Value at Risk

a practical introduction

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Outline of the talk

- Some mathematical background
- Value at Risk
- Backtesting
- Stresstesting

Some mathematical background

Random Variables

Dummy Definition:

A probability is a non deterministic [0,1] valued function which is additif on disjoint sets.

Dummy Definition:

A random variable X is a real valued non deterministic function. If its image (the set of all possible values) Ω_X is finite or countable X is called a discrete random variable, otherwise X is called a continuous random variable.

Definition:

The law or distribution of a random variable X is the probabiltity P_X defined for all subset A of Ω_X by

$$P_X(A) = P(\{X \in A\}) = P[X^{-1}(A)].$$

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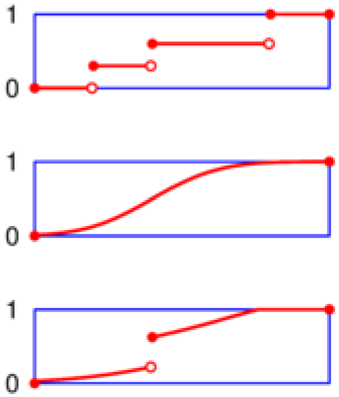
Cumulative density function

Definition:

The cumulative distribution function of a real-valued random variable X is defined for every real number x by



$$F(x) = P(X \leq x).$$

Examples

As an example, suppose X is uniformly distributed on the unit interval [0, 1]. Then the CDF of X is given by

$$F(x) = \begin{cases} 0 & : x < 0 \\ x & : 0 \leq x \leq 1 \\ 1 & : 1 < x \end{cases}$$

Take another example, suppose X takes only the discrete values 0 and 1, with equal probability. Then the CDF of X is given by

$$F(x) = \begin{cases} 0 & : x < 0 \\ 1/2 & : 0 \leq x < 1 \\ 1 & : 1 \leq x \end{cases}$$

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Quantile function

If the cdf F is strictly increasing and continuous then $F^{-1}(y)$ is the unique real number x such that $F(x) = y$.

Unfortunately, the distribution does not, in general, have an inverse. One may define

$$F^{-1}(y) = \inf_{r \in \mathbb{R}} \{F(r) > y\}$$

Example 1: The median is $F^{-1}(0.5)$.

Example 2: Put $\tau = F^{-1}(0,95)$. Then we call τ the 95% quantile.

The inverse of the cdf is called the quantile function.

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Density function

Definition:

A random variable X admits a density function f (with respect to Lebesgue measure) if there exists a measurable function f such that

$$\forall x \in \mathbb{R}, \;\; F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Property:

If X is a continuous random variable with density f and cdf F, then

$$\forall a, b \in \mathbb{R} \;\; a < b \;\Rightarrow\; \int_a^b f(t)\,dt = F(b) - F(a) = P(a < X < b).$$

Expectation

Definition:

If X denotes a discrete random variable taking the values x_i with probabilities p_i its expectation or mean value is the real number E(X) defined by

$$E(X) = \sum_{i \geq 1} x_i \, p_i .$$

Proposition: For any continuous function g, $\quad E \;\, g(X) \;\, = \sum_{i \geq 1} g(x_i)\, p_i .$

Example: Let X be the result of throwing a dice.

$$E(X) = \sum_{i=1}^{6} i \cdot P(X = i) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = 3{,}5.$$

$$E(X^2) = \sum_{i=1}^{6} i^2 \cdot P(X = i) = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + \ldots + 36 \cdot \frac{1}{6} = \frac{91}{6} = 15{,}17.$$

Expectation

Definition:

If X denotes a continuous random variable with density function f its expectation or mean value is the real number E(X) defined by

$$E(X) = \int_{-\infty}^{+\infty} x\, f(x)\, dx.$$

Proposition: For any continuous function g, $E\left[g(X)\right] = \int_{-\infty}^{+\infty} g(x)\, f(x)\, dx.$

Proposition: If X and Y are two random variables and a and b two real numbers,

$$E(aX + bY) = a\,E(X) + b\,E(Y)$$

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Variance

Definition:

If X denotes a random variable its variance is the real number V(X) defined by

$$V(X) = E\left(X - E(X)\right)^2 .$$

The standard deviation σ(X) of X is the square root of its variance.

Proposition: If X and Y are two random variables and a and b two real numbers,

$$V(aX + b) = a^2 V(X)$$

$$V(X+Y) = V(X) + V(Y) + 2\ E(XY) - E(X)E(Y) .$$

Definition:
If X and Y are two random variables the quantity E(XY) - E(X)E(Y) is called the covariance of X and Y and denotes by Cov (X,Y).

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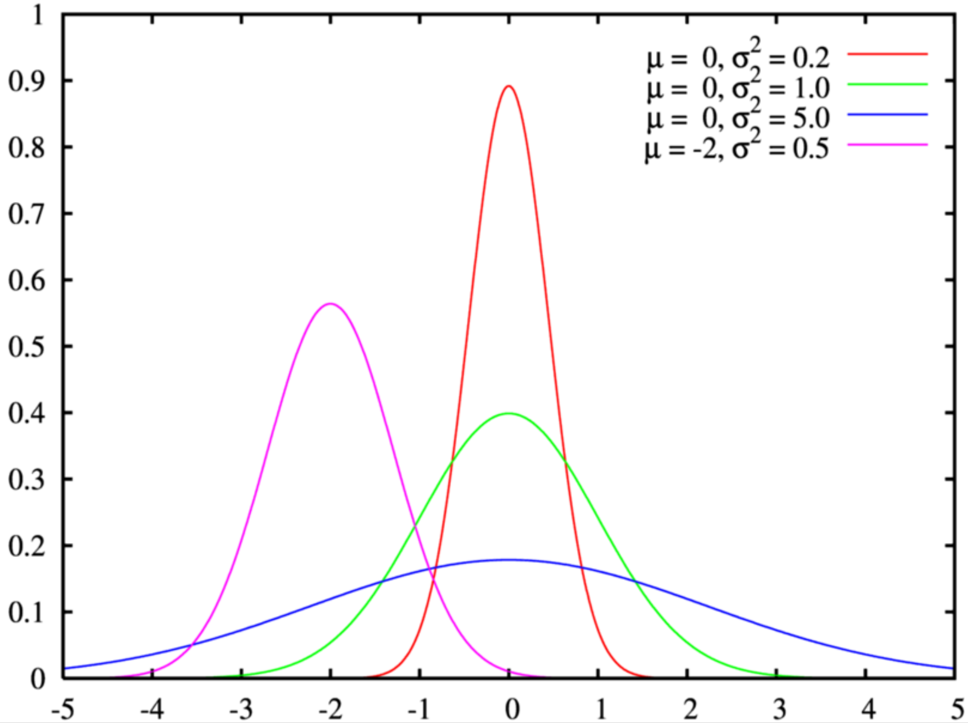
Normal distribution

