Aggregation Functions
for Multicriteria Decision Aid

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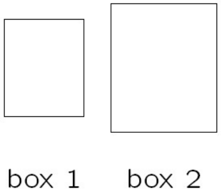
The aggregation problem

Combining several numerical values into a single one

Example (voting theory)
Several individuals form quantifiable judgements about the measure of an object.



$$\frac{\text{area(box 2)}}{\text{area(box 1)}} = ?$$

$$x_1, \ldots, x_n \longrightarrow F(x_1, \ldots, x_n) = x$$

where F = arithmetic mean
geometric mean
median
...

The aggregation problem

Decision making (voters $\rightarrow$ criteria)

$$x_1, \ldots, x_n \;=\; \text{satisfaction degrees (for instance)}$$

	math.	physics	literature	global
student a	18	16	10	?
student b	10	12	18	?
student c	14	15	15	?

Aggregation in multicriteria decision making

- Alternatives $A = \{a, b, c, \dots\}$
- Criteria $N = \{1, 2, \dots, n\}$
- Profile $a \in A \longrightarrow \quad \mathbf{x}^a = (x_1^a, \dots, x_n^a) \in \mathbb{R}^n$

$$\swarrow \qquad \searrow$$

commensurate partial scores

- Aggregation function $F : \mathbb{R}^n \to \mathbb{R}$

$$F : E^n \to \mathbb{R} \quad (E \subseteq \mathbb{R})$$

Alternative	crit. 1	$\cdots$	crit. n	global score
a	x_1^a	$\cdots$	x_n^a	$F(x_1^a, \dots, x_n^a)$
b	x_1^b	$\cdots$	x_n^b	$F(x_1^b, \dots, x_n^b)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Aggregation in multicriteria decision making

Non-commensurate scales :

	price (to minimize)	consumption (to minimize)	comfort (to maximize)	global
car a	$\$10,000$	$0.15\ \ell pm$	good	?
car b	$\$20,000$	$0.17\ \ell pm$	excellent	?
car c	$\$30,000$	$0.13\ \ell pm$	very good	?
car d	$\$20,000$	$0.16\ \ell pm$	good	?

Scoring approach

For each $i \in N$, one can define a net score :

$$S_i(a) = \big|\{b \in A \mid b \preccurlyeq_i a\}\big| - \big|\{b \in A \mid b \succcurlyeq_i a\}\big|$$

$$\overline{S}_i(a) = \frac{S_i(a) + (|A| - 1)}{2(|A| - 1)} \quad \in [0, 1]$$

Aggregation in multicriteria decision making

Non-commensurate scales :

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car *a*	$\$10,000$	$0.15\ \ell pm$	good	?
car *b*	$\$20,000$	$0.17\ \ell pm$	excellent	?
car *c*	$\$30,000$	$0.13\ \ell pm$	very good	?
car *d*	$\$20,000$	$0.16\ \ell pm$	good	?

↓

	price	cons.	comf.	global
car *a*	1.00	0.66	0.16	?
car *b*	0.50	0.00	1.00	?
car *c*	0.00	1.00	0.66	?
car *d*	0.50	0.33	0.16	?

(satisfaction degrees)

Aggregation properties

- **Symmetry.** $F(x_1, \ldots, x_n)$ is symmetric
- **Increasing monotonicity.** $F(x_1, \ldots, x_n)$ is nondecreasing in each variable
- **Strict increasing monotonicity.** $F(x_1, \ldots, x_n)$ is strictly increasing in each variable
- **Idempotency.** $F(x, \ldots, x) = x$ for all x
- **Internality.** $\min x_i \leqslant F(x_1, \ldots, x_n) \leqslant \max x_i$
  Note : id. + inc. $\Rightarrow$ int. $\Rightarrow$ id.

