Associative and preassociative functions

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Associative functions

Let X be a nonempty set

$G\colon X^2 \to X$ is *associative* if

$$G(x, G(y,z)) \;=\; G(G(x,y),z)$$

Example: $G(x,y) = x + y$ on $X = \mathbb{R}$

Associative functions of multiple arities

Let

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

$F\colon X^* \to X$ is *associative* if

$$\begin{aligned} & F(x_1, \ldots, x_p, \quad y_1, \ldots, y_q, \quad z_1, \ldots, z_r) \\ = \; & F(x_1, \ldots, x_p, F(y_1, \ldots, y_q), z_1, \ldots, z_r) \end{aligned}$$

Example: $F(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ on $X = \mathbb{R}$

Notation

We regard n-tuples $\mathbf{x}$ in X^n as *n-strings* over X

0-string: ε
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

X^* is endowed with concatenation

Example: $\mathbf{x} \in X^n$, $y \in X$, $\mathbf{z} \in X^m \quad \Rightarrow \quad \mathbf{x}y\mathbf{z} \in X^{n+1+m}$

$$|\mathbf{x}| \; = \; \text{length of } \mathbf{x}$$

Functions of multiple arities

Let

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

$$\boxed{F \colon X^* \to X}$$

Components of F :

$$F_n \colon X^n \to X$$

$$\boxed{F_n = F|_{X^n}}$$

F is described by its components $F_1, F_2, F_3, \ldots, F_n, \ldots$

Associative functions of multiple arities

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Theorem (Couceiro and M.)

$F\colon X^* \to X$ is associative if and only if

$$F(\mathbf{xy}) \;=\; F(F(\mathbf{x})F(\mathbf{y})) \qquad \forall\ \mathbf{xy} \in X^*$$

Associative functions of multiple arities

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Theorem

We can assume that $|\mathbf{xz}| \leqslant 1$ in the definition above

That is, $F\colon X^* \to X$ is associative if and only if

$$\begin{aligned} F(\mathbf{y}) &= F(F(\mathbf{y})) \\ F(x\mathbf{y}) &= F(xF(\mathbf{y})) \\ F(\mathbf{y}z) &= F(F(\mathbf{y})z) \end{aligned}$$

Associative functions of multiple arities

Associative functions are completely determined by their unary and binary components

$$F_n(x_1 \cdots x_n) \;=\; F_2(F_{n-1}(x_1 \cdots x_{n-1})x_n) \qquad n \geqslant 3$$

Proposition

Let $F\colon X^* \to X$ and $G\colon X^* \to X$ be two associative functions such that $F_1 = G_1$ and $F_2 = G_2$. Then $F = G$.

Associative functions of multiple arities

Link with binary associative functions ?

Proposition

A binary function $G\colon X^2 \to X$ is associative if and only if there exists an associative function $F\colon X^* \to X$ such that $F_2 = G$.

Does F_1 really play a role ?

$$F_1(F(\mathbf{x})) = F(\mathbf{x})$$
$$F(\mathbf{x}y\mathbf{z}) = F(\mathbf{x}F_1(y)\mathbf{z})$$

Associative functions of multiple arities

Theorem

$F\colon X^* \to X$ is associative if and only if

(i) $F_1(F_1(x)) = F_1(x), \quad F_1(F_2(xy)) = F_2(xy)$

(ii) $F_2(xy) = F_2(F_1(x)y) = F_2(xF_1(y))$

(iii) $F_2(F_2(xy)z) = F_2(xF_2(yz))$

(iv) $F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})x_n) \qquad n \geqslant 3$

Suppose F_2 satisfying (iii) is given. What could be F_1 ?

