A Complete Description of Comparison Meaningful Functions

Jean-Luc MARICHAL*, Radko MESIAR,
and Tatiana RÜCKSCHLOSSOVÁ

* University of Luxembourg

Let F be an aggregation function from $\mathbb{R}^n$ to $\mathbb{R}$:

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ are the independent variables and x_{n+1} is the dependent variable.

The general form of F is restricted if we know the *scale type* of the variables $x_1, \ldots, x_n$ and x_{n+1} (Luce 1959).

A scale type is defined by the class of *admissible transformations*, transformations which change the scale into an alternative acceptable scale.

x_i defines an *ordinal scale* if the class of admissible transformations consists of the increasing bijections of $\mathbb{R}$ onto $\mathbb{R}$.

Principle of theory construction (Luce 1959)

Admissible transformations of the independent variables should lead to an admissible transformation of the dependent variable.

Suppose that

$$x_{n+1} = F(x_1, \ldots, x_n)$$

where x_{n+1} is an ordinal scale and $x_1, \ldots, x_n$ are independent ordinal scales.

Let $A(\mathbb{R})$ be the set of increasing bijections of $\mathbb{R}$ onto $\mathbb{R}$.

For any $\phi_1, \ldots, \phi_n \in A(\mathbb{R})$, there is $\Phi_{\phi_1,\ldots,\phi_n} \in A(\mathbb{R})$ such that

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] = \Phi_{\phi_1,\ldots,\phi_n}[F(x_1, \ldots, x_n)]$$

Assume $x_1, \ldots, x_n$ define the *same* ordinal scale.
Then the functional equation simplifies into

$$F[\phi(x_1), \ldots, \phi(x_n)] = \Phi_\phi[F(x_1, \ldots, x_n)]$$

Equivalently, F fulfills the condition (Orlov 1981)

$$\begin{gathered} F(x_1, \ldots, x_n) \leqslant F(x'_1, \ldots, x'_n) \\ \Updownarrow \\ F[\phi(x_1), \ldots, \phi(x_n)] \leqslant F[\phi(x'_1), \ldots, \phi(x'_n)] \end{gathered}$$

F is said to be *comparison meaningful* (Ovchinnikov 1996)

Assume $x_1, \ldots, x_n$ are *independent* ordinal scales.
Recall that the functional equation is

$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] = \Phi_{\phi_1,\ldots,\phi_n}[F(x_1, \ldots, x_n)]$$

Equivalently, F fulfills the condition

$$F(x_1, \ldots, x_n) \leqslant F(x'_1, \ldots, x'_n)$$
$$\Updownarrow$$
$$F[\phi_1(x_1), \ldots, \phi_n(x_n)] \leqslant F[\phi_1(x'_1), \ldots, \phi_n(x'_n)]$$

We say that F is *strongly comparison meaningful*

Purpose of the presentation

To provide a complete description of comparison meaningful functions

To provide a complete description of strongly comparison meaningful functions

The continuous case

First result (Osborne 1970, Kim 1990)

$F : \mathbb{R}^n \to \mathbb{R}$ is continuous and strongly comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \quad \text{- continuous} \\ \qquad\qquad\qquad \text{- strictly monotonic or constant} \\ \text{such that} \\ \qquad F(x_1,\ldots,x_n) = g(x_k) \end{cases}$$

$+$ idempotent, i.e., $F(x,\ldots,x) = x$

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \text{ such that} \\ \qquad F(x_1,\ldots,x_n) = x_k \end{cases}$$

The nondecreasing case

Second result (Marichal & Mesiar & Rückschlossová 2005)
$F : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing and strongly comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \ \text{ strictly increasing or constant} \\ \text{such that} \\ \qquad F(x_1,\ldots,x_n) = g(x_k) \end{cases}$$

+ idempotent

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \text{ such that} \\ \qquad F(x_1,\ldots,x_n) = x_k \end{cases}$$

The general case

Third result (Marichal & Mesiar & Rückschlossová 2005)

