Aggregation on finite ordinal scales by scale independent functions

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Abstract

We give an interpretation of order invariant functions as scale independent functions for the aggregation on finite ordinal scales. More precisely, we show how order invariant functions can act, through discrete representatives, on ordinal scales represented by finite chains. In particular, this interpretation allows us to justify the continuity property for certain order invariant functions in a natural way.

Keywords: Aggregation functions; Finite ordinal scales; Order invariant functions; Smooth discrete functions; Lattice polynomials.

1 Introduction

A finite ordinal scale can be defined in two equivalent ways; one is symbolical and the other is numerical. Symbolically, a finite ordinal scale is a finite chain $(S, \preccurlyeq)$, that is a totally ordered finite set, whose elements are ranked according to some criterion. For example [6, 7], a scale of evaluation of a commodity by a consumer such as

$$S = \{B \prec RB \prec A \prec MLG \prec G\}$$

is a finite ordinal scale, whose elements B, RB, A, MLG, G might refer to the following linguistic terms: *bad*, *rather bad*, *acceptable*, *more or less good*, *good*. Numerically, a finite ordinal scale is a finite and strictly increasing sequence of real numbers defined up to order and representing the possible rating benchmarks defined along some criterion; see e.g. [18]. For example, the sequences

$$(1, 2, 3, 4, 5) \text{ and } (-6.5, -1.2, 8.7, 205.6, 750)$$

represent two equivalent versions of the scale defined above.

The equivalence between these two definitions follows immediately from the fact that the order defined on any finite chain $(S, \preccurlyeq)$ can always be numerically represented in a real interval $E \subseteq \mathbb{R}$ by means of an order-preserving utility function $f : S \to E$, which is defined up to an increasing bijection $\phi : E \to E$; see e.g. [9].

Now, suppose that we have n evaluations expressed in the same ordinal scale $(S, \preccurlyeq)$ of cardinality $k = |S|$ and suppose we want to aggregate these evaluations and obtain a representative overall evaluation in the same scale. Of course, we can use a discrete aggregation function $G : S^n \to S$, that is, a ranking function sorting k^n n-tuples into k classes. Alternatively, we can use a universal n-place aggregation function *independent* of the ordinal scale used. In this latter case, since no scale can be specified, the aggregation function must be a numerical function $M : E^n \to E$. For instance, the classical median function, which gives the middle value of an odd-length sequence of ordered values, is a scale independent function able to aggregate numerical values expressed on any ordinal scale.

In this paper we investigate two types of scale independent functions for the aggregation on finite ordinal scales. First, we consider the functions mapping n copies of the same ordinal scale into itself (see Definition 4.1). Then, we consider the functions mapping n copies of the same ordinal scale into an ordinal scale (see Definition 4.3). It appears that these functions, known in the literature as *order invariant functions*, have already been investigated and described in a pure numerical setting [12] (see also [10, 14]). Our contribution here is to interpret them as scale independent functions, that is, numerical functions that always have symbolical representatives when acting upon specified ordinal scales.

We also show that, even though at first glance it seems unappropriate to ask any order invariant function to be continuous, the continuity property can be interpreted in a very natural way for those order invariant functions of the first type.

The organization of the paper is as follows. In §2 we introduce the notation and the assumptions that we adopt in this work. In §3 we recall the concept of invariant subsets, which is necessary to describe the scale independent functions. In §4, we present separately the two types of scale independent functions mentioned above. Finally, in §5 we investigate the continuity property for those functions.

2 Preliminaries and notation

Let E be any real interval, bounded or not, and let $e_0 := \inf E$, $e_1 := \sup E$, and $E^\circ := E \setminus \{e_0, e_1\}$. We denote by $B(E)$ the set of included boundaries of E, that is

$$B(E) := \{e_0, e_1\} \cap E.$$

The automorphism group of E, that is the group of all increasing bijections ϕ of E onto itself, is denoted by $A(E)$. For the sake of simplicity, we also denote the index set $\{1, \ldots, n\}$ by $[n]$ and the minimum and maximum operations by $\wedge$ and $\vee$, respectively.

For any $k \geqslant 2$, a k-point ordinal scale $(S, \preccurlyeq)$ will be denoted by

$$S = \{s_1 \prec s_2 \prec \cdots \prec s_k\}$$

where $s_1 = s_*$ (resp. $s_k = s^*$) is the bottom element (resp. top element) of the scale and $\prec$ represents the asymmetric part of $\preccurlyeq$.