Example: $F_2(xy) = x + y$

By (i), we have

$$F_1(x+y) \;=\; F_1(F_2(xy)) \;=\; F_2(xy) \;=\; x+y$$

$\Rightarrow \quad F_1(x) = x$

Associative functions of multiple arities

Theorem

$F\colon X^* \to X$ is associative if and only if

(i) $F_1(F_1(x)) = F_1(x), \quad F_1(F_2(xy)) = F_2(xy)$

(ii) $F_2(xy) = F_2(F_1(x)y) = F_2(xF_1(y))$

(iii) $F_2(F_2(xy)z) = F_2(xF_2(yz))$

(iv) $F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})x_n) \qquad n \geqslant 3$

Example: $F_n(x_1 \cdots x_n) = \big(|x_1|^2 + \cdots + |x_n|^2\big)^{1/2}$

$$F_1(x) \;=\; x$$

$$F_1(x) \;=\; |x|$$

Preassociative functions

Let Y be a nonempty set

Definition. We say that $F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) \; = \; F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \; = \; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Example: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \qquad (X = Y = \mathbb{R})$

Proposition

$F\colon X^* \to Y$ is preassociative if and only if

$$F(\mathbf{x}) \; = \; F(\mathbf{x}') \quad \text{and} \quad F(\mathbf{y}) \; = \; F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}) \; = \; F(\mathbf{x}'\mathbf{y}')$$

Preassociative functions

Remark. If $F\colon X^* \to X$ is associative, then it is preassociative

Proof. Suppose $F(\mathbf{y}) = F(\mathbf{y}')$
Then $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{x}F(\mathbf{y}')\mathbf{z}) = F(\mathbf{xy}'\mathbf{z})$ □

Proposition

$F\colon X^* \to X$ is associative if and only if it is preassociative and $F_1(F(\mathbf{x})) = F(\mathbf{x})$

Proof. (Necessity) OK.
(Sufficiency) We have $F(\mathbf{y}) = F(F(\mathbf{y}))$
Hence, by preassociativity, $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$ □

Preassociative functions

Proposition

If $F\colon X^* \to Y$ is preassociative, then so is $F \circ (g, \ldots, g)$ for every function $g\colon X \to X$, where

$$F \circ (g, \ldots, g) \;:\; \qquad x_1 \cdots x_n \;\mapsto F_n(g(x_1) \cdots g(x_n))$$

Example: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

Proposition

If $F\colon X^* \to Y$ is preassociative, then so is $g \circ F$ for every function $g\colon Y \to Y$ such that $g|_{\mathrm{ran}(F)}$ is constant or one-to-one

Example: $F_n(\mathbf{x}) = \exp(x_1^2 + \cdots + x_n^2) \quad (X = Y = \mathbb{R})$

Preassociative functions

Proposition

Assume $F\colon X^* \to Y$ is preassociative
If F_n is constant, then so is F_{n+1}

Proof. If $F_n(\mathbf{y}) = F_n(\mathbf{y}')$ for all $\mathbf{y}, \mathbf{y}' \in X^n$, then $F_{n+1}(x\mathbf{y}) = F_{n+1}(x\mathbf{y}')$ and hence F_{n+1} depends only on its first argument... □

Proposition

Assume $F\colon X^* \to Y$ is preassociative
If F_n and F_{n+1} are the same constant c, then $F_m = c$ for all $m \geqslant n$

Proof. If $c = F_n(\mathbf{x}) = F_{n+1}(\mathbf{x}y)$, then $c = F_{n+1}(\mathbf{x}z) = F_{n+2}(\mathbf{x}yz)$.
So $F_{n+2} = c \ldots$ □

Preassociative functions

We have seen that $F\colon X^* \to X$ is associative if and only if it is preassociative and $F_1(F(\mathbf{x})) = F(\mathbf{x})$

Relaxation of $F_1(F(\mathbf{x})) = F(\mathbf{x})$:

$$\mathrm{ran}(F_1) \ = \ \mathrm{ran}(F)$$

We now focus on preassociative functions $F\colon X^* \to Y$ satisfying $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

Proposition

Let $F\colon X^* \to Y$ and $G\colon X^* \to Y$ be two preassociative functions such that $\mathrm{ran}(F_1) = \mathrm{ran}(F)$ and $\mathrm{ran}(G_1) = \mathrm{ran}(G)$.
If $F_1 = G_1$ and $F_2 = G_2$, then $F = G$.