$F : \mathbb{R}^n \to \mathbb{R}$ is strongly comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \;\; \text{strictly monotonic or constant} \\ \text{such that} \\ \qquad F(x_1,\ldots,x_n) = g(x_k) \end{cases}$$

$+$ idempotent

$$\Leftrightarrow \begin{cases} \exists\, k \in \{1,\ldots,n\} \text{ such that} \\ \qquad F(x_1,\ldots,x_n) = x_k \end{cases}$$

Comparison meaningful functions

First result (Orlov 1981)

$F : \mathbb{R}^n \to \mathbb{R}$ is
- symmetric
- continuous
- internal, i.e., $\min_i x_i \leqslant F(x_1, \ldots, x_n) \leqslant \max_i x_i$
- comparison meaningful

$$\Leftrightarrow \left\{ \begin{array}{c} \exists\, k \in \{1, \ldots, n\} \text{ such that} \\ F(x_1, \ldots, x_n) = x_{(k)} \end{array} \right.$$

where $x_{(1)}, \ldots, x_{(n)}$ denote the *order statistics* resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order.

Next step : suppress symmetry and relax internality into idempotency

Lattice polynomials

Definition (Birkhoff 1967)

An n-variable *lattice polynomial* is any expression involving n variables $x_1, \ldots, x_n$ linked by the lattice operations

$$\wedge = \min \quad \text{and} \quad \vee = \max$$

in an arbitrary combination of parentheses.

For example,

$$L(x_1, x_2, x_3) = (x_1 \vee x_3) \wedge x_2$$

is a 3-variable lattice polynomial.

Lattice polynomials

Proposition (Ovchinnikov 1998, Marichal 2002)

A lattice polynomial on $\mathbb{R}^n$ is symmetric iff it is an order statistic.

We have

$$x_{(k)} = \bigvee_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = n-k+1}} \bigwedge_{i \in T} x_i = \bigwedge_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = k}} \bigvee_{i \in T} x_i$$

The nonsymmetric case

Second result (Yanovskaya 1989)

$F : \mathbb{R}^n \to \mathbb{R}$ is
- continuous
- idempotent
- comparison meaningful

$\Leftrightarrow \quad \exists$ a lattice polynomial $L : \mathbb{R}^n \to \mathbb{R}$ such that $F = L$.

$+$ symmetric

$\Leftrightarrow \quad \exists\ k \in \{1, \ldots, n\}$ such that $F = \mathrm{OS}_k$ (kth order statistic).

Next step : suppress idempotency

The nonidempotent case

Third result (Marichal 2002)

$F:\mathbb{R}^n\to\mathbb{R}$ is - continuous
- comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, L:\mathbb{R}^n\to\mathbb{R} \text{ lattice polynomial} \\ \exists\, g:\mathbb{R}\to\mathbb{R} \quad \text{- continuous} \\ \qquad\qquad\qquad \text{- strictly monotonic or constant} \\ \text{such that} \\ \qquad F = g\circ L \end{cases}$$

$+$ symmetric

$$F = g\circ \mathrm{OS}_k$$

Towards the noncontinuous case

Fourth result (Marichal 2002)

$F : \mathbb{R}^n \to \mathbb{R}$ is
- nondecreasing
- idempotent
- comparison meaningful

$\Leftrightarrow \quad \exists$ a lattice polynomial $L : \mathbb{R}^n \to \mathbb{R}$ such that $F = L$.

Note : These functions are continuous !

$+$ symmetric

$$F = \mathrm{OS}_k$$

Next step : suppress idempotency

The nondecreasing case

Fifth result (Marichal & Mesiar & Rückschlossová 2005)

$F : \mathbb{R}^n \to \mathbb{R}$ is - nondecreasing
- comparison meaningful

$$\Leftrightarrow \begin{cases} \exists\, L : \mathbb{R}^n \to \mathbb{R} \text{ lattice polynomial} \\ \exists\, g : \mathbb{R} \to \mathbb{R} \text{ strictly increasing or constant} \\ \text{such that} \\ \qquad F = g \circ L \end{cases}$$

These functions are continuous up to possible discontinuities of function g

Final step : suppress nondecreasing monotonicity (a hard task !)

