A COMPLETE DESCRIPTION OF COMPARISON MEANINGFUL FUNCTIONS

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Summary

Comparison meaningful functions acting on some real interval E are completely described as transformed coordinate projections on minimal invariant subsets. The case of monotone comparison meaningful functions is further specified. Several already known results for comparison meaningful functions and invariant functions are obtained as consequences of our description.

Keywords: Comparison meaningful function, Invariant function, Ordinal scale.

1 INTRODUCTION

Measurement theory (see e.g. [6, 14]) studies, among others, the assignments to each measured object of a real number so that the ordinal structure of discussed objects is preserved. When aggregating several observed objects, their aggregation is often also characterized by a real number, which can be understood as a function of numerical characterizations of fused objects. A sound approach to such aggregation cannot lead to contradictory results depending on the actual scale (numerical evaluation of objects) we are dealing with. This fact was a key motivation for Orlov [11] when introducing comparison meaningful functions. Their strengthening to invariant functions (scale independent functions) was proposed by Marichal and Roubens [9]. The general structure of invariant functions (and of monotone invariant functions) is now completely known from recent works of Ovchinnikov [12], Ovchinnikov and Dukhovny [13], Marichal [7], Bartlomiejczyk and Drewniak [2], and Mesiar and Rückschlossová [10]. Moreover, comparison meaningful functions were already characterized in some special cases, e.g., when they are continuous; see Yanovskaya [16] and Marichal [7]. However, a complete description of all comparison meaningful functions was still missing. This gap is now filled by the present paper, which is organized as follows. In the next section, we give some preliminaries and recall some known results. In Section 3, a complete description of comparison meaningful functions is given, while in Section 4 we describe all monotone comparison meaningful functions.

2 PRELIMINARIES

Let $E \subseteq \mathbb{R}$ be a nontrivial convex set and set $e_0 := \inf E$, $e_1 := \sup E$, and $E^\circ := E \setminus \{e_0, e_1\}$. Let $n \in \mathbb{N}$ be fixed and set $[n] := \{1, \ldots, n\}$. Denote also by $\Phi(E)$ the class of all automorphisms (nondecreasing bijections) $\phi : E \to E$, and for $x = (x_1, \ldots, x_n) \in E^n$ put $\phi(x) := (\phi(x_1), \ldots, \phi(x_n))$.

Following the earlier literature, we introduce the next notions and recall a few results.

Definition 2.1 ([9])**.** A function $f : E^n \to E$ is invariant if, for any $\phi \in \Phi(E)$ and any $x \in E^n$, we have $f(\phi(x)) = \phi(f(x))$.

Definition 2.2 ([1, 11, 16])**.** A function $f : E^n \to \mathbb{R}$ is comparison meaningful if, for any $\phi \in \Phi(E)$ and any $x, y \in E^n$, we have

$$f(x) \begin{Bmatrix} < \\ = \end{Bmatrix} f(y) \quad \Rightarrow \quad f(\phi(x)) \begin{Bmatrix} < \\ = \end{Bmatrix} f(\phi(y)). \qquad (1)$$

Definition 2.3 ([1, 5])**.** A function $f : E^n \to \mathbb{R}$ is strongly comparison meaningful if, for any $\phi_1, \ldots, \phi_n \in \Phi(E)$ and any $x, y \in E^n$, we have

$$f(x) \begin{Bmatrix} < \\ = \end{Bmatrix} f(y) \quad \Rightarrow \quad f(\phi(x)) \begin{Bmatrix} < \\ = \end{Bmatrix} f(\phi(y)),$$

where here the notation $\phi(x)$ means $(\phi_1(x_1), \ldots, \phi_n(x_n))$.

Definition 2.4 ([2]). A nonempty subset B of E^n is called invariant if $\phi(B) \subseteq B$ for any $\phi \in \Phi(E)$, where $\phi(B) = \{\phi(x) \mid x \in B\}$. Moreover, an invariant subset B of E^n is called minimal invariant if it does not contain any proper invariant subset.

It can be easily proved that $B \subseteq E^n$ is invariant if and only if its characteristic function $\mathbf{1}_B : E^n \to \mathbb{R}$ is comparison meaningful (or invariant if $E = [0,1]$).