Aggregation properties

- **Associativity.**

$$\begin{aligned} F(x_1, x_2, x_3) &= F(F(x_1, x_2), x_3) \\ &= F(x_1, F(x_2, x_3)) \end{aligned}$$

- **Decomposability.**

$$\begin{aligned} F(x_1, x_2, x_3) &= F(F(x_1, x_2), F(x_1, x_2), x_3) \\ &= F(x_1, F(x_2, x_3), F(x_2, x_3)) \\ &= F(F(x_1, x_3), x_2, F(x_1, x_3)) \end{aligned}$$

- **Bisymmetry.**

$$F(F(x_1, x_2), F(x_3, x_4)) = F(F(x_1, x_3), F(x_2, x_4))$$

Quasi-arithmetic means

Theorem 1 (Kolmogorov-Nagumo, 1930)
The functions $F_n : E^n \to \mathbb{R}$ $(n \geqslant 1)$ are

- symmetric
- continuous
- strictly increasing
- idempotent
- **decomposable**

if and only if there exists a continuous and strictly monotonic function $f : E \to \mathbb{R}$ such that

$$F_n(\mathbf{x}) = f^{-1}\Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big] \qquad (n \geqslant 1)$$

Proposition 1 (Marichal, 2000)
Symmetry can be removed in the K-N theorem

Quasi-arithmetic means

$f(x)$	$F_n(\mathbf{x})$	name
x	$\frac{1}{n}\sum_{i=1}^{n} x_i$	arithmetic
$\log x$	$\sqrt[n]{\prod_{i=1}^{n} x_i}$	geometric
x^{-1}	$\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}}$	harmonic
$x^\alpha \ (\alpha \in \mathbb{R}_0)$	$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}}$	root-power

Quasi-arithmetic means

Theorem 2 (Fodor-Marichal, 1997)
The functions $F_n : [a,b]^n \to \mathbb{R}$ ($n \geqslant 1$) are

- symmetric
- continuous
- **increasing**
- idempotent
- **decomposable**

if and only if there exist $\alpha, \beta \in \mathbb{R}$ fulfilling $a \leqslant \alpha \leqslant \beta \leqslant b$ and a continuous and strictly monotonic function $f : [\alpha, \beta] \to \mathbb{R}$ such that, for any $n \geqslant 1$,

$$F_n(\mathbf{x}) = \begin{cases} G_n(\mathbf{x}) & \text{if } \mathbf{x} \in [a,\alpha]^n \\ H_n(\mathbf{x}) & \text{if } \mathbf{x} \in [\beta, b]^n \\ f^{-1}\left[\frac{1}{n}\sum_i f(\mathrm{median}[\alpha, x_i, \beta])\right] & \text{otherwise} \end{cases}$$

where G_n and H_n are defined by...

Open problem : remove symmetry !

Quasi-arithmetic means

Theorem 3 (Aczél, 1948)
The function $F : E^n \to \mathbb{R}$ is

- symmetric
- continuous
- strictly increasing
- idempotent
- **bisymmetric**

if and only if there exists a continuous and strictly monotonic function $f : E \to \mathbb{R}$ such that

$$F(\mathbf{x}) = f^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right]$$

When symmetry is removed :
There exist $w_1, \ldots, w_n > 0$ fulfilling $\sum_i w_i = 1$ such that

$$F(\mathbf{x}) = f^{-1}\left[\sum_{i=1}^{n} w_i f(x_i)\right]$$

Quasi-arithmetic means

$f(x)$	$F_n(\mathbf{x})$	name
x	$\sum_{i=1}^{n} w_i x_i$	arithmetic
$\log x$	$\prod_{i=1}^{n} x_i^{w_i}$	geometric
x^{-1}	$\frac{1}{\sum_{i=1}^{n} w_i \frac{1}{x_i}}$	harmonic
$x^\alpha \ (\alpha \in \mathbb{R}_0)$	$\left(\sum_{i=1}^{n} w_i x_i^\alpha\right)^{\frac{1}{\alpha}}$	root-power

Associative functions

Theorem 4 (Aczél, 1948)
The functions $F_n : E^n \to E$ $(n \geqslant 1)$ are

- continuous
- strictly increasing
- **associative**

if and only if there exists a continuous and strictly monotonic function $f : E \to \mathbb{R}$ such that

$$F_n(\mathbf{x}) = f^{-1}\Big[\sum_{i=1}^{n} f(x_i)\Big] \qquad (n \geqslant 1)$$

+ idempotency : $\varnothing$

Open problem : replace strict increasing monotonicity with nondecreasing monotonicity

