Aggregation on finite ordinal scales by scale independent functions

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What is a finite ordinal scale ?

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Two equivalent approaches :

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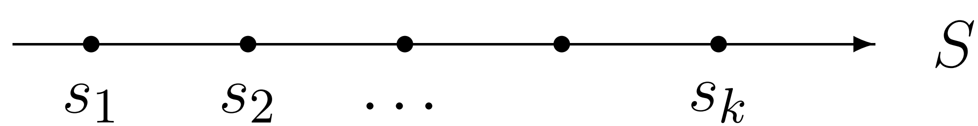

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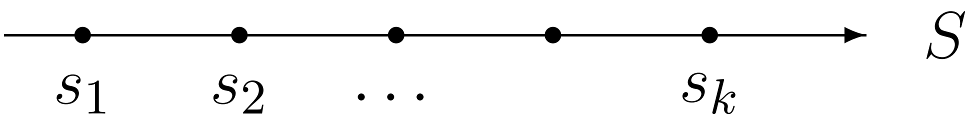

Example: evaluation of a product by a consumer

$$S = \{B \prec RB \prec A \prec MLG \prec G\}$$

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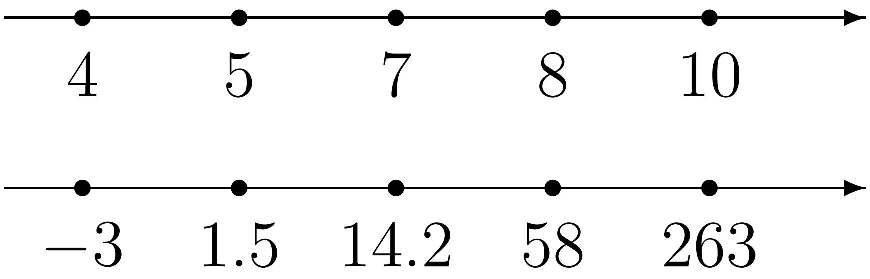

rating benchmarks ↗

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$\exists$ isomorphism $f : S \rightarrow E \subseteq \mathbb{R}$ such that

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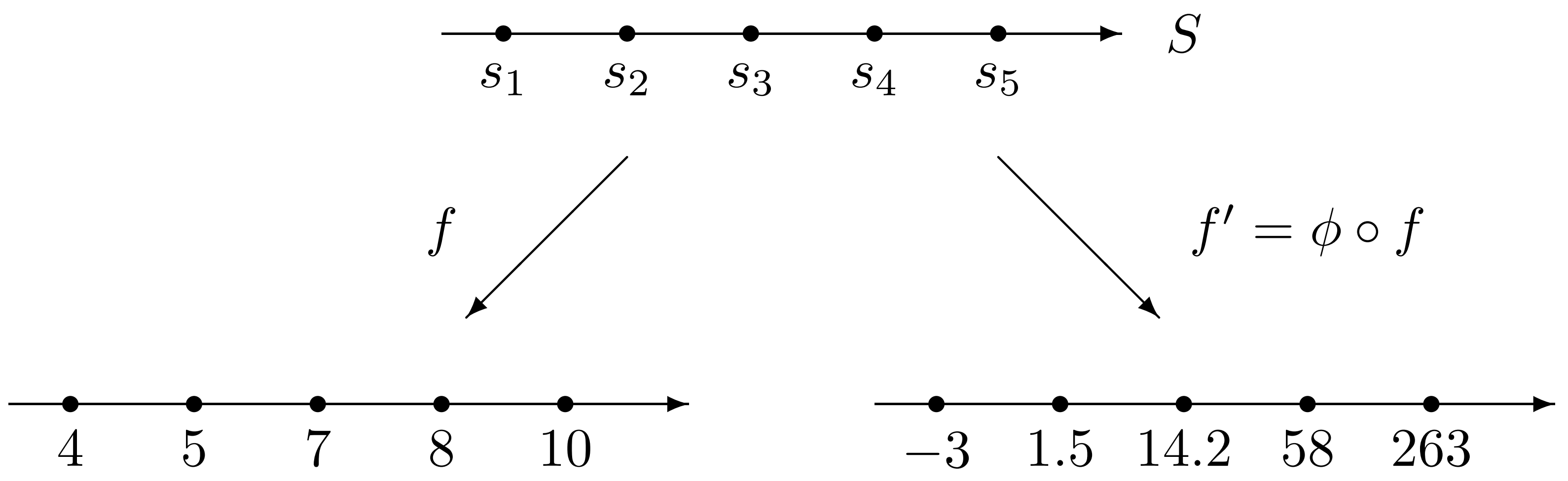