Definition:

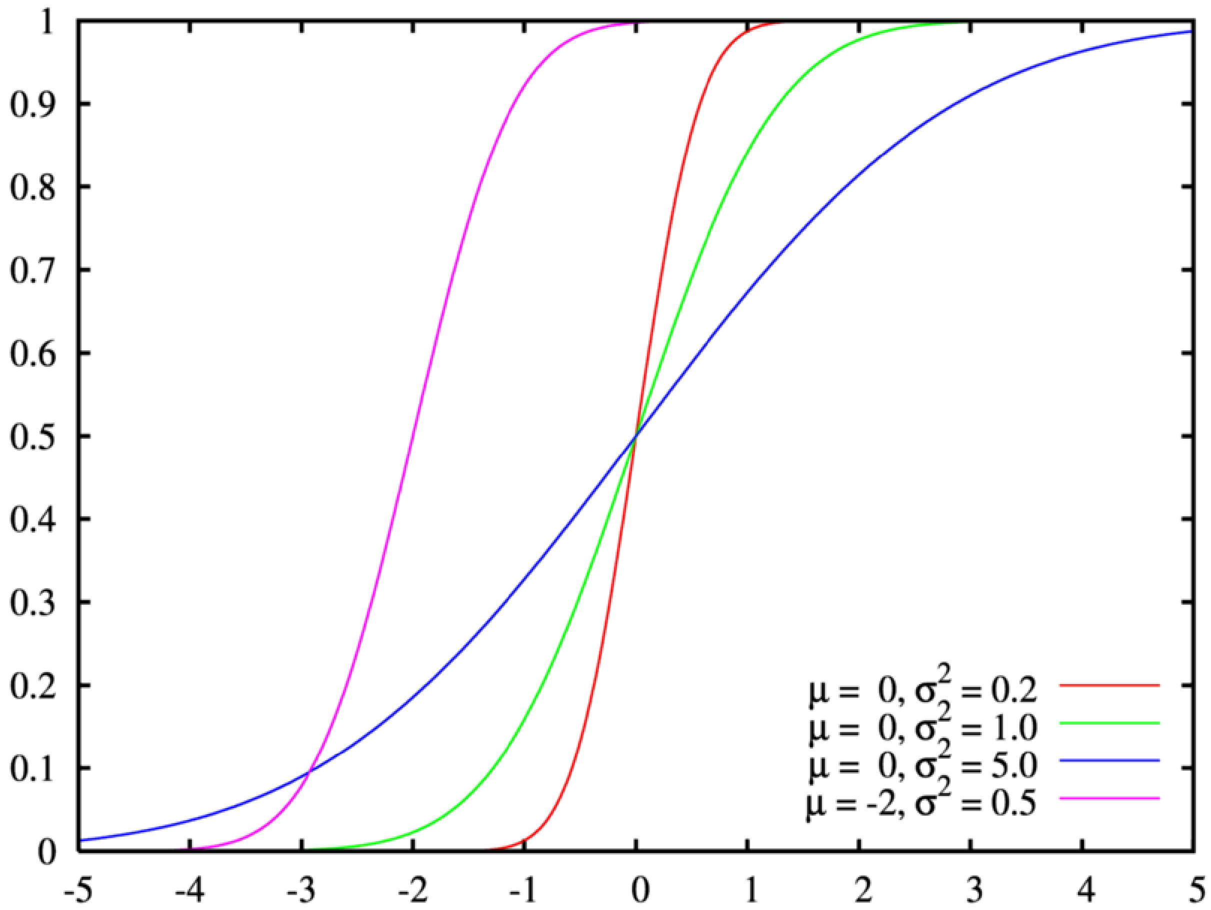
The normal distribution of mean m and standard deviation σ is the continuous random variable X defined by the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{m-x}{\sigma}\right)^2}.$$



Density functions of different normal random variables.

Normal distribution



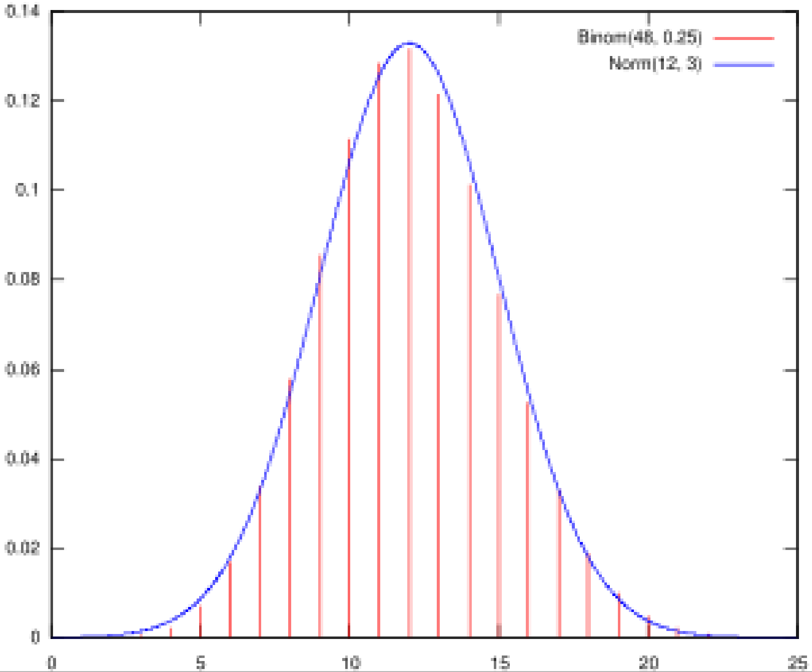
Cumulative distribution functions of different normal random variables.