Since the binary relation $\preccurlyeq$ is a total order on a finite set S, it can always be numerically represented by a strictly increasing utility function $f : S \to E$ such that

$$s_i \left\{ \begin{matrix} \prec \\ = \end{matrix} \right\} s_j \quad \Leftrightarrow \quad f(s_i) \left\{ \begin{matrix} < \\ = \end{matrix} \right\} f(s_j),$$

see [9, Chapter 1]. Such a utility function is defined up to an automorphism $\phi \in A(E)$; that is, with f all functions $f' = \phi \circ f$ (and only these) represent the same order on S. Thus, $A(E)$ represents the set of all *admissible scale transformations*, i.e., transformations of E that lead from one numerical scale to an equivalent one; see e.g. [18].

Throughout, we will assume that f is *endpoint-preserving*, that is, if $e_0 \in E$ (resp. $e_1 \in E$) then $f(s_*) = e_0$ (resp. $f(s^*) = e_1$) for all ordinal scale $(S, \preccurlyeq)$. This amounts to assuming that the ordinal scales all have a common bottom element s_* (resp. a common top element s^*) whose numerical representation is e_0 (resp. e_1). This assumption is why we consider numerical representations in a subset E of $\mathbb{R}$, possibly non-open, rather than $\mathbb{R}$ itself. For example, if $E = [0, 1]$, all the ordinal scales we can consider have fixed endpoints.

To avoid a heavy notation, we will write $\phi(x)$ and $f(a)$ instead of

$$(\phi(x_1), \ldots, \phi(x_n)) \text{ and } (f(a_1), \ldots, f(a_n)),$$

respectively.

Finally, the range of any function f will be denoted by $\operatorname{ran}(f)$.

3 Background on invariant subsets

In this section we recall the concept of invariant subset, which will be useful throughout this paper. For theoretical developments, see e.g. [1, 12, 14].

Definition 3.1. A nonempty subset $I \subseteq E^n$ is said to be invariant if

$$x \in I \;\Rightarrow\; \phi(x) \in I \qquad (\phi \in A(E)).$$

An invariant set I is said to be minimal if it has no proper invariant subset.

The family $\mathcal{I}(E^n)$ of all minimal invariant subsets of E^n provides a partition of E^n into equivalence classes, where $x, y \in E^n$ are equivalent if there exists $\phi \in A(E)$ such that $y = \phi(x)$. A complete description of elements of $\mathcal{I}(E^n)$ is given in the following proposition [14]:

Proposition 3.1. *We have $I \in \mathcal{I}(E^n)$ if and only if there exists a permutation π on $[n]$ and a sequence $\{\lhd_i\}_{i=0}^n$ of symbols $\lhd_i \in \{<, =\}$, not all equality if $e_0 \in E$ and $e_1 \in E$, such that*

$$I = \{x \in E^n | e_0 \lhd_0 x_{\pi(1)} \lhd_1 \cdots \lhd_{n-1} x_{\pi(n)} \lhd_n e_1\},$$

where $\lhd_0$ is $<$ if $e_0 \notin E$ and $\lhd_n$ is $<$ if $e_1 \notin E$.

Example 3.1. The unit square $[0,1]^2$ contains exactly eleven minimal invariant subsets, namely the open triangles $\{(x_1, x_2) \mid 0 < x_1 < x_2 < 1\}$ and $\{(x_1, x_2) \mid 0 < x_2 < x_1 < 1\}$, the open diagonal $\{(x_1, x_2) \mid 0 < x_1 = x_2 < 1\}$, the four square vertices, and the four open line segments joining neighboring vertices.

4 Scale independent functions

In the present section we investigate the two kinds of scale independent functions we have mentioned in the introduction. Actually, we will see that these functions are nothing else than the so-called *order invariant functions*, namely: invariant functions and comparison meaningful functions.

4.1 Uniscale independent functions

The first scale independent functions we investigate are n-place numerical aggregation functions whose input and output values are expressed in the same ordinal scale. We call them *uniscale independent functions*.

Definition 4.1. A function $M : E^n \to E$ is said to be uniscale independent if, for any finite ordinal scale $(S, \preccurlyeq)$, there exists an aggregation function $G : S^n \to S$ such that, for any endpoint-preserving numerical representation $f : S \to E$ of $\preccurlyeq$, we have

$$M[f(a)] = f[G(a)] \qquad (a \in S^n). \qquad (1)$$

We then say that G represents M in $(S, \preccurlyeq)$.