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Consider two independent ordinal scales $(S, \preccurlyeq_S)$ and $(T, \preccurlyeq_T)$

- If $E =]0,1[$, those scales have nothing in common
- If $E = [0,1]$, those scales have fixed endpoints

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Example: $n = 2$ and $|S| = 3$

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There are $|S|^{|S|^n} = 19\,683$ such functions G !

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Alternative approach

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Scale independent function $F : E^n \longrightarrow E$

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Example: $F =$ median

$$\text{median}(0.1, 0.3, 0.6) = 0.3$$

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$A(E) =$ automorphism group of E

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Definition (Marichal and Roubens 1993)
$F : E^n \to E$ is an *invariant* function if, for any $\phi \in A(E)$, we have

$$F[\phi(x_1), \ldots, \phi(x_n)] = \phi[F(x_1, \ldots, x_n)]$$

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median : yes
arithmetic mean : no !

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A subset $I \subseteq E^n$ is *invariant* if

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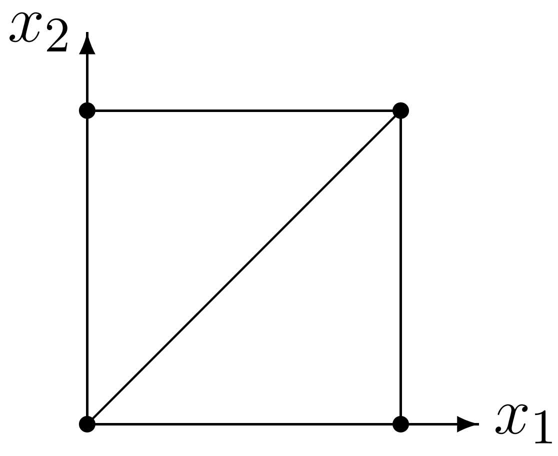

11 minimal invariant subsets !

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- *or* $\exists\, i \in \{1, \ldots, n\}$ *such that* $F|_I = P_i$ *(coord. proj.)*

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$$F : E^n \to E \textit{ is invariant}$$

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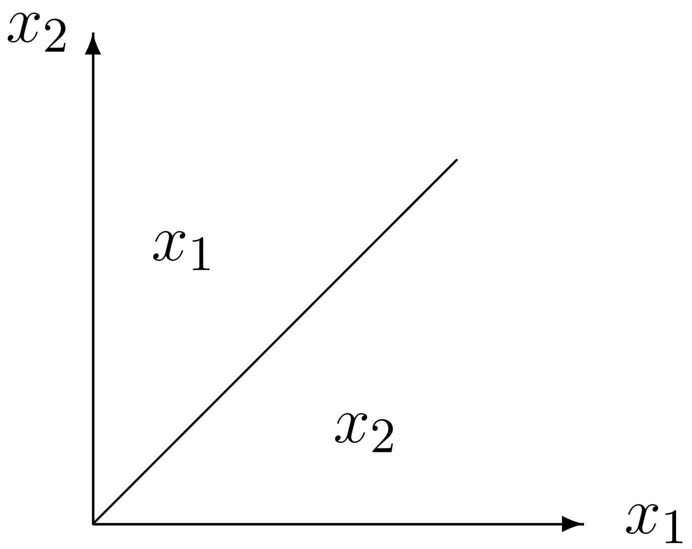