Preassociative functions

Theorem

Let $F\colon X^* \to Y$ be a function. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

(ii) F can be factorized into $F = f \circ H$,
where $H\colon X^* \to X$ is associative
and $f\colon \mathrm{ran}(H) \to Y$ is one-to-one.
In this case, we have $f = F_1|_{\mathrm{ran}(H)}$ and $F = F_1 \circ H$

Open problems

(1) Suppress the condition $\mathrm{ran}(F_1) = \mathrm{ran}(F)$ in this theorem

(2) Find necessary and sufficient conditions on F_1 for a function F of the form $F = F_1 \circ H$, where H is associative, to be preassociative.

Axiomatizations of function classes

Theorem (Aczél 1949)

$H\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, one-to-one in each argument, and associative if and only if there exists a continuous and strictly monotone function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that

$$H(xy) \;=\; \varphi^{-1}(\varphi(x) + \varphi(y))$$

Axiomatizations of function classes

Theorem

Let $F\colon \mathbb{R}^* \to \mathbb{R}$ be a function. The following assertions are equivalent:

(i) F is preassociative and satisfies $\operatorname{ran}(F_1) = \operatorname{ran}(F)$, and F_1 and F_2 are continuous and one-to-one in each argument

(ii) there exist continuous and strictly monotone functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ such that

$$F_n(\mathbf{x}) \;=\; \psi\big(\varphi(x_1) + \cdots + \varphi(x_n)\big)$$

Axiomatizations of function classes

Recall that a *triangular norm* is a function $T\colon [0,1]^2 \to [0,1]$ which is nondecreasing in each argument, symmetric, associative, and such that $T(1x) = x$

Theorem

Let $F\colon [0,1]^* \to \mathbb{R}$ be such that F_1 is strictly increasing. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, and F_2 is symmetric, nondecreasing, and satisfies $F_2(1x) = F_1(x)$

(ii) there exists a strictly increasing function $f\colon [0,1] \to \mathbb{R}$ and a triangular norm $T\colon [0,1]^* \to [0,1]$ such that

$$F = f \circ T.$$

Axiomatizations of function classes

Theorem

Let $H\colon \mathbb{R}^* \to \mathbb{R}$ be a function. The following assertions are equivalent:

(i) H is associative and satisfies $H(H(x)H(x)) = H(x)$, and H_1 and H_2 are symmetric, continuous, and nondecreasing

(ii) there exist $a, b, c \in \mathbb{R}$, $a \leqslant c \leqslant b$, such that

$$H_n(\mathbf{x}) \ = \ \mathrm{med}\Big(a, \mathrm{med}\Big(\bigwedge_{i=1}^{n} x_i, c, \bigvee_{i=1}^{n} x_i\Big), b\Big)$$

Axiomatizations of function classes

Theorem

Let $F\colon \mathbb{R}^* \to \mathbb{R}$ be a function and let $[a, b]$ be a closed interval. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, there exists a continuous and strictly increasing function $f\colon [a, b] \to \mathbb{R}$ such that $F_1(x) = (f \circ \mathrm{med})(a, x, b)$, F_2 is continuous, nondecreasing and satisfies $F_2(xx) = F_1(x)$

(ii) there exist $c \in [a, b]$ such that

$$F_n(\mathbf{x}) \;=\; (f \circ \mathrm{med})\biggl(a, \mathrm{med}\Bigl(\bigwedge_{i=1}^{n} x_i, c, \bigvee_{i=1}^{n} x_i\Bigr), b\biggr)$$

Strongly preassociative functions

Definition. We say that $F\colon X^* \to Y$ is *strongly preassociative* if

$$F(\mathbf{xz}) \;=\; F(\mathbf{x}'\mathbf{z}') \quad \Rightarrow \quad F(\mathbf{xyz}) \;=\; F(\mathbf{x}'\mathbf{y}\mathbf{z}')$$

Theorem

$F\colon X^* \to Y$ is strongly preassociative if and only if it is preassociative and F_n is symmetric for every $n \in \mathbb{N}$

Open problems

(1) Find new axiomatizations of classes of preassociative functions from existing axiomatizations of classes of associative functions

(2) Find interpretations of preassociativity in fuzzy logic, artificial intelligence,...

Thank you for your attention !