Let $\mathcal{B}(E^n)$ be the class of all minimal invariant subsets of E^n, and define

$$B_x(E) := \{\phi(x) \mid \phi \in \Phi(E)\}$$

for all $x \in E^n$. Then, we have

$$\mathcal{B}(E^n) = \{B_x(E) \mid x \in E^n\},$$

which clearly shows that the elements of $\mathcal{B}(E^n)$ partition E^n into equivalence classes, where $x, y \in E^n$ are equivalent if there exists $\phi \in \Phi(E)$ such that $y = \phi(x)$. A complete description of elements of $\mathcal{B}(E^n)$ is given in the following proposition:

Proposition 2.1 ([2, 10]). *We have $B \in \mathcal{B}(E^n)$ if and only if there exists a permutation π on $[n]$ and a sequence $\{\lhd_i\}_{i=0}^n$ of symbols $\lhd_i \in \{<, =\}$, containing at least one symbol $<$ if $e_0 \in E$ and $e_1 \in E$, such that*

$$B = \{x \in E^n \mid e_0 \lhd_0 x_{\pi(1)} \lhd_1 \cdots \lhd_{n-1} x_{\pi(n)} \lhd_n e_1\},$$

where $\lhd_0$ is $<$ if $e_0 \notin E$ and $\lhd_n$ is $<$ if $e_1 \notin E$.

Example 2.1. The unit square $[0,1]^2$ contains exactly eleven minimal invariant subsets, namely the open triangles $\{(x_1, x_2) \mid 0 < x_1 < x_2 < 1\}$ and $\{(x_1, x_2) \mid 0 < x_2 < x_1 < 1\}$, the open diagonal $\{(x_1, x_2) \mid 0 < x_1 = x_2 < 1\}$, the four square vertices, and the four open line segments joining neighboring vertices.

We also have the following important result:

Proposition 2.2 ([2, 7, 10]). *Consider a function $f : E^n \to E$.*

i) If f is idempotent, (i.e., $f(x, \ldots, x) = x$ for all $x \in E$) and comparison meaningful then it is invariant.

ii) If f is invariant, then it is comparison meaningful.

iii) If E is open, then f is idempotent and comparison meaningful if and only if it is invariant.

iv) f is invariant if and only if, for any $B \in \mathcal{B}(E^n)$, either $f|_B \equiv c$ is a constant $c \in \{e_0, e_1\} \cap E$ (if this constant exists) or there is $i \in [n]$ so that $f|_B = P_i|_B$ is the projection on the ith coordinate.

For nondecreasing invariant functions, a crucial role in their characterization is played by an equivalence relation $\sim$ acting on $\mathcal{B}(E^n)$, namely $B \sim C$ if and only if $P_i(B) = P_i(C)$ for all $i \in [n]$. Note that projections $P_i(B)$ of minimal invariant subsets are necessarily either $\{e_0\} \cap E$ or $\{e_1\} \cap E$ or E°. Further, for any $B \in \mathcal{B}(E^n)$, the set

$$B^* = \bigcup_{\substack{C \in \mathcal{B}(E^n) \\ C \sim B}} C = P_1(B) \times \cdots \times P_n(B)$$

is an invariant subset of E^n, and

$$\mathcal{B}^*(E^n) = \{B^* \mid B \in \mathcal{B}(E^n)\}$$

is a partition of E^n coarsening $\mathcal{B}(E^n)$. We also have $\mathrm{card}(\mathcal{B}^*(E^n)) = k^n$, where $k = 1 + \mathrm{card}(E \cap \{e_0, e_1\})$.

Notice that any subset B^* can also be regarded as a minimal "strongly" invariant subset of E^n in the sense that

$$\{(\phi_1(x_1), \ldots, \phi_n(x_n)) \mid x \in B^*\} \subseteq B^*$$

for all $\phi_1, \ldots, \phi_n \in \Phi(E)$. Equivalently, the characteristic function $\mathbf{1}_{B^*} : E^n \to \mathbb{R}$ is strongly comparison meaningful.

From the natural order

$$\{e_0\} \prec E^\circ \prec \{e_1\}$$

we can straightforwardly derive a partial order $\preceq$ on $\mathcal{B}(E^n)$, namely $B \preceq C$ if and only if $P_i(B) \preceq P_i(C)$ for all $i \in [n]$. A partial order on $\mathcal{B}^*(E^n)$ can be defined similarly.