Associative functions

Theorem 5 (Fung-Fu, 1975)
The functions $F_n : E^n \to \mathbb{R}$ ($n \geqslant 1$) are

- symmetric
- continuous
- nondecreasing
- idempotent
- **associative**

if and only if there exists $\alpha \in E$ such that

$$F_n(\mathbf{x}) = \mathrm{median}\Big[\bigwedge_{i=1}^{n} x_i, \bigvee_{i=1}^{n} x_i, \alpha\Big] = \mathrm{median}[x_1, \ldots, x_n, \underbrace{\alpha, \ldots, \alpha}_{n-1}]$$

where

$$\mathrm{median}[x_1, \ldots, x_{2n-1}] = x_{(n)} \qquad (x_{(1)} \leqslant \cdots \leqslant x_{(2n-1)})$$

Associative functions

Without symmetry :

Theorem 6 (Marichal, 2000)
The functions $F_n : E^n \to \mathbb{R}$ $(n \geqslant 1)$ are

- continuous
- nondecreasing
- idempotent
- **associative**

if and only if there exists $\alpha, \beta \in E$ such that

$$F_n(\mathbf{x}) = (\alpha \wedge x_1) \vee \Big(\bigvee_{i=1}^{n} (\alpha \wedge \beta \wedge x_i)\Big) \vee (\beta \wedge x_n) \vee \Big(\bigwedge_{i=1}^{n} x_i\Big)$$

Without symmetry and idempotency : Open problem

Interval scales

Example : grades obtained by students
- on a $[0, 20]$ scale : 16, 11, 7, 14
- on a $[0, 1]$ scale : 0.80, 0.55, 0.35, 0.70
- on a $[-1, 1]$ scale : 0.60, 0.10, -0.30, 0.40

Definition. $F : \mathbb{R}^n \to \mathbb{R}$ is stable for the positive linear transformations if

$$F(rx_1 + s, \ldots, rx_n + s) = r\, F(x_1, \ldots, x_n) + s$$

for all $x_1, \ldots, x_n \in \mathbb{R}$ and all $r > 0$, $s \in \mathbb{R}$.

Interval scales

Theorem 8 (Aczél-Roberts-Rosenbaum, 1986)
The function $F : \mathbb{R}^n \to \mathbb{R}$ is stable for the positive linear transformations if and only if

$$F(\mathbf{x}) = S(\mathbf{x})\, G\Big(\frac{x_1 - A(\mathbf{x})}{S(\mathbf{x})}, \ldots, \frac{x_n - A(\mathbf{x})}{S(\mathbf{x})}\Big) + A(\mathbf{x})$$

where $A(\mathbf{x}) = \frac{1}{n}\sum_i x_i$, $S(\mathbf{x}) = \sqrt{\sum_i [x_i - A(\mathbf{x})]^2}$, and $G : \mathbb{R}^n \to \mathbb{R}$ is arbitrary.

Interesting unsolved problem :
Describe nondecreasing and stable functions

Interval scales

Theorem 9 (Marichal-Mathonet-Tousset, 1999)
The function $F : E^n \to \mathbb{R}$ is

- nondecreasing
- stable for the positive linear transformations
- **bisymmetric**

if and only if it is of the form

$$F(\mathbf{x}) = \bigvee_{i \in S} x_i \quad \text{or} \quad \bigwedge_{i \in S} x_i \quad \text{or} \quad \sum_{i=1}^{n} w_i x_i$$

where $S \subseteq N$, $S \neq \varnothing$, $w_1, \ldots, w_n > 0$, and $\sum_i w_i = 1$.