As any admissible scale transformation of the input values must lead to the same transformation of the output values, it seems that the uniscale independent functions are *invariant functions* in the following sense:

Definition 4.2. $M : E^n \to E$ is said to be an invariant function if

$$M[\phi(x)] = \phi[M(x)]$$

for all $x \in E^n$ and all $\phi \in A(E)$.

The invariant functions have been investigated extensively by several authors; see e.g. [10, 13, 14, 17]. Moreover, the full description of those functions has been given very recently as follows [12, 14]:

Theorem 4.1. *$M : E^n \to E$ is an invariant function if and only if, for any $I \in \mathcal{I}(E^n)$ either $M|_I \equiv c \in B(E)$ (if this constant exists) or there exists $i \in [n]$ such that $M|_I = P_i|_I$ is the ith coordinate projection.*

Thus, an invariant function $M : E^n \to E$ reduces to a constant or a coordinate projection on every minimal invariant subset of E^n. In particular, we have

$$M(x) \in \{x_1, \ldots, x_n\} \cup B(E) \qquad (x \in E^n).$$

We now have the following result:

Proposition 4.1. *The function $M : E^n \to E$ is uniscale independent if and only if it is invariant.*

According to Proposition 4.1, an invariant function $M : E^n \to E$ can always be represented by a discrete aggregation function $G : S^n \to S$ on any ordinal scale $(S, \preccurlyeq)$, regardless of the cardinality of this scale. Moreover, it is clear from Eq. (1) that G is uniquely determined and isomorphic to the "restriction" of M to S^n.

Example 4.1. Let $n = 2$ and let $M(x) = x_1 \wedge x_2$. Then, the unique representative G of M is defined by $G(a) = a_1 \wedge a_2$ for all $a \in S^2$.

In fact, for a given ordinal scale $(S, \preccurlyeq)$, the set of functions $G : S^n \to S$ representing invariant functions in $(S, \preccurlyeq)$ is described exactly as the discrete version of Theorem 4.1, where E is replaced with S and the family of "discrete" minimal invariant subsets of S^n is simply defined either as

$$\{f^{-1}(I) \mid I \in \mathcal{I}(E^n)\},$$

for any fixed f, or independently of any f, by means of Proposition 3.1. Clearly, to have a one-to-one correspondence between M and G we need that $f^{-1}(I) \neq \varnothing$ for all $I \in \mathcal{I}(E^n)$, a condition that holds if and only if

$$|S| \geqslant n + |B(E)|.$$

In this case, given $I \in \mathcal{I}(E^n)$ and $i \in [n]$, we have $M|_I = P_i|_I$ (resp. $M|_I \equiv e_0$, $M|_I \equiv e_1$) if and only if $G(a) = a_i$ (resp. $G(a) = s_*$, $G(a) = s^*$) for all $a \in f^{-1}(I)$, f being fixed. On the other hand, if $|S| < n + |B(E)|$, several M's may lead to the same G. For example, if $n = 2$, $|S| = 3$, $E = [0, 1]$, and $I \in \mathcal{I}([0, 1]^2)$ is either of the two open triangles, then $f^{-1}(I) = \varnothing$ and, for a given G, the invariant function M can take on any value in I.

4.2 Input-uniscale independent functions

We now investigate scale independent functions whose input values are expressed in the same ordinal scale and the output values in an ordinal scale. We call these functions *input-uniscale independent functions*.

Definition 4.3. A function $M : E^n \to \mathbb{R}$ is said to be input-uniscale independent if, for any finite ordinal scale $(S, \preccurlyeq_S)$, there exists a finite ordinal scale $(T, \preccurlyeq_T)$ and a surjective aggregation function $G : S^n \to T$ such that, for any endpoint-preserving numerical representation $f : S \to E$ of $\preccurlyeq_S$, there is a numerical representation $g_f : T \to \mathbb{R}$ of $\preccurlyeq_T$ such that

$$M[f(a)] = g_f[G(a)] \qquad (a \in S^n). \qquad (2)$$

We then say that G represents M in $(S, \preccurlyeq_S)$.

As we have seen that the uniscale independent functions are exactly the invariant functions, we will see in this subsection that the input-uniscale independent functions are exactly the *comparison meaningful functions*.