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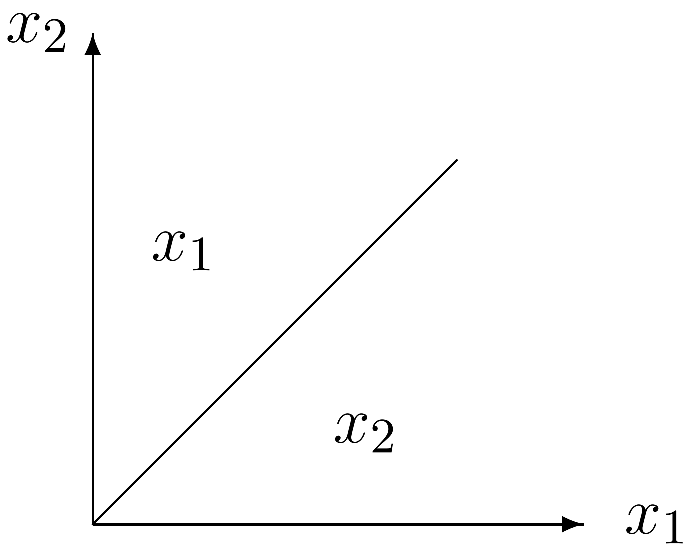

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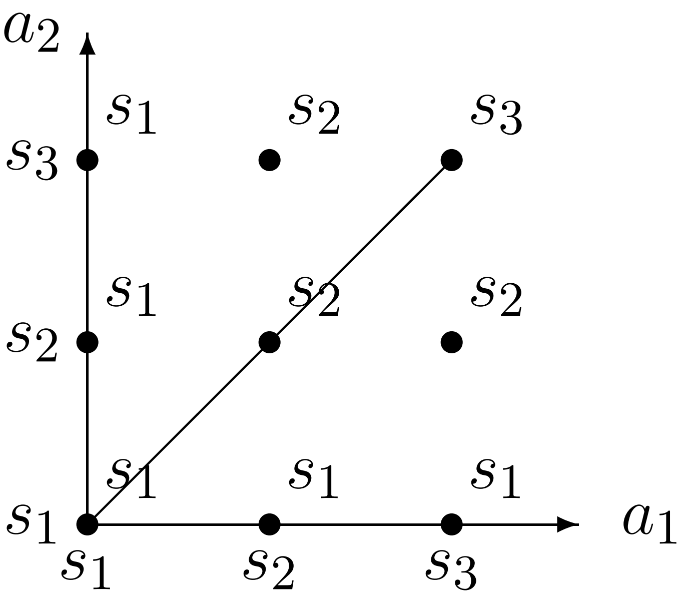

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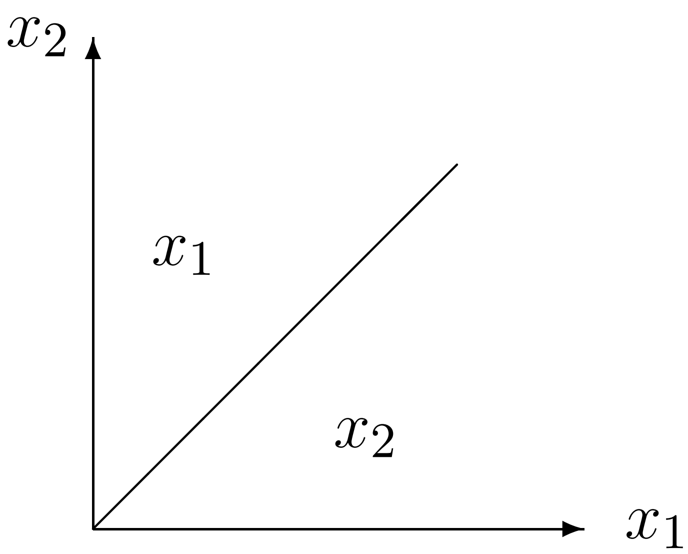