Denote by $\mathcal{M}_n$ the system of all nondecreasing functions $\mu : \{0,1\}^n \to \{0,1\}$, and let

$$\mathcal{M}_n(E) := \mathcal{M}_n \setminus \{\mu_j \mid j \in \{0,1\},\ e_j \notin E\},$$

where $\mu_j \in \mathcal{M}_n$ is the constant set function $\mu_j \equiv j$. Clearly $\mathcal{M}_n(E)$ is partially ordered through the order defined as

$$\mu \preceq \mu' \quad \Leftrightarrow \quad \mu(x) \leq \mu'(x) \ \ \forall x \in \{0,1\}^n.$$

For $\mu \in \mathcal{M}_n(E)$, we define a function $L_\mu : E^n \to E$ by

$$L_\mu(x_1, \ldots, x_n) = \bigvee_{\substack{t \in \{0,1\}^n \\ \mu(t)=1}} \bigwedge_{t_i=1} x_i$$

with obvious conventions

$$\bigvee_{\varnothing} = e_0 \quad \text{and} \quad \bigwedge_{\varnothing} = e_1.$$

Observe that for any $\mu \in \mathcal{M}_n(E)$, L_μ is a continuous invariant function which is also idempotent whenever $\mu(0, \ldots, 0) < \mu(1, \ldots, 1)$, that is, whenever $\mu(0, \ldots, 0) = 0$ and $\mu(1, \ldots, 1) = 1$.

Remark. Functions $\mu \in \mathcal{M}_n(E)$ with $\mu(0,\ldots,0) < \mu(1,\ldots,1)$ are called also $\{0,1\}$-valued fuzzy measures (when an element $t \in \{0,1\}^n$ is taken as the characteristic vector of a subset of $[n]$). For any such μ, the corresponding function L_μ is exactly the Choquet integral with respect to μ [4, 13], but also the Sugeno integral with respect to μ [15, 13]. These functions are called also lattice polynomials [3] or Boolean max-min functions [8].

We also have the following result:

Proposition 2.3 ([7, 10])**.** *Consider a function $f : E^n \to E$. Then we have*

i) f is continuous and invariant if and only if $f = L_\mu$ for some $\mu \in \mathcal{M}_n(E)$.

ii) f is nondecreasing and invariant if and only if there exists a nondecreasing mapping $\xi : \mathcal{B}^*(E^n) \to \mathcal{M}_n(E)$ so that

$$f(x) = L_{\xi(B^*)}(x) \qquad (x \in B^* \in \mathcal{B}^*(E^n)).$$

3 COMPARISON MEANINGFUL FUNCTIONS

Following Definition 2.1, the invariance of a function $f : E^n \to E$ can be reduced to the invariance of $f|_B$ for all minimal invariant subsets $B \in \mathcal{B}(E^n)$. This observation is a key point in the description of invariant functions as given in Proposition 2.2, *iv*). However, in the case of comparison meaningful functions, the situation is more complicated. In fact, we have to examine property (1) for $x \in B$, $y \in C$, with $B, C \in \mathcal{B}(E^n)$, to be able to describe comparison meaningful functions. We start first with the case when $B = C$, i.e., when $y = \phi(x)$ for some $\phi \in \Phi(E)$.

Proposition 3.1. *Let $f : E^n \to \mathbb{R}$ be a comparison meaningful function. Then, for any $B \in \mathcal{B}(E^n)$, there is an index $i_B \in [n]$ and a strictly monotone or constant function $g_B : P_{i_B}(B) \to \mathbb{R}$ such that*

$$f(x) = g_B(x_{i_B}) \qquad (x = (x_1,\ldots,x_n) \in B).$$

As an easy corollary of Proposition 3.1 we obtain the characterization of invariant functions stated in Proposition 2.2, *iv*); see also [2]. Indeed, for a fixed $B \in \mathcal{B}(E^n)$, we should have $f(x) = g(x_i)$ and hence, for all $\phi \in \Phi(E)$ with fixed point x_i, we have

$$\phi(g(x_i)) = \phi(f(x)) = f(\phi(x)) = g(x_i),$$

which implies that $g(x_i)$ is a fixed point of all such ϕ's, that is,

$$g(x_i) = x_i \quad \text{or} \quad e_0 \quad \text{or} \quad e_1.$$

As we have already observed, the structure of invariant functions on a given minimal invariant subset is completely independent of their structure on any other minimal invariant subset. This fact is due to the invariance property: $\phi(x) \in B$ for all $x \in B$, $\phi \in \Phi(E)$ and $B \in \mathcal{B}(E^n)$. However, in the case of comparison meaningful functions we are faced a quite different situation, in which we should take into account all minimal invariant subsets.

Observe first that for a given comparison meaningful function $f : E^n \to \mathbb{R}$ and a given $B \in \mathcal{B}(E^n)$, the corresponding index i_B need not be determined univocally. This happens for instance when g_B is constant or when B is defined with equalities on coordinates (see Proposition 2.1). On the other hand, given i_B, the function g_B is necessarily unique.

Now, we are ready to give a complete description of all comparison meaningful functions.