The general case

… is much more complicated to describe

- We loose the concept of lattice polynomial
- The description of F is done through a partition of the domain $\mathbb{R}^n$ into particular subsets, called *invariant subsets*

Invariant subsets

Let us consider the subsets of $\mathbb{R}^n$ of the form

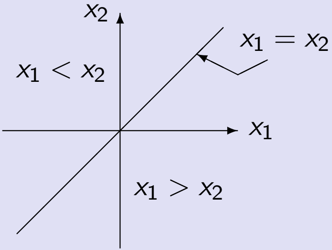
$$I = \{x \in \mathbb{R}^n \mid x_{\pi(1)} \lhd_1 \cdots \lhd_{n-1} x_{\pi(n)}\}$$

where π is any permutation on $\{1, \ldots, n\}$ and $\lhd_i \in \{<, =\}$.

Denote this class of subsets by $\mathcal{I}(\mathbb{R}^n)$.

Example : $\mathbb{R}^2$

Description of $\mathcal{I}(\mathbb{R}^2)$:
$I_1 = \{(x_1, x_2) \mid x_1 = x_2\}$
$I_2 = \{(x_1, x_2) \mid x_1 < x_2\}$
$I_3 = \{(x_1, x_2) \mid x_1 > x_2\}$



Invariant subsets

Proposition (Bartłomiejczyk & Drewniak 2004)

The class $\mathcal{I}(\mathbb{R}^n)$ consists of the *minimal invariant* subsets of $\mathbb{R}^n$.

That is,

- Each subset $I \in \mathcal{I}(\mathbb{R}^n)$ is *invariant* in the sense that

$$(x_1, \ldots, x_n) \in I \quad \Rightarrow \quad (\phi(x_1), \ldots, \phi(x_n)) \in I \qquad \forall\, \phi \in A(\mathbb{R})$$

- Each subset $I \in \mathcal{I}(\mathbb{R}^n)$ is *minimal* in the sense that it has no proper invariant subset

The family $\mathcal{I}(\mathbb{R}^n)$ partitions $\mathbb{R}^n$ into equivalence classes :

$$x \sim y \quad \Leftrightarrow \quad \exists\, \phi \in A(\mathbb{R}) \;:\; y_i = \phi(x_i) \;\; \forall\, i$$

The general case

Sixth result (Marichal & Mesiar & Rückschlossová 2005)

$F : \mathbb{R}^n \to \mathbb{R}$ is comparison meaningful

$$\Leftrightarrow \quad \forall\, I \in \mathcal{I}(\mathbb{R}^n), \quad \begin{cases} \exists\, k_I \in \{1, \ldots, n\} \\ \exists\, g_I : \mathbb{R} \to \mathbb{R} \ \text{ strictly monotonic or constant} \\ \text{such that} \\ \qquad F|_I(x_1, \ldots, x_n) = g_I(x_{k_I}) \\ \\ \quad \text{where } \forall\, I, I' \in \mathcal{I}(\mathbb{R}^n), \end{cases}$$

- either $g_I = g_{I'}$
- or $ran(g_I) = ran(g_{I'})$ is a singleton
- or $ran(g_I) < ran(g_{I'})$
- or $ran(g_I) > ran(g_{I'})$

Conclusion

We have described all the possible merging functions

$$F : \mathbb{R}^n \to \mathbb{R},$$

which map n ordinal scales into an ordinal scale.

These results hold true when F is defined on E^n, where E is any open real interval.

The cases where E is a non-open real interval all have been described and can be found in

J.-L. Marichal, R. Mesiar, and T. Rückschlossová,
A Complete Description of Comparison Meaningful Functions,
Aequationes Mathematicae 69 (2005) 309–320.

Thank you for your attention