Definition 4.4. $M : E^n \to \mathbb{R}$ is said to be a comparison meaningful function (from an ordinal scale) if

$$M(x) \begin{Bmatrix} < \\ = \end{Bmatrix} M(x')$$
$$\Rightarrow \quad M[\phi(x)] \begin{Bmatrix} < \\ = \end{Bmatrix} M[\phi(x')]$$

for any $x, x' \in E^n$ and any $\phi \in A(E)$.

The comparison meaningful functions have been studied by various authors; see e.g. [10, 12, 15, 16, 19]. Moreover, the full description of those functions has been given very recently as follows [12]:

Theorem 4.2. *$M : E^n \to \mathbb{R}$ is a comparison meaningful function if and only if, for any $I \in \mathcal{I}(E^n)$, there exists an index $i_I \in [n]$ and a constant or strictly monotonic function $g_I : P_{i_I}(I) \to \mathbb{R}$ such that*

$$M|_I = g_I \circ P_{i_I}|_I,$$

where, for any $I, J \in \mathcal{I}(E^n)$, either $g_I = g_J$, or $\mathrm{ran}(g_I) = \mathrm{ran}(g_J)$ is a singleton, or $\mathrm{ran}(g_I) < \mathrm{ran}(g_J)$, or $\mathrm{ran}(g_I) > \mathrm{ran}(g_J)$.

Thus, a comparison meaningful function $M : E^n \to \mathbb{R}$ reduces to a constant or a transformed coordinate projection on every minimal invariant subset of E^n.

The following result clearly shows that comparison meaningfulness generalizes invariantness:

Proposition 4.2. *$M : E^n \to \mathbb{R}$ is a comparison meaningful function if and only if, for any $\phi \in A(E)$, there is a strictly increasing mapping $\psi_\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$M[\phi(x)] = \psi_\phi[M(x)] \qquad (x \in E^n). \qquad (3)$$

Note that it is clear from Eq. (3) that the restriction of ψ_ϕ to the range of M is uniquely determined.

We now have the following result:

Proposition 4.3. *The function $M : E^n \to \mathbb{R}$ is input-uniscale independent if and only if it is comparison meaningful.*

According to Proposition 4.3 a comparison meaningful function $M : E^n \to \mathbb{R}$ can always be represented by a discrete aggregation function $G : S^n \to T$ on any ordinal scale $(S, \preccurlyeq_S)$, regardless of the cardinality of this scale. Moreover, the necessary steps to determine the output scale T and the functions $G : S^n \to T$ and $g_f : T \to \mathbb{R}$ are:

Step 1. Fix a particular endpoint-preserving numerical representation $f^* : S \to E$ of $\preccurlyeq_S$.

Step 2. We have $T = \{t_1 \prec \cdots \prec t_{|R|}\}$, where $R := \{M[f^*(a)] \mid a \in S^n\}$.

Step 3. We have $G(a) = \sigma^{-1}(M[f^*(a)])$, where $\sigma : T \to R$ is defined as $\sigma(t_i) = r_i$ for all $1 \leqslant i \leqslant |R|$.

Step 4. Determine the unique function $\psi_\phi : \mathrm{ran}(M) \to \mathrm{ran}(M)$ of Proposition 4.2.

Step 5. We have $g_f = \psi_{f\circ f^{*-1}} \circ \sigma$.

We clearly observe that, given $M : E^n \to \mathbb{R}$ and $(S, \preccurlyeq_S)$, the scale T and the functions $G : S^n \to T$ and $g_f : T \to \mathbb{R}$ are uniquely determined and do not depend upon the choice of f^*.

Example 4.2. Let $M : [0,1]^2 \to \mathbb{R}$ be defined by

$$M(x) = x_1 \wedge x_2 + 2\,\mathrm{sign}(x_2 - x_1).$$

Then, given a 3-point ordinal scale $(S, \preccurlyeq_S)$ and an endpoint-preserving numerical representation $f^* : S \to E$, we have

$$R = \{-2 < z - 2 < 0 < z < 1 < 2 < z + 2\},$$

where $z = f^*(s_2)$. Then we have $|T| = 7$ and the function $G : S^n \to T$ is given by

$$G(a) = \sigma^{-1}[f^*(a_1 \wedge a_2) + 2\,\mathrm{sign}(a_2 - a_1)],$$

or equivalently by the following table

$a_2 \backslash a_1$	s_1	s_2	s_3
s_1	t_3	t_1	t_1
s_2	t_6	t_4	t_2
s_3	t_6	t_7	t_5

Finally, we have

$$\psi_\phi(x) = \begin{cases} \phi(x), & \text{if } x \in [0,1], \\ \phi(x-2)+2, & \text{if } x \in [2,3), \\ \phi(x+2)-2, & \text{if } x \in [-2,-1). \end{cases}$$

and

$$g_f(t_i) = \begin{cases} (f \circ f^{*-1})[\sigma(t_i)], & \text{if } i = 3,4,5, \\ (f \circ f^{*-1})[\sigma(t_i) - 2] + 2, & \text{if } i = 6,7, \\ (f \circ f^{*-1})[\sigma(t_i) + 2] - 2, & \text{if } i = 1,2. \end{cases}$$