Theorem 3.1. *The function $f : E^n \to \mathbb{R}$ is comparison meaningful if and only if, for any $B \in \mathcal{B}(E^n)$, there exist an index $i_B \in [n]$ and a strictly monotone or constant mapping $g_B : P_{i_B}(B) \to \mathbb{R}$ such that*

$$f(x) = g_B(x_{i_B}) \qquad (x \in B), \tag{2}$$

where, for any $B, C \in \mathcal{B}(E^n)$, either $g_B = g_C$, or $\mathrm{Ran}(g_B) = \mathrm{Ran}(g_C)$ is singleton, or $\mathrm{Ran}(g_B) < \mathrm{Ran}(g_C)$, or $\mathrm{Ran}(g_B) > \mathrm{Ran}(g_C)$. (Note that $\mathrm{Ran}(g_B) < \mathrm{Ran}(g_C)$ means that for all $r \in \mathrm{Ran}(g_B)$ and all $s \in \mathrm{Ran}(g_C)$, we have $r < s$.)

Example 3.1. Put $E = [0,1]$ and $n = 2$. Then there are eleven minimal invariant subsets in $\mathcal{B}([0,1]^2)$, namely $B_1 = \{(0,0)\}$, $B_2 = \{(1,0)\}$, $B_3 = \{(1,1)\}$, $B_4 = \{(0,1)\}$, $B_5 =]0,1[\times\{0\}$, $B_6 = \{1\}\times]0,1[$, $B_7 =]0,1[\times\{1\}$, $B_8 = \{0\}\times]0,1[$, $B_9 = \{(x_1,x_2) \mid 0 < x_1 = x_2 < 1\}$, $B_{10} = \{(x_1,x_2) \mid 0 < x_1 < x_2 < 1\}$, $B_{11} = \{(x_1,x_2) \mid 0 < x_2 < x_1 < 1\}$. Let $i_{B_j} = 1$ and $g_{B_j}(x) = 1 - x$ for $j \in \{1,2,3,5,6,9,11\}$, and $i_{B_j} = 2$ and $g_{B_j}(x) = 2x - 3$ for $j \in \{4,7,8,10\}$, where always $x \in P_{i_{B_j}}(B_j)$. Then the relevant comparison meaningful function $f : [0,1]^2 \to [0,1]$ is given by

$$f(x_1,x_2) = \begin{cases} 1 - x_1, & \text{if } x_1 \geq x_2, \\ 2x_2 - 3, & \text{if } x_1 < x_2. \end{cases}$$

Theorem 3.1 enables us to characterize strong comparison meaningful functions, too. Observe that while in the case of comparison meaningful functions, for any point $x \in E^n$ the set of all $\phi(x) = (\phi(x_1),\ldots,\phi(x_n))$, with $\phi \in \Phi(E)$, gives some minimal invariant set B, in the case of strong comparison meaningful functions we are faced to the set of all points $(\phi_1(x_1),\ldots,\phi_n(x_n))$, with $\phi_1,\ldots,\phi_n \in \Phi(E)$, which is exactly the invariant set B^* linked to the previous B, which together with Theorem 3.1 results in the next corollary.

Corollary 3.1. *The function* $f : E^n \to \mathbb{R}$ *is strongly comparison meaningful if and only if, for any* $B^* \in \mathcal{B}^*(E^n)$*, there exist an index* $i_{B^*} \in [n]$ *and a strictly monotone or constant mapping* $g_{B^*} : P_{i_{B^*}}(B^*) \to \mathbb{R}$ *such that*

$$f(x) = g_{B^*}(x_{i_{B^*}}) \qquad (x \in B^*),$$

where, for any $B^*, C^* \in \mathcal{B}^*(E^n)$*, either* $g_{B^*} = g_{C^*}$*, or* $\mathrm{Ran}(g_{B^*}) = \mathrm{Ran}(g_{C^*})$ *is singleton, or* $\mathrm{Ran}(g_{B^*}) < \mathrm{Ran}(g_{C^*})$*, or* $\mathrm{Ran}(g_{B^*}) > \mathrm{Ran}(g_{C^*})$.

4 MONOTONE COMPARISON MEANINGFUL FUNCTIONS

In this section we will examine monotone comparison meaningful functions. Note that the monotonicity of a fusion function is a rather natural property.

Now, for any strictly monotone or constant real function $h : \mathbb{R} \to \mathbb{R}$, and any comparison meaningful function $f : E^n \to \mathbb{R}$, also the composite $h \circ f : E^n \to \mathbb{R}$ is comparison meaningful. Consequently, to get a complete description of monotone comparison meaningful functions it is enough to examine nondecreasing comparison meaningful functions only.