The central limit theorem

Theorem:

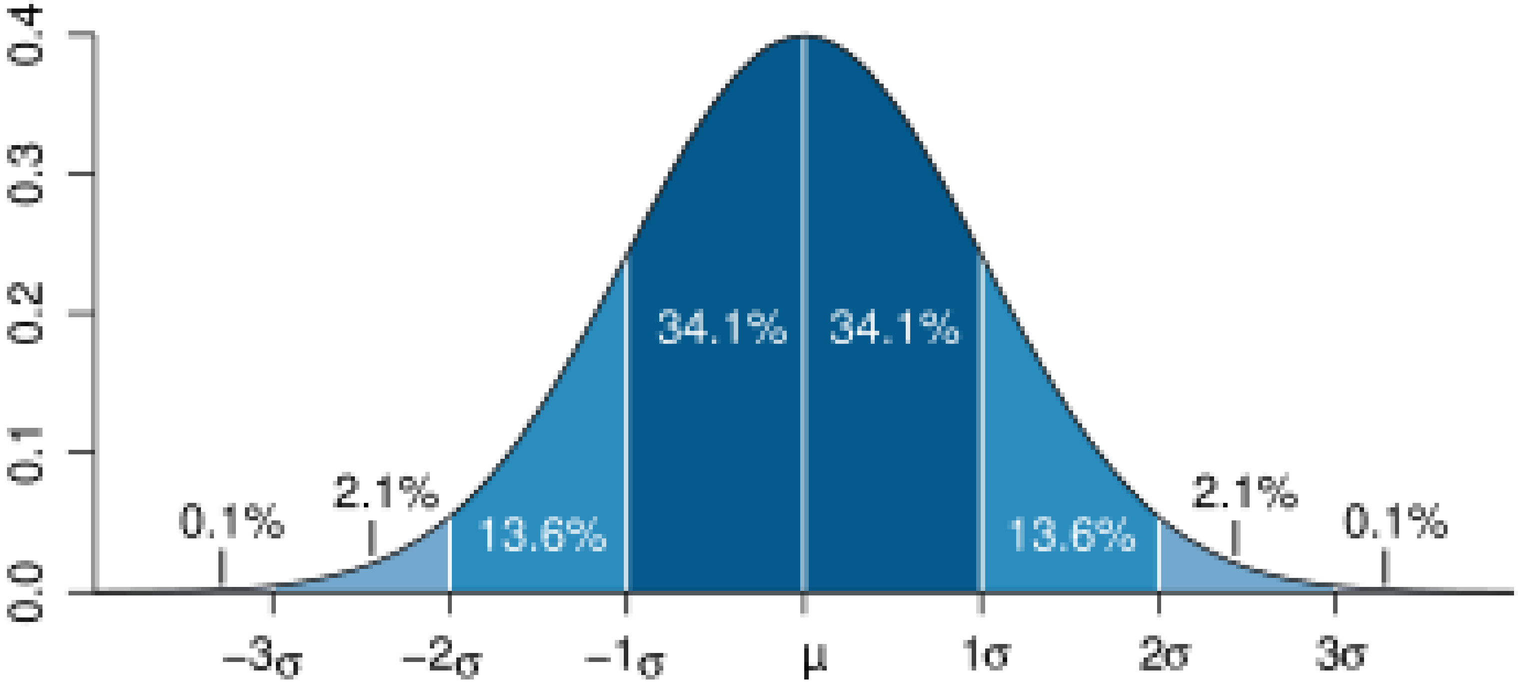
Consider an n-dimensional random vector with independent and identically distributed components X_i of mean m and standard deviation σ. Then,

$$\frac{X_1 + X_2 + \ldots + X_n - nm}{\sigma\sqrt{n}} \xrightarrow{\text{in law}} \mathcal{N}(0,1).$$



Plot of the pdf of a normal distribution with $\mu = 12$ and $\sigma = 3$, approximating the pdf of a binomial distribution with $n = 48$ and $p = 1/4$

Standard deviation and confidence intervals



Estimators

Definition:

An estimator is a function of the observable sample data that is used to estimate an unknown population parameter.

Example:

Let $\{x_1, \ldots, x_n\}$ be a sample of n independent realizations of a random variable X.

The mean of X is estimated by the random variable

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} x_k .$$

The variance of X is estimated by the random variable

$$S^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(x_k - \bar{X}\right)^2 .$$

Monte Carlo Methods

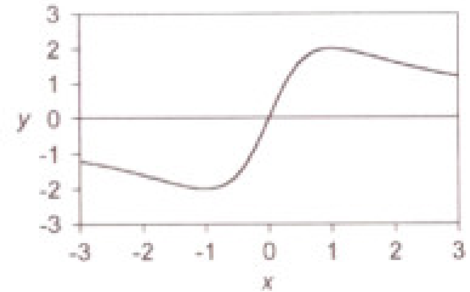
Technique of numerical integration which allows high dimensionality

Simulation method relying on repeated random sampling

Monte Carlo Methods

Example : Approximating a standard deviation

Let X be a standard normal distributed and $f(x) = \dfrac{4x}{x^2+1}$



Consider $Y = f(X)$. What is the variance of Y?