Notice that the relationship between M and G is not as clear as in the case of uniscale independent functions. Particularly, reconstructing M from G (or characterizing G arising from the M's) seems a difficult task. We then propose the following interesting problem:

Open Problem 1. Describe all the comparison meaningful functions having the same discrete representative.

Notice also that, from Eq. (2), we immediately have the following result, which will be useful in the next section:

Proposition 4.4. *Let $M : E^n \to \mathbb{R}$ be an input-uniscale independent function, with discrete representative $G : S^n \to T$. Then, for any strictly increasing (resp. strictly decreasing) function $g : \mathrm{ran}(M) \to \mathbb{R}$, the discrete representation of $g \circ M$ is $\eta \circ G : S^n \to T'$, where T' is order isomorphic to T and $\eta : T \to T'$ is defined by $\eta(t_i) = t'_i$ (resp. $\eta(t_i) = t'_{|T|-i+1}$) for all $i = 1, \ldots, |T|$.*

5 Continuous order invariant functions

In this final section we examine the case of continuous order invariant functions, namely: continuous invariant functions and continuous comparison meaningful functions.

Until recently, it was thought that coupling continuity with any order invariance property was somewhat awkward since the classical definition of continuity uses distance between numerical values and hence makes use of the cardinal properties of these values while any order invariance implies that the cardinal properties of the numerical values should not be used.

In fact, as we will now see, continuity makes sense for invariant functions and can even be interpreted in a very natural way. More precisely, we yield an interpretation of continuity for invariant functions by imposing a *smoothness* property on their discrete representatives.

We shall also see that such an interpretation fails to hold for comparison meaningful functions and that continuity is a rather restrictive condition for these functions.

First, let us describe the continuous order invariant functions. A typical example of whose is given by a *lattice polynomial* [2]:

Definition 5.1. An n-place lattice polynomial is any expression involving n variables $x_1, \ldots, x_n$ linked by the lattice operations $\wedge = \min$ and $\vee = \max$ in an arbitrary combination of parentheses.

It can be shown (see e.g. [2, Chapter 2, §5]) that any n-place lattice polynomial in $\mathbb{R}^n$ can be put in the following *disjunctive normal form*:

$$L_\gamma(x) = \bigvee_{\substack{A \subseteq [n] \\ \gamma(A)=1}} \bigwedge_{i \in A} x_i \qquad (x \in \mathbb{R}^n),$$

where $\gamma : 2^{[n]} \to \{0,1\}$ is a nonconstant nondecreasing set function. We will denote by Γ_n the family of those set functions.

The complete description of continuous order invariant functions are given in the following two theorems [10, 12]:

Theorem 5.1. *$M : E^n \to E$ is a continuous invariant function if and only if $M \equiv c \in B(E)$ (if this constant exists) or there exists $\gamma \in \Gamma_n$ such that $M = L_\gamma$.*

Theorem 5.2. *$M : E^n \to \mathbb{R}$ is a continuous comparison meaningful function if and only if there exists $\gamma \in \Gamma_n$ and a continuous strictly monotonic or constant function $g : E \to \mathbb{R}$ such that $M = g \circ L_\gamma$.*

We will now give an interpretation of the continuity property for invariant functions through their discrete representatives. For this purpose we use the concept of *smoothness* [5] for discrete functions.