Theorem 4.1. *Let* $f : E^n \to \mathbb{R}$ *be a nondecreasing function. Then* f *is comparison meaningful if and only if it has the representation*

$$\{(i_B, g_B) \mid B \in \mathcal{B}(E^n)\},$$

as stated in Theorem 3.1, such that any g_B *is either constant or strictly increasing,* $\mathrm{Ran}(g_B) = \mathrm{Ran}(g_C)$ *if* $B \sim C$*, and* $\mathrm{Ran}(g_B) \not> \mathrm{Ran}(g_C)$ *if* $B \nsim C$ *and* $B \preceq C$.

Now, several results mentioned in Section 2 are immediate corollaries of Theorems 3.1 and 4.1. Interesting seems to be also the next result, in which $\mathcal{G}(E)$ means the system of all strictly increasing or constant real functions g defined either on E° or on singleton $\{e_0\} \cap E$ or on $\{e_1\} \cap E$ (if these singletons exist) and for $g_1, g_2 \in \mathcal{G}(E)$ we put $g_1 \preceq g_2$ if either $g_1 = g_2$, or $\mathrm{Ran}(g_1) = \mathrm{Ran}(g_2)$ is a singleton, or $\mathrm{Ran}(g_1) < \mathrm{Ran}(g_2)$.

Corollary 4.1. *A nondecreasing function* $f : E^n \to \mathbb{R}$ *is comparison meaningful if and only if there are nondecreasing mappings* $\xi : \mathcal{B}^*(E^n) \to \mathcal{M}_n(E)$ *and* $\gamma : \mathcal{B}^*(E^n) \to \mathcal{G}(E)$ *so that*

$$f(x) = \gamma(B^*)(L_{\xi(B^*)}(x)) \qquad (x \in B^* \in \mathcal{B}^*(E^n)). \tag{3}$$

Observe also that whenever B^* is not singleton then the relevant function $\gamma(B^*)$ from the representation (3) can be obtained (for all $z \in E^\circ$) by

$$\gamma(B^*)(z) = f(z_1, \ldots, z_n),$$

where

$$z_i = \begin{cases} e_0, & \text{if } P_i(B^*) = \{e_0\}, \\ e_1, & \text{if } P_i(B^*) = \{e_1\}, \\ z, & \text{otherwise.} \end{cases}$$

For example, if E is open, then $\mathcal{B}^*(E^n) = \{E^n\}$ and then necessarily each monotone comparison meaningful $f : E^n \to \mathbb{R}$ is given by $f = g \circ L_\mu$, where $\mu \in \mathcal{M}_n(E)$ and $g(z) = f(z, \ldots, z)$ is strictly monotone or constant (see also [7]).

Based on Corollaries 3.1 and 4.1, we can characterize nondecreasing strong comparison meaningful functions as follows:

Corollary 4.2. *A nondecreasing function* $f : E^n \to \mathbb{R}$ *is strongly comparison meaningful if and only if there is a mapping* $\delta : \mathcal{B}^*(E^n) \to [n]$ *and a nondecreasing mapping* $\gamma : \mathcal{B}^*(E^n) \to \mathcal{G}(E)$ *such that*

$$f(x) = \gamma(B^*)(x_{\delta(B^*)}) \qquad (x \in B^* \in \mathcal{B}^*(E^n)),$$

where, if $\gamma(B^*) = \gamma(C^*)$*, then also* $\delta(B^*) = \delta(C^*)$ *(unless* $\gamma(B^*) = \gamma(C^*)$ *is constant).*

Continuity of a comparison meaningful function is even more restrictive and it forces the monotonicity. From Theorem 3.1 we have the next result (see also [7]).

Corollary 4.3. *A continuous function* $f : E^n \to \mathbb{R}$ *is comparison meaningful if and only if there is a continuous, strictly monotone or constant mapping* $g : E \to \mathbb{R}$ *and a function* $\mu \in \mathcal{M}_n(E)$ *such that*

$$f = g \circ L_\mu. \tag{4}$$

Note that in trivial cases when f is constant, f admits also representations different from (4), however, always in the form $f = g \circ f^*$, where g is a constant function on E and $f^* : E^n \to E$ is an arbitrary function. In all other cases the representation (4) is unique.

Corollary 4.4. *A continuous function* $f : E^n \to \mathbb{R}$ *is strongly comparison meaningful if and only if there is a continuous, strictly monotone or constant mapping* $g : E \to \mathbb{R}$ *and an index* $i \in [n]$ *so that*

$$f = g \circ P_i.$$

5 CONCLUSIONS

We have described the structure of a general comparison meaningful function. As corollaries, some results concerning special cases (monotone and/or continuous operators) were characterized. Moreover, our characterization can be understood also as a hint how to construct comparison meaningful operators.

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