Define a sample

$\{X^{[1]}, X^{[2]}, \ldots, X^{[100]}\}$ for X and construct a realization

$\{x^{[1]}, x^{[2]}, \ldots, x^{[100]}\}$.

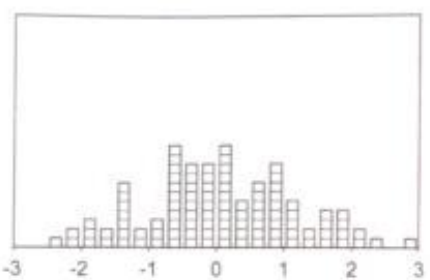


Exhibit 5.5 Histogram of the realization $\{x^{[1]}, x^{[2]}, \ldots, x^{[100]}\}$.

0.23002	−0.46834	0.64139	−0.08692	−0.15508
0.82457	−2.09127	−2.43988	−0.86112	1.68951
−0.24670	0.27326	0.75721	0.80790	−1.29161
1.36672	1.78588	2.36952	−0.27323	−1.41296
0.86917	−0.06726	−0.62629	0.67229	0.83437
−1.34663	−0.61277	0.30748	0.14596	0.84569
2.96975	0.67757	−0.84644	1.50476	−0.06666
−0.52096	1.38464	−0.53059	−2.01150	−0.38679
1.66196	−0.32325	1.16994	−0.89059	0.46550
−0.51023	0.81236	−1.22809	−0.19939	−1.18054
2.21762	−0.50815	−1.52396	−1.99767	0.39806
−0.45419	1.10424	1.05993	−1.31552	−0.04703
−0.52572	0.23828	0.61519	0.24278	0.16820
0.82893	−0.31305	−1.43708	1.87178	1.73266
−0.05739	0.17749	0.24117	−1.49394	−0.73508
−0.38452	1.06686	0.04340	0.14024	−0.67774
−0.41968	1.85422	0.52743	−0.40165	0.58535
0.80286	1.78565	−1.99099	−0.06844	1.21061
−1.82357	−0.66618	0.12124	−0.63079	−1.46235
0.39613	0.59384	0.10556	−1.73320	2.01235

Exhibit 5.4 A realization of a sample $\{X^{[1]}, X^{[2]}, \ldots, X^{[100]}\}$ for X.

Monte Carlo Methods

Example : Approximating a standard deviation

Compute $y^{[k]} = f(x^{[k]})$ for each k to get a

realization $\{y^{[1]}, y^{[2]}, \ldots, y^{[100]}\}$.

Apply sample estimator

$$S^2 = \frac{1}{n}\sum_{k=1}^{n}\left(y^{[k]} - \bar{Y}\right)^2 = \frac{1}{99}\sum_{k=1}^{100} y^{[k]2}.$$

to get an estimation of the variance of Y.

0.87384	−1.53637	1.81776	−0.34509	−0.60574
1.96336	−1.55675	−1.40364	−1.97785	1.75331
−0.93020	1.01709	1.92507	1.95534	−1.93626
1.90622	1.70515	1.43290	−1.01701	−1.88618
1.98050	−0.26782	−1.79938	1.85207	1.96765
−1.91458	−1.78197	1.12369	0.57167	1.97224
1.20974	1.85750	−1.97252	1.84390	−0.26547
−1.63901	1.89857	−1.65613	−1.59449	−1.34581
1.76705	−1.17068	1.97562	−1.98665	1.53039
−1.61935	1.95758	−1.95852	−0.76707	−1.97276
1.49894	−1.61546	−1.83474	−1.60112	1.37446
−1.50608	1.99021	1.99662	−1.92708	−0.18770
−1.64754	0.90191	1.78515	0.91705	0.65427
1.96531	−1.14045	−1.87535	1.66249	1.73175
−0.22883	0.68828	0.91166	−1.84902	−1.90887
−1.33996	1.99582	0.17329	0.55013	−1.85768
−1.42733	1.67117	1.65056	−1.38343	1.74389
1.95274	1.70527	−1.60433	−0.27250	1.96402
−1.68638	−1.84563	0.47793	−1.80497	−1.86377
1.36962	1.75608	0.41758	−1.73148	1.59408

Exhibit 5.6 Values $y^{[k]} = \frac{4x^{[k]}}{x^{[k]2}+1}$.

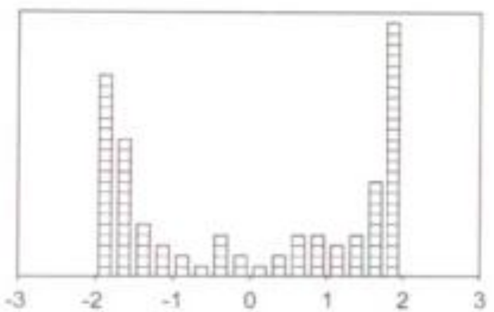


Exhibit 5.7 Histogram of values $y^{[k]}$.

Monte Carlo Methods

Realizations of samples

Conditions :

• the sample mean of $\{x^{[1]}, x^{[2]}, \ldots, x^{[n]}\}$ should be close to the mean of X.

• the sample covariance matrix of $\{x^{[1]}, x^{[2]}, \ldots, x^{[n]}\}$ should be close to the covariance matrix of X.

• to satisfy the independent identically distributed condition, sample autocorrelations between lagged values $x^{[i]}$ and $x^{[j]}$ should be approximately 0.