Interval scales

Theorem 10 (Marichal-Mathonet-Tousset, 1999)
The functions $F_n : E^n \to \mathbb{R}$ $(n \geqslant 1)$ are

- nondecreasing
- stable for the positive linear transformations
- **decomposable**

if and only if they are of the form

$$F_n(\mathbf{x}) = \bigvee_{i=1}^{n} x_i \quad \text{or} \quad \bigwedge_{i=1}^{n} x_i \quad \text{or} \quad \frac{1}{n}\sum_{i=1}^{n} x_i$$

Interval scales

Theorem 11 (Marichal-Mathonet-Tousset, 1999)
The functions $F_n : E^n \to \mathbb{R}$ $(n \geqslant 1)$ are

- nondecreasing
- stable for the positive linear transformations
- **associative**

if and only if they are of the form

$$F_n(\mathbf{x}) = \bigvee_{i=1}^{n} x_i \quad \text{or} \quad \bigwedge_{i=1}^{n} x_i \quad \text{or} \quad x_1 \quad \text{or} \quad x_n$$

An illustrative example (Grabisch, 1996)

Evaluation of students w.r.t. three subjects :
mathematics, physics, and literature.

student	M	P	L	global
a	0.90	0.80	0.50	?
b	0.50	0.60	0.90	?
c	0.70	0.75	0.75	?

(grades are expressed on a scale from 0 to 1)

Often used : the weighted arithmetic mean

$$\mathrm{WAM}_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$$

with $\sum_i w_i = 1$ and $w_i \geqslant 0$ for all $i \in N$

An illustrative example (Grabisch, 1996)

$$\left.\begin{aligned} w_M &= 0.35 \\ w_P &= 0.35 \\ w_L &= 0.30 \end{aligned}\right\} \quad \Rightarrow$$

student	global
a	0.74
b	0.65
c	0.73

$$\boxed{a \succ c \succ b}$$

An illustrative example (Grabisch, 1996)

Suppose we want to favor student c

student	M	P	L	global
a	0.90	0.80	0.50	0.74
b	0.50	0.60	0.90	0.65
c	0.70	0.75	0.75	0.73

No weight vector (w_M, w_P, w_L) satisfying

$$w_M = w_P > w_L$$

is able to provide $c \succ a$

Proof.

$$\begin{aligned}
c \succ a \quad &\Leftrightarrow \quad 0.70w_M + 0.75w_P + 0.75w_L > 0.90w_M + 0.80w_P + 0.50w_L \\
&\Leftrightarrow \quad -0.20w_M - 0.05w_P + 0.25w_L > 0 \\
&\Leftrightarrow \quad -0.25w_M + 0.25w_L > 0 \\
&\Leftrightarrow \quad w_L > w_M
\end{aligned}$$

An illustrative example (Grabisch, 1996)

What's wrong ?

$$
\begin{aligned}
\mathrm{WAM}_{\mathbf{w}}(1,0,0) &= w_M = 0.35\\
\mathrm{WAM}_{\mathbf{w}}(0,1,0) &= w_P = 0.35\\
\mathrm{WAM}_{\mathbf{w}}(1,1,0) &= 0.70 \text{ !!!}
\end{aligned}
$$

What is the importance of $\{M, P\}$?

The Choquet integral

Definition (Choquet, 1953 ; Sugeno, 1974)
A fuzzy measure on N is a set function $v : 2^N \to [0,1]$ such that

$i)$ $v(\varnothing) = 0$, $v(N) = 1$

$ii)$ $S \subseteq T \ \Rightarrow \ v(S) \leqslant v(T)$

$$\begin{aligned} v(S) &= \text{weight of } S \\ &= \text{degree of importance of } S \end{aligned}$$

A fuzzy measure is additive if

$$v(S \cup T) = v(S) + v(T) \quad \text{if } S \cap T = \varnothing$$

$\to$ independent criteria

$$v(M,P) = v(M) + v(P) \ \ (= 0.70)$$

The Choquet integral

Question : How can we extend the weighted arithmetic mean by taking into account the interaction among criteria ?

