_\mu(x)\,\mathrm{d}x = \sum_{S\subseteq[n]} \int_0^{\mu(S)} y^{n-|S|}(1-y)^{|S|}\,\mathrm{d}y$$

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Consider a system made up of n indep. components $C_1, \ldots, C_n$

Each component C_i has

- a *lifetime* X_i
- a *reliability* $r_i(t)$ at time $t > 0$

$$r_i(t) := \Pr[X_i > t] = 1 - F_i(t)$$

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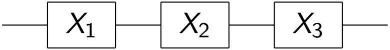

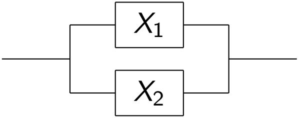

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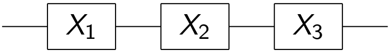

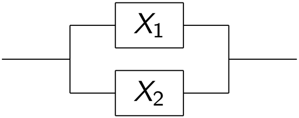

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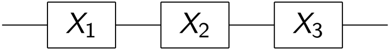

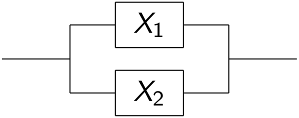

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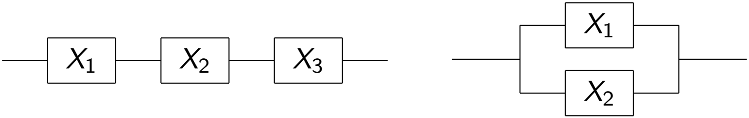

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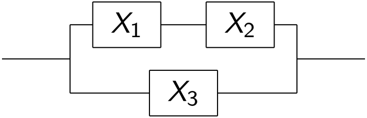

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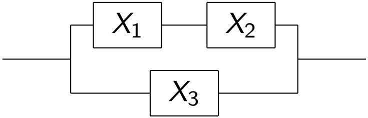

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For a system mixing series and parallel connections :

System lifetime :

$$Y_p = p(X_1, \ldots, X_n)$$

where p is

- an n-ary l.p.f.
- an n-ary w.l.p.f. if some X_i's are constant

We then have explicit formulas for

- the c.d.f. of Y_p
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Application : reliability of systems

System reliability at time $t > 0$

$$R_p(t) := \Pr[Y_p > t] = 1 - F_p(t)$$

For any measurable function $g : [0, \infty[\to \mathbb{R}$ such that

$$g(\infty) r_i(\infty) = 0 \qquad (i = 1, \ldots, n)$$

we have

$$\mathbf{E}\big[g(Y_p)\big] = g(0) + \int_0^\infty R_p(t)\, \mathrm{d}g(t)$$

Mean time to failure :

$$\mathbf{E}[Y_p] = \int_0^\infty R_p(t)\, \mathrm{d}t$$

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Example. Assume $r_i(t) = e^{-\lambda_i t}$ $(i = 1, \ldots, n)$

$$\mathbf{E}[Y_p] = \sum_{\substack{S \subseteq [n] \\ S \neq \varnothing}} m_{v_p}(S) \, \frac{1}{\sum_{i \in S} \lambda_i}$$

- Series system

$$\mathbf{E}[Y_p] = \frac{1}{\sum_{i \in [n]} \lambda_i}$$

- Parallel system

$$\mathbf{E}[Y_p] = \sum_{\substack{S \subseteq [n] \\ S \neq \varnothing}} (-1)^{|S|-1} \, \frac{1}{\sum_{i \in S} \lambda_i}$$

(Barlow & Proschan 1981)

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Thanks for your attention !