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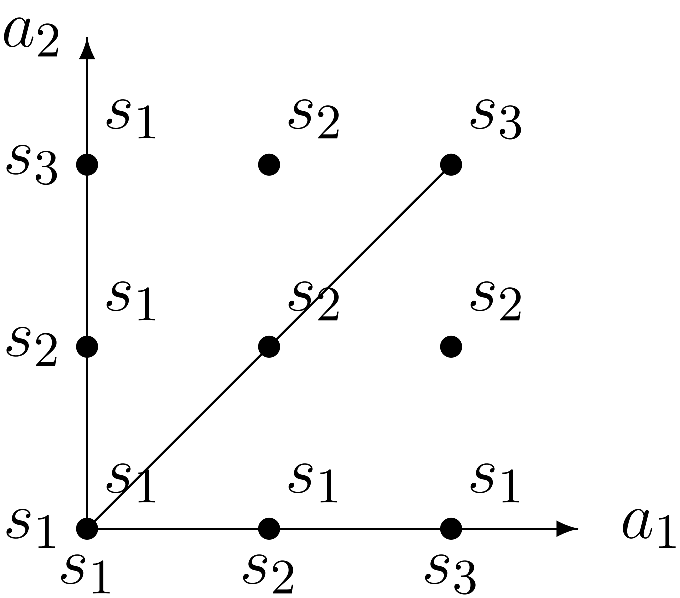

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F and G are always isomorphic !

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Note: If F invariant then

$$\boxed{F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\} \cup \{\inf E, \sup E\}}$$

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$$\left.\begin{array}{l}\text{continuity}\\ \text{order invariance}\end{array}\right\}\ \text{meaning ??}$$

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Not smooth !

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Proposition
An invariant function is continuous iff it is represented only by smooth discrete functions

A more general case

Two finite ordinal scales $(S, \preccurlyeq_S)$ and $(T, \preccurlyeq_T)$

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Example: $n = 2$, $|S| = 3$, and $|T| = 5$

$a_2 \backslash a_1$	s_1	s_2	s_3
s_1	t_4	t_1	t_2
s_2	t_4	t_3	t_5
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$F : E^n \to \mathbb{R}$ is *weakly invariant* if, for any $\phi \in A(E)$, there is a strictly increasing $\psi_\phi : \mathbb{R} \to \mathbb{R}$ such that

$$F[\phi(x_1), \ldots, \phi(x_n)] = \psi_\phi[F(x_1, \ldots, x_n)]$$

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\begin{aligned}
i)\quad & i \in \{1, \ldots, n\} \\
ii)\quad & g : \mathbb{R} \to \mathbb{R}, \textit{ constant or strictly monotonic}
\end{aligned}
$$

Description of weakly invariant functions

Proposition
$F : E^n \to \mathbb{R}$ is weakly invariant if and only if, for any minimal invariant subset $I \subseteq E^n$, there exist

$$\begin{aligned} i)\quad & i \in \{1, \ldots, n\} \\ ii)\quad & g : \mathbb{R} \to \mathbb{R}, \text{ constant or strictly monotonic} \end{aligned}$$

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(+ some extra conditions)

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G represents F in $(S, \preccurlyeq)$

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G represents F in $(S, \preccurlyeq)$
G and g_f are uniquely determined

Continuous weakly invariant functions

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Proposition (Yanovskaya 1989, Marichal 2002)
$F : E^n \to \mathbb{R}$ *is continuous and weakly invariant if and only if there exist*

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$i)$ $L : E^n \to E$*, lattice polynomial*

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or continuous and strictly monotonic

such that

$$F = g \circ L$$

Continuous weakly invariant functions

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A continuous weakly invariant function is represented only by smooth discrete functions

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The converse is false !!!