Definition. Let $v \in \mathcal{F}_N$. The Choquet integral of $\mathbf{x} \in \mathbb{R}^n$ w.r.t. v is defined by

$$\mathcal{C}_v(\mathbf{x}) := \sum_{i=1}^{n} x_{(i)} \left[v\big((i), \ldots, (n)\big) - v\big((i+1), \ldots, (n)\big) \right]$$

with the convention that $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$

Example : If $x_3 \leqslant x_1 \leqslant x_2$, we have

$$\begin{aligned} \mathcal{C}_v(x_1, x_2, x_3) \quad = \quad & \quad x_3\,[v(3,1,2) - v(1,2)] \\ & + x_1\,[v(1,2) - v(2)] \\ & + x_2\,v(2) \end{aligned}$$

The Choquet integral

Special case :

$$\boxed{v \text{ additive} \quad \Rightarrow \quad \mathcal{C}_v = \mathrm{WAM}_{\mathbf{w}}}$$

Proof.

$$\begin{aligned}
\mathcal{C}_v(\mathbf{x}) &= \sum_{i=1}^{n} x_{(i)} \left[v\big((i),\ldots,(n)\big) - v\big((i+1),\ldots,(n)\big) \right] \\
&= \sum_{i=1}^{n} x_{(i)}\, v\big((i)\big) \\
&= \sum_{i=1}^{n} x_i \underbrace{v(i)}_{w_i}
\end{aligned}$$

Properties of the Choquet integral

- **Linearity w.r.t. the fuzzy measures**
  There exist 2^n functions $f_T : \mathbb{R}^n \to \mathbb{R}$ $(T \subseteq N)$ such that

$$\mathcal{C}_v(\mathbf{x}) = \sum_{T \subseteq N} v(T)\, f_T$$

Indeed, one can show that

$$\mathcal{C}_v(\mathbf{x}) = \sum_{T \subseteq N} v(T) \underbrace{\sum_{K \supseteq T} (-1)^{|K|-|T|} \bigwedge_{i \in K} x_i}_{f_T(\mathbf{x})}$$

Properties of the Choquet integral

- **Stability w.r.t. positive linear transformations**
  For any $\mathbf{x} \in \mathbb{R}^n$, and any $r > 0$, $s \in \mathbb{R}$,

$$\mathcal{C}_v(rx_1 + s, \ldots, rx_n + s) = r\,\mathcal{C}_v(x_1, \ldots, x_n) + s$$

Example : grades obtained by students
- on a $[0, 20]$ scale : 16, 11, 7, 14
- on a $[0, 1]$ scale : 0.80, 0.55, 0.35, 0.70
- on a $[-1, 1]$ scale : 0.60, 0.10, -0.30, 0.40

Remark : The grades may be embedded in $[0, 1]$

Properties of the Choquet integral

- **Increasing monotonicity**
  For any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, one has

$$x_i \leqslant x_i' \;\; \forall i \in N \quad \Rightarrow \quad \mathcal{C}_v(\mathbf{x}) \leqslant \mathcal{C}_v(\mathbf{x}')$$

Properties of the Choquet integral

- $\mathcal{C}_v$ **is properly weighted by** v

$$\mathcal{C}_v(e_S) = v(S) \qquad (S \subseteq N)$$

e_S = characteristic vector of S in $\{0,1\}^n$

Example : $e_{\{1,3\}} = (1,0,1,0,\ldots)$

Independent criteria	Dependent criteria
$\mathrm{WAM}_{\mathbf{w}}(e_{\{i\}}) = w_i$	$\mathcal{C}_v(e_{\{i\}}) = v(i)$
$\mathrm{WAM}_{\mathbf{w}}(e_{\{i,j\}}) = w_i + w_j$	$\mathcal{C}_v(e_{\{i,j\}}) = v(i,j)$

Example :

$$\begin{array}{ccccc} v(M,P) & < & v(M) & + & v(P) \\ \| & & \| & & \| \\ \mathcal{C}_v(1,1,0) & & \mathcal{C}_v(1,0,0) & & \mathcal{C}_v(0,1,0) \end{array}$$

Axiomatization of the class of Choquet integrals

Theorem (Marichal, 2000)
The functions $F_v : \mathbb{R}^n \to \mathbb{R}$ $(v \in \mathcal{F}_N)$ are