$\Rightarrow$ use of a good pseudorandom generator

Monte Carlo Methods

Monte Carlo estimator

A Monte Carlo method can always be seen as a statistical estimator $H(X^{[1]}, X^{[2]}, \ldots, X^{[n]}\}$ of an (not necessarily probabilistic) integral

$$\Psi = \int f(u)du.$$

We write as $\Psi = E[f(X)]$ for some random variable X and estimate it by

$$H = f(X^{[1]})$$

with standard error σ.

If we take a larger sample $\{X^{[1]}, X^{[2]}, \ldots, X^{[n]}\}$,

$$H = \frac{1}{n}\sum_{k=1}^{n} f(X^{[k]})$$

is an estimator of Ψ with standard error $\frac{\sigma}{\sqrt{n}}$.

Monte Carlo Methods

Variance reduction

Consider a Monto Carlo estimator $H = \frac{1}{n}\sum_{k=1}^{n} f(X^{[k]})$ for some quantity Ψ = E[f(X)].

Let ξ be function for which the mean E[ξ(X)] is known. ξ(X) is called a control variate.

Consider the random function $f^*(X) = f(X) - c\,[\xi(X) - E[\xi(X)]]$ for some constant c.

$f^*(X)$ is then an unbiased estimator of Ψ since E[$f^*(X)$]= Ψ.

We can estimate it with the Monte Carlo estimator

$$H^* = \frac{1}{m}\sum_{k=1}^{n} f^*(X^{[k]}) = \frac{1}{n}\sum_{k=1}^{n}\Big(f(X^{[k]}) - c\big[\xi(X^{[k]}) - E\big[\xi(X)\big]\big]\Big).$$

It has a lower standard error which can be minimized by a clever choice of the constant c.

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Value at Risk

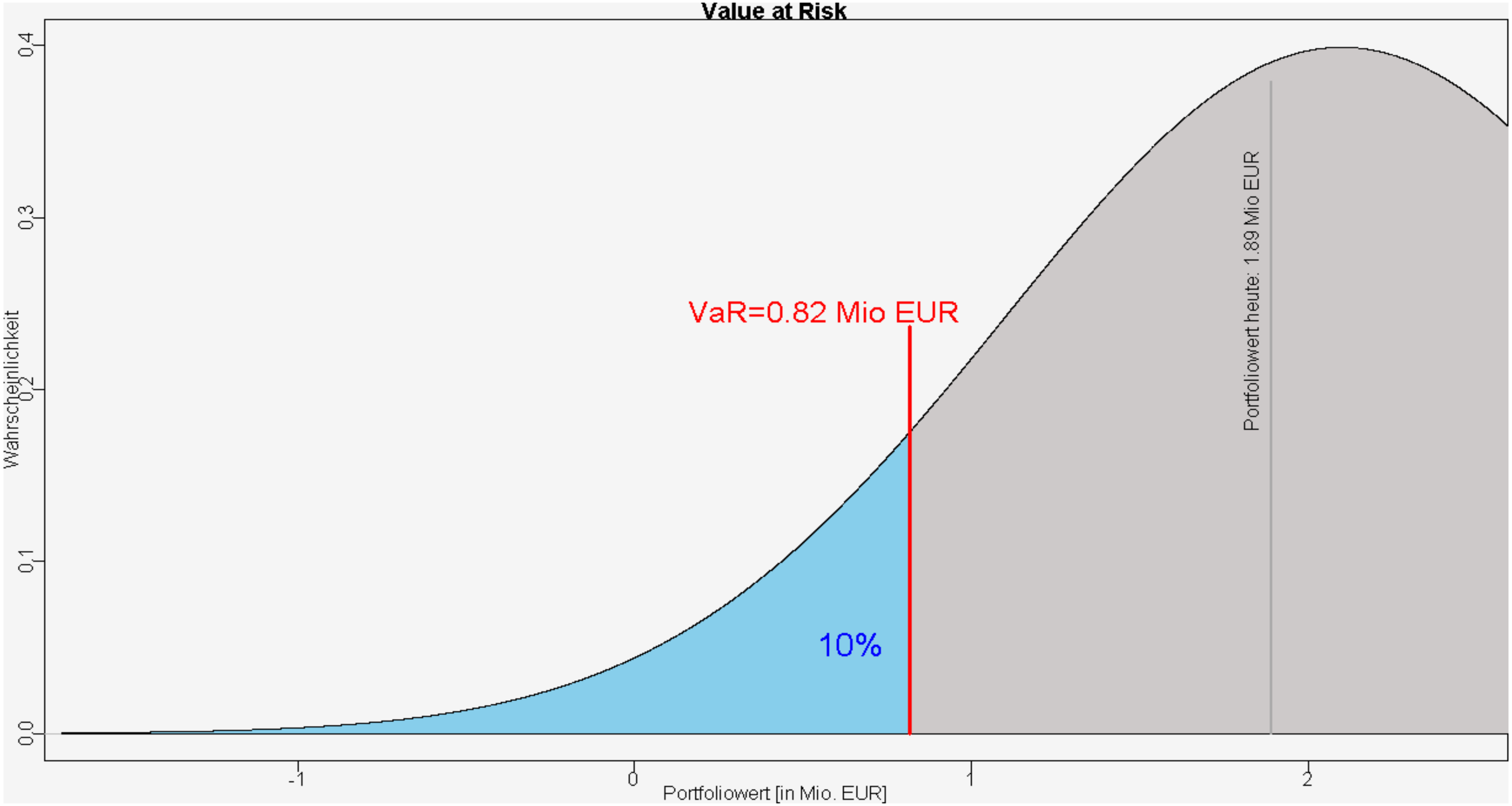
Definition

Value at risk (VAR) summarizes the worst loss over a target horizon that will not be exceeded with a given level of confidence under normal market conditions.