Let $(S, \preccurlyeq) = \{s_1 \prec \cdots \prec s_k\}$ be a k-point ordinal scale and let $a \in S$. In order to locate a in S we define an index mapping $\mathrm{ind} : S \to \{1, \ldots, k\}$ as

$$\mathrm{ind}(a) = r \;\Leftrightarrow\; a = s_r \qquad (1 \leqslant r \leqslant k).$$

Definition 5.2. A discrete function $G : \times_{i=1}^n S^{(i)} \to T$ is said to be smooth if, for any $a, b \in \times_{i=1}^n S^{(i)}$, we have

$$\sum_{i=1}^n |\mathrm{ind}(a_i) - \mathrm{ind}(b_i)| \leqslant 1 \;\Rightarrow\; |\mathrm{ind}[G(a)] - \mathrm{ind}[G(b)]| \leqslant 1.$$

The smoothness property, which was initially introduced only for nondecreasing discrete functions (see [5]), clearly represents the discrete counterpart of continuity. Moreover, it can be proved (see [4, Theorem 2] for a proof in a particular case) that this property is equivalent to the discrete counterpart of the intermediate value theorem. The result is the following:

Proposition 5.1. *The smoothness property for $G : \times_{i=1}^n S^{(i)} \to T$ is equivalent to the following condition: For any $j \in [n]$ and any $a, b \in \times_{i=1}^n S^{(i)}$ differing only on coordinate j, the element $t \in T$ lies between $G(a)$ and $G(b)$ inclusive if and only if there exists $c \in \times_{i=1}^n S^{(i)}$ differing from a and b only on coordinate j, such that c_j is an element between a_j and b_j inclusive and $t = G(c)$.*

We will now see that any invariant function is continuous if and only if it is represented only by smooth discrete aggregation functions. This makes continuity sensible and even appealing for invariant functions. We will also see that this result does not hold for comparison meaningful functions. More precisely, we will see that continuity is only a sufficient condition for those functions to be represented only by smooth discrete functions.

Proposition 5.2. *An invariant function $M : E^n \to E$ is continuous if and only if it is represented only by smooth discrete aggregation functions.*

Let us now examine the case of continuous comparison meaningful functions. By Proposition 4.4, we observe that any continuous invariant function of the form L_γ and any nonconstant and continuous comparison meaningful function of the form $g \circ L_\gamma$, where g is strictly increasing (resp. strictly decreasing), both lead to the representatives $L_\gamma : S^n \to S$ and $\eta \circ L_\gamma : S^n \to T$, respectively, where T is order isomorphic to S, and $\eta : S \to T$ is the index-preserving (resp. index-reversing) mapping.

Proposition 5.3. *A continuous comparison meaningful function* $M : E^n \to \mathbb{R}$ *is represented only by smooth discrete aggregation functions.*

Back to Example 4.2, we can immediately see from the table describing the function G that this function is not smooth. This is in accordance with the noncontinuity of M.

Notice that, contrary to the case of invariant functions, the converse of Proposition 5.3 is not true. There are noncontinuous comparison meaningful functions having smooth representatives. Indeed, starting from a strictly monotonic (but not necessarily continuous) $g : \mathbb{R} \to \mathbb{R}$, we can always transform a continuous invariant function $M : E^n \to \mathbb{R}$ into the (not necessarily continuous) comparison meaningful function $g \circ M$, which has a similar smooth representative as M (cf. Proposition 4.4). More precisely, for any strictly increasing (resp. strictly decreasing, constant) function $g : \mathbb{R} \to \mathbb{R}$, the unique representative in $(S, \preccurlyeq_S)$ of $M = g \circ L_\gamma$ is the smooth function $G = \eta \circ L_\gamma$, where $\eta : S \to T = \mathrm{ran}(G)$ is index-preserving (resp. index-reversing, constant) and $|T| = |S|$ (resp. $|T| = |S|$, $|T| = 1$).

The following interesting problem naturally arises from this analysis:

Open Problem 2. Describe (or characterize) all the comparison meaningful functions that are represented only by smooth discrete aggregation functions.

6 Concluding remarks

We have shed light on the meaning of invariant functions by interpreting them as scale independent functions, that is, functions that have discrete representatives on any finite ordinal scale.

In particular, this interpretation shows that considering a discrete function $G : S^n \to S$, where $(S, \preccurlyeq)$ is a given ordinal scale, is not equivalent to considering an invariant function $M : E^n \to E$. Indeed, the latter form is much more restrictive since M is independent of any scale. For instance, if $n = 2$ and E is open, we see by Theorem 4.1 that there are only 4 invariant functions (since E^2 has only three minimal invariant subsets and there is only one possibility on the diagonal) while the number of discrete functions $G : S^2 \to S$ is clearly $|S|^{|S|^2}$.

We have also interpreted the comparison meaningful functions in a similar way. In this case, describing all the order invariant functions leading to the same discrete representative remains an interesting open problem.

Finally, we have observed that these interpretations make the continuity property very sensible for invariant functions and, however, rather restrictive for comparison meaningful functions.

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