- **linear w.r.t. the underlying fuzzy measures** v
  F_v is of the form
  $$F_v(\mathbf{x}) = \sum_{T \subseteq N} v(T)\, f_T \qquad (v \in \mathcal{F}_N)$$
  where f_T's are independent of v
- **stable for the positive linear transformations**
  $$F_v(rx_1 + s, \ldots, rx_n + s) = r\, F_v(x_1, \ldots, x_n) + s$$
  for all $\mathbf{x} \in \mathbb{R}^n$, and all $r > 0$, $s \in \mathbb{R}$, $v \in \mathcal{F}_N$
- **Nondecreasing**
- **Properly weighted by** v
  $$F_v(e_S) = v(S) \qquad (S \subseteq N,\, v \in \mathcal{F}_N)$$

if and only if $F_v = \mathcal{C}_v$ for all $v \in \mathcal{F}_N$

Back to the example

Assumptions :
- M and P are more important than L
- M and P are somewhat substitutive

Non-additive model : $\mathcal{C}_v$

$$v(M) = 0.35$$
$$v(P) = 0.35$$
$$v(L) = 0.30$$

$$v(M, P) = 0.60 \quad \text{(redundancy)}$$
$$v(M, L) = 0.80 \quad \text{(complementarity)}$$
$$v(P, L) = 0.80 \quad \text{(complementarity)}$$

$$v(\varnothing) = 0$$
$$v(M, P, L) = 1$$

Back to the example

student	M	P	L	WAM	Choquet
a	0.90	0.80	0.50	0.74	0.71
b	0.50	0.60	0.90	0.65	0.67
c	0.70	0.75	0.75	0.73	0.74

Now : $\boxed{c \succ a \succ b}$

An alternative example (Marichal, 2000)

student	M	P	L	global
a	0.90	0.70	0.80	?
b	0.90	0.80	0.70	?
c	0.60	0.70	0.80	?
d	0.60	0.80	0.70	?

Behavior of the decision maker :
When a student is good at M (0.90), it is preferable that (s)he is better at L than P, so

$$\boxed{a \succ b}$$

When a student is not good at M (0.60), it is preferable that (s)he is better at P than L, so

$$\boxed{d \succ c}$$

An alternative example (Marichal, 2000)

Additive model : $\mathrm{WAM}_{\mathbf{w}}$

$$\left.\begin{array}{lcl} a \succ b & \Leftrightarrow & w_L > w_P \\ d \succ c & \Leftrightarrow & w_L < w_P \end{array}\right\} \quad \text{No solution !}$$

Non additive model : $\mathcal{C}_v$

student	M	P	L	global
a	0.90	0.70	0.80	0.81
b	0.90	0.80	0.70	0.79
c	0.60	0.70	0.80	0.71
d	0.60	0.80	0.70	0.72

Special cases of Choquet integrals

- **Weighted arithmetic mean**

$$\mathrm{WAM}_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i \,, \qquad \sum_{i=1}^{n} w_i = 1\,, \qquad w_i > 0$$

Proposition

Let $v \in \mathcal{F}_N$. The following assertions are equivalent :

i) v is additive

ii) $\exists$ a weight vector $\mathbf{w}$ such that $\mathcal{C}_v = \mathrm{WAM}_{\mathbf{w}}$

iii) $\mathcal{C}_v$ is additive : $\mathcal{C}_v(\mathbf{x}+\mathbf{x}') = \mathcal{C}_v(\mathbf{x}) + \mathcal{C}_v(\mathbf{x}')$

Special cases of Choquet integrals

- **Ordered weighted averaging (Yager, 1988)**

$$\mathrm{OWA}_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)}\,, \qquad \sum_{i=1}^{n} w_i = 1\,, \qquad w_i > 0$$

with the convention that $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$.

Proposition (Grabisch-Marichal, 1995)
Let $v \in \mathcal{F}_N$. The following assertions are equivalent :
i) v is cardinality-based
ii) $\exists$ a weight vector $\mathbf{w}$ such that $\mathcal{C}_v = \mathrm{OWA}_{\mathbf{w}}$
iii) $\mathcal{C}_v$ is a symmetric function.