Mathematically, if c denotes the confidence level, t the target horizon and L_t the loss at time t, VAR is defined by

$$P(L_t > \mathrm{VAR}) \leq 1 - c$$

Example



10% Value at Risk with normally distibuted portfoliovalue

Steps in computing VAR

- Mark to market the current portfolio
- Set the time horizon
- Set the confidence level
- Measure the variability of the risk factor
- Compute the probability

Example : VAR of a $100 million equity portfolio over 10 days at the 99 % confidence level.

Quantile of the confidence level

TAB. A.1 – Fonction de répartition de la loi normale centrée réduite

Probabilité d'avoir une valeur inférieure à t : $F(t) = P(X \leq t)$

t	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952

Quantile of the confidence level

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0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964

Quantile of the confidence level

Tab. A.1 – Fonction de répartition de la loi normale centrée réduite

Probabilité d'avoir une valeur inférieure à t : $F(t) = P(X \leq t)$

t	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952

Time adjustment

Suppose that the portfolio has an annual variability of 15 %.

We are interested in a 10 days VaR. Since the trading year is constituted of 250 days, we have to adjust the volatility to ten days.

The central limit theorem tells us that for independent and identically distributed random variables, variances are additif over time, which implies that volatility grows with the square root of time.

Finally we get :

$$VaR = \$100MM \bullet 15\% \bullet \sqrt{10/250} \bullet 2,33 = \$7MM.$$

Example 2

Compute a portfolio's 90 % 1 week USD VaR. We need to compute the value at time one (after a week) of the portfolio. Define

$$
{}^1S = \begin{pmatrix} {}^1S_1 \\ {}^1S_2 \\ {}^1S_3 \\ {}^1S_4 \\ {}^1S_5 \\ {}^1S_6 \end{pmatrix} = \begin{pmatrix} \text{value of a ton of aluminium at time 1} \\ \text{value of a ton of copper at time 1} \\ \text{value of a ton of lead at time 1} \\ \text{value of a ton of nickel at time 1} \\ \text{value of a ton of tin at time 1} \\ \text{value of a ton of zinc at time 1} \end{pmatrix}
$$

Current values in USD/ton are

$$
{}^0S = \begin{pmatrix} {}^0S_1 \\ {}^0S_2 \\ {}^0S_3 \\ {}^0S_4 \\ {}^0S_5 \\ {}^0S_6 \end{pmatrix} = \begin{pmatrix} 1516,0 \\ 1719,5 \\ 476,0 \\ 7945,0 \\ 5715,0 \\ 1165,0 \end{pmatrix}
$$

Example 2

The portfolio holdings ω are reprsented as a row vector.

Suppose $\omega = (1000, 2000, 500, 250, 1000, 100)$.

The current portfolio value is ${}^0P = \omega\, {}^0S = 13{,}011$ MM USD.

It's value at time one ${}^1P$ is random:

$${}^1P = \omega\, {}^1S.$$

Let ${}^{1|0}\sigma$ and ${}^{1|0}\Sigma$ be the standard deviation of ${}^1P$ and the covariance matrix of ${}^1S$.

We have :

$${}^{1|0}\sigma = \sqrt{\omega \Sigma \omega'}.$$

How do we compute the value of ${}^{1|0}\Sigma$?

Example 2

Date	Time t	Aluminum ${}^t s_1$	Copper ${}^t s_2$	Lead ${}^t s_3$	Nickel ${}^t s_4$	Tin ${}^t s_5$	Zinc ${}^t s_6$
12/10/99	–29	1,516.0	1,719.5	476.0	7,945.0	5,715.0	1,165.0
12/17/99	–28	1,580.5	1,796.0	482.0	8,155.0	5,730.0	1,216.0
12/24/99	–27	1,609.0	1,834.0	474.0	8,380.0	5,700.0	1,200.0
12/30/99	–26	1,630.5	1,846.0	478.0	8,450.0	6,105.0	1,239.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5/12/00	–7	1,455.0	1,808.5	418.0	10,040.0	5,490.0	1,170.5
5/19/00	–6	1,498.0	1,815.0	403.0	10,600.0	5,480.0	1,156.0
5/26/00	–5	1,464.0	1,793.5	432.0	10,435.0	5,405.0	1,158.0
6/2/00	–4	1,464.0	1,770.0	423.0	10,020.0	5,440.0	1,118.5
6/9/00	–3	1,456.5	1,722.5	421.0	8,480.0	5,450.0	1,099.0
6/16/00	–2	1,555.0	1,768.0	422.0	8,230.0	5,525.0	1,125.5
6/23/00	–1	1,544.5	1,767.0	416.0	7,925.0	5,515.0	1,124.0
6/30/00	0	1,564.0	1,773.5	440.5	8,245.0	5,465.0	1,148.0

Exhibit 1.4 Thirty weekly historical prices for the indicated metals. All prices are in USD per ton. Source: London Metals Exchange (LME).

Time series analysis gives us

$$\Sigma = \begin{pmatrix} 1709 & 1227 & 8 & 3557 & 774 & 275 \\ 1227 & 1746 & 65 & 6274 & 574 & 469 \\ 8 & 65 & 128 & -270 & -49 & 69 \\ 3557 & 6274 & -270 & 137361 & -2459 & 1764 \\ 774 & 574 & -49 & -2459 & 13621 & 952 \\ 275 & 469 & 69 & 1764 & 952 & 544 \end{pmatrix}.$$

Example 2

Conditional standard deviation of ${}^{1}P$:

$$ {}^{1|0}\sigma = \sqrt{\omega \Sigma \omega'} = 271.400 \text{ USD} $$

To conclude we need a modelization for the probability law of ${}^{1}P$.

Suppose that ${}^{1}P$ follows a normal law with mean ${}^{0}P$ and standard deviation ${}^{1|0}\sigma$. The loss at time 1 denoted by L_1, then follows a centered normal law with the same standard deviation.

Denote the cumulative distribution function of L_1 by F_{L_1}.