Ordinal scales

Example : Evaluation of a scientific journal paper on importance

1=Poor, 2=Below average, 3=Average,
4=Very Good, 5=Excellent

$$\begin{aligned} \text{Values :} \quad & 1, 2, 3, 4, 5 \\ \text{or :} \quad & 2, 7, 20, 100, 246 \\ \text{or :} \quad & -46, -3, 0, 17, 98 \end{aligned}$$

Numbers assigned to an ordinal scale are defined up an increasing bijection $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

Means on ordered sets

Definition. A function $F : E^n \to \mathbb{R}$ is comparison meaningful if, for any increasing bijection $\phi : E \to E$ and any $\mathbf{x}, \mathbf{x}' \in E^n$,

$$F(x_1, \ldots, x_n) \leqslant F(x'_1, \ldots, x'_n)$$
$$\Updownarrow$$
$$F(\phi(x_1), \ldots, \phi(x_n)) \leqslant F(\phi(x'_1), \ldots, \phi(x'_n))$$

Example. The arithmetic mean is not comparison meaningful
Consider

$$4 = \frac{3+5}{2} < \frac{1+8}{2} = 4.5$$

and any bijection ϕ such that $\phi(1) = 1$, $\phi(3) = 4$, $\phi(5) = 7$, $\phi(8) = 8$. We have

$$5.5 = \frac{4+7}{2} \not< \frac{1+8}{2} = 4.5$$

Means on ordered sets

Theorem 12 (Ovchinnikov, 1996)
The function $F : E^n \to \mathbb{R}$ is

- symmetric
- continuous
- internal
- comparison meaningful

if and only if there exists $k \in N$ such that

$$F(\mathbf{x}) = x_{(k)}$$

Note : $x_{(k)} = \mathrm{median}[\mathbf{x}]$ if $n = 2k - 1$

Lattice polynomials

Definition. A lattice polynomial function in $\mathbb{R}^n$ is defined from any well-formed expression constructed from the variables $x_1, \ldots, x_n$ and the symbols $\wedge, \vee$.

Example : $(x_2 \vee (x_1 \wedge x_3)) \wedge (x_4 \vee x_2)$

It can be proved that a lattice polynomial can always be put in the form

$$L_c(\mathbf{x}) = \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{i \in T} x_i$$

where $c : 2^N \to \{0, 1\}$ is a nonconstant set function such that $c(\varnothing) = 0$.

In particular

$$x_{(k)} = \bigvee_{\substack{T \subseteq N \\ |T|=n-k+1}} \bigwedge_{i \in T} x_i$$

Axiomatization of lattice polynomials in $\mathbb{R}^n$

Theorem 13 (Marichal-Mathonet, 2001)
The function $F : E^n \to \mathbb{R}$ is

- continuous
- idempotent
- comparison meaningful

if and only if there exists a nonconstant set function $c : 2^N \to \{0, 1\}$, with $c(\varnothing) = 0$, such that $F = L_c$

Note : If E is open, continuity can be replaced with nondecreasing monotonicity

Complete description of comparison meaningful functions :
see Marichal-Mesiar-Rückschlossová, 2005

Connection with Choquet integral

Proposition 2 (Murofushi-Sugeno, 1993)
If $v \in \mathcal{F}_N$ is $\{0,1\}$-valued then $\mathcal{C}_v = L_v$

Conversely, we have $L_c = \mathcal{C}_c$.

Proposition 3 (Radojević, 1998)
A function $F : E^n \to \mathbb{R}$ is a Choquet integral if and only if it is a weighted arithmetic mean of lattice polynomials

$$\mathcal{C}_v = \sum_{i=1}^{q} w_i \, L_{c_i}$$

This decomposition is not unique !

$$0.2x_1 + 0.6x_2 + 0.2(x_1 \wedge x_2) = 0.4x_2 + 0.4(x_1 \wedge x_2) + 0.2(x_1 \vee x_2)$$

Connection with Choquet integral

Proposition 4 (Marichal, 2001)
Any Choquet integral can be expressed as a lattice polynomial of weighted arithmetic means

$$\mathcal{C}_v(\mathbf{x}) = L_c(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$$

Example (continued)

$$0.2x_1 + 0.6x_2 + 0.2(x_1 \wedge x_2) = (0.4x_1 + 0.6x_2) \wedge (0.2x_1 + 0.8x_2)$$

The converse is not true : $\left(\frac{x_1+x_2}{2}\right) \wedge x_3$ is not a Choquet integral

Unsolved problem : Give conditions under which a lattice polynomial of weighted arithmetic means is a Choquet integral