Then the 90 % VaR of our portfolio after one week can be computed by

$$
\begin{aligned}
& P(L_1 > VaR) = 0{,}9 = 1 - P(L_1 \leq VaR) \\
& \Rightarrow 1 - F_{L_1}(VaR) = 0{,}9 \\
& \Rightarrow F_{L_1}(VaR) = 0{,}1 \\
& \Rightarrow VaR = 1{,}282\ {}^{1|0}\sigma = 278.000 \text{ USD.}
\end{aligned}
$$

Example 3

Compute a portfolio's 95 % 1 day GBP VaR if the historical prices are in another currency. We need to compute the value at time one (after a day) of the portfolio. Define

$$^1S = \begin{pmatrix} ^1S_1 \\ ^1S_2 \\ ^1S_3 \end{pmatrix} = \begin{pmatrix} \text{GPB value of a share of National Australia Bank} \\ \text{GPB value of a share of Westpac Banking Corp.} \\ \text{GPB value of a share of Goodman Fielder} \end{pmatrix}.$$

The portfolio holdings ω are represented as a row vector.

Suppose ω = (10.000, 30.000, -15.000).

The current portfolio value is $^0P = \omega\ ^0S$ = 198.000 GPB

It's value at time one ^1P is random:

$$^1P = \omega\ ^1S.$$

Problem : The historical data of the stock prices are in Australian dollars.

Example 3

We introduce a change of variables ${}^1S = \varphi({}^1R)$, where

$$ {}^1R = \begin{pmatrix} {}^1R_1 \\ {}^1R_2 \\ {}^1R_3 \\ {}^1R_4 \end{pmatrix} = \begin{pmatrix} \text{AUD value of a share of National Australia Bank} \\ \text{AUD value of a share of Westpac Banking Corp.} \\ \text{AUD value of a share of goodman Fielder} \\ \text{GBP/AUD exchange rate} \end{pmatrix} \text{ and} $$

$$ \varphi({}^1R) = {}^1R_4 \begin{pmatrix} {}^1R_1 \\ {}^1R_2 \\ {}^1R_3 \end{pmatrix}. $$

Hence, ${}^1P = \theta({}^1R) = {}^1R_4(10.000\ {}^1R_1 + 30.000\ {}^1R_2 - 15.000\ {}^1R_3)$

This is not a linear function of ${}^1R$!!!

Example 3

Let ${}^{1|0}\sigma$ and ${}^{1|0}\Sigma$ be the standard deviation of ${}^{1}P$ and the covariance matrix of ${}^{1}R$.

Date	Time t	National Australia Bank ${}^{t}r_1$	Westpac Banking Corp. ${}^{t}r_2$	Goodman Fielder ${}^{t}r_3$	GBP/AUD ${}^{t}r_4$
1/10/00	–42	22.200	10.207	1.400	0.4007
1/11/00	–41	21.800	10.215	1.410	0.3990
1/12/00	–40	21.630	10.220	1.380	0.3995
1/13/00	–39	21.430	10.310	1.370	0.4057
⋮	⋮	⋮	⋮	⋮	⋮
2/29/00	–7	21.400	10.400	1.170	0.3901
3/1/00	–6	22.106	10.767	1.184	0.3828
3/2/00	–5	22.273	10.580	1.200	0.3855
3/3/00	–4	21.442	10.410	1.170	0.3847
3/6/00	–3	20.950	10.410	1.140	0.3824
3/7/00	–2	21.340	10.414	1.080	0.3826
3/8/00	–1	20.830	10.500	1.130	0.3844
3/9/00	0	20.080	10.800	1.150	0.3892

Exhibit 1.5 Two months of historical data for the GBP/AUD exchange rate and AUD prices for the indicated stocks. None of the stocks had ex-dividend dates during the period indicated. Source: Federal Reserve Bank of Chicago and Dow Jones.

Time series analysis gives us

$$\Sigma = \begin{pmatrix} 0,156644 & 0,030382 & -0,00135 & -0,000213 \\ 0,030382 & 0,029574 & 0,000157 & 0,000053 \\ -0,000135 & 0,000157 & 0,000739 & -0,000010 \\ -0,000213 & 0,000053 & -0,000010 & 0,000015 \end{pmatrix}.$$

Example 3

Solutions in case of a nonlinear portfolio mapping:

- apply the Monte Carlo method to approximate the desired quantile
- approximate the quadratic polynomial θ with a linear polynomial
- assume ${}^{1}R$ is conditionally joint-normal and apply probabilistic techniques

Example 3

Use of the Monte Carlo method

k	National Australia Bank $^1r_1^{[k]}$	Westpac Banking Corp. $^1r_2^{[k]}$	Goodman Fielder $^1r_3^{[k]}$	GBP/AUD $^1r_4^{[k]}$	Portfolio $^1p^{[k]}$
1	20.290	10.913	1.150	0.3823	196,135
2	19.392	10.333	1.153	0.3870	188,327
3	20.088	10.744	1.164	0.3917	198,117
4	20.620	11.083	1.124	0.3920	204,538
5	19.660	10.811	1.154	0.3909	196,855
6	19.973	10.806	1.162	0.3823	193,639
7	19.732	10.867	1.158	0.3902	197,437
8	19.655	10.925	1.200	0.3889	196,902
9	20.101	10.886	1.122	0.3909	199,665
10	21.136	11.064	1.129	0.3801	200,037
11	19.968	10.839	1.180	0.3881	196,804
12	20.112	10.750	1.119	0.3906	197,961
⋮	⋮	⋮	⋮	⋮	⋮
9998	20.240	10.565	1.166	0.3846	193,033
9999	19.531	10.378	1.186	0.3930	192,149
10000	20.078	11.215	1.154	0.3936	204,619

Exhibit 1.6 Results of the Monte Carlo analysis.

Compute $^1p^{[k]}=\theta(^1r^{[k]})$ for all *k*.

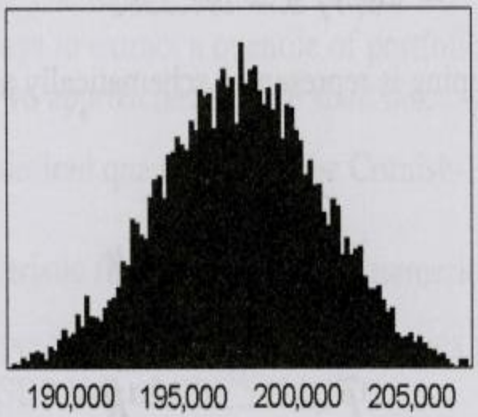


Exhibit 1.7 Histogram of realizations $^1p^{[k]}$ of the portfolio's value ^1P.

Example 3

Use of the Monte Carlo method

The sample 5% - quantile of our realizations is GPB 191.614.

The loss at time 1 L_1 is the difference of the initial portfolio value and the portfolio value at time 1.

$$F_{L_1}(VaR) = F_{{}^0P - {}^1P}(VaR) = 0,05$$

$$\Rightarrow VaR = {}^0P - F_{{}^1P} = 197.539 - 191.614$$

$$\Rightarrow VaR = 5.925 \text{ GPB.}$$

Luxemburgish definition

CSSF criteria : *c = 1 %*

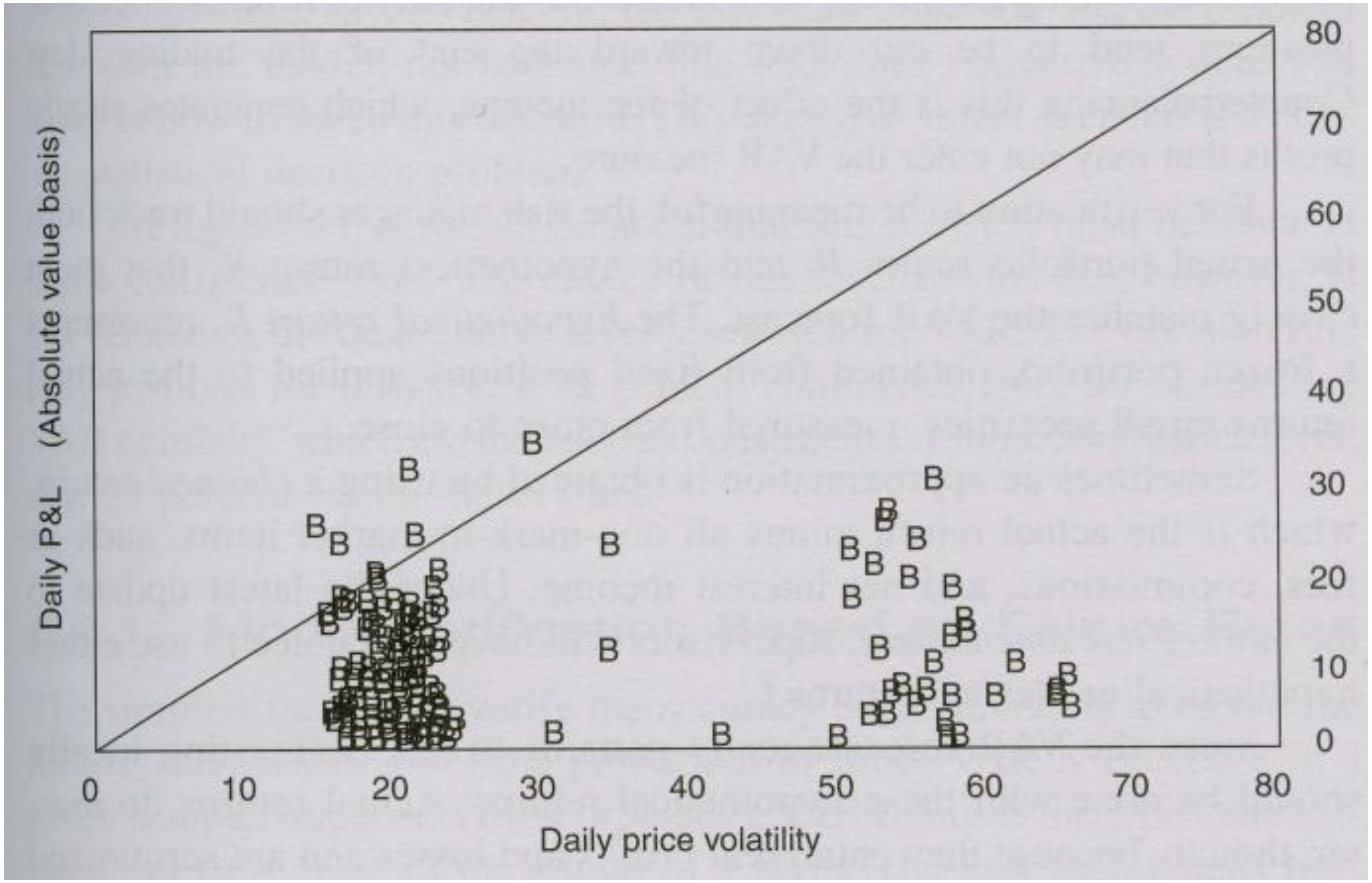
t = 20 days (1 month)

$$P(L_{20} > \mathrm{VAR}) \leq 99\ \%$$

Moreover the observation of the risk factors used in the computations have to be done on a time horizon of at least a year (250 days) and the VaR computations have to be done every day.

Backtesting

Example



Daily P&L (Absolute value basis)
Daily price volatility
0
10
20
30
40
50
60
70
80

Model verification based on failure rates

Define *N* as the number of exceptions for a total of T days and N/T as the failure rate.

This rate should converge to p = 1- c as the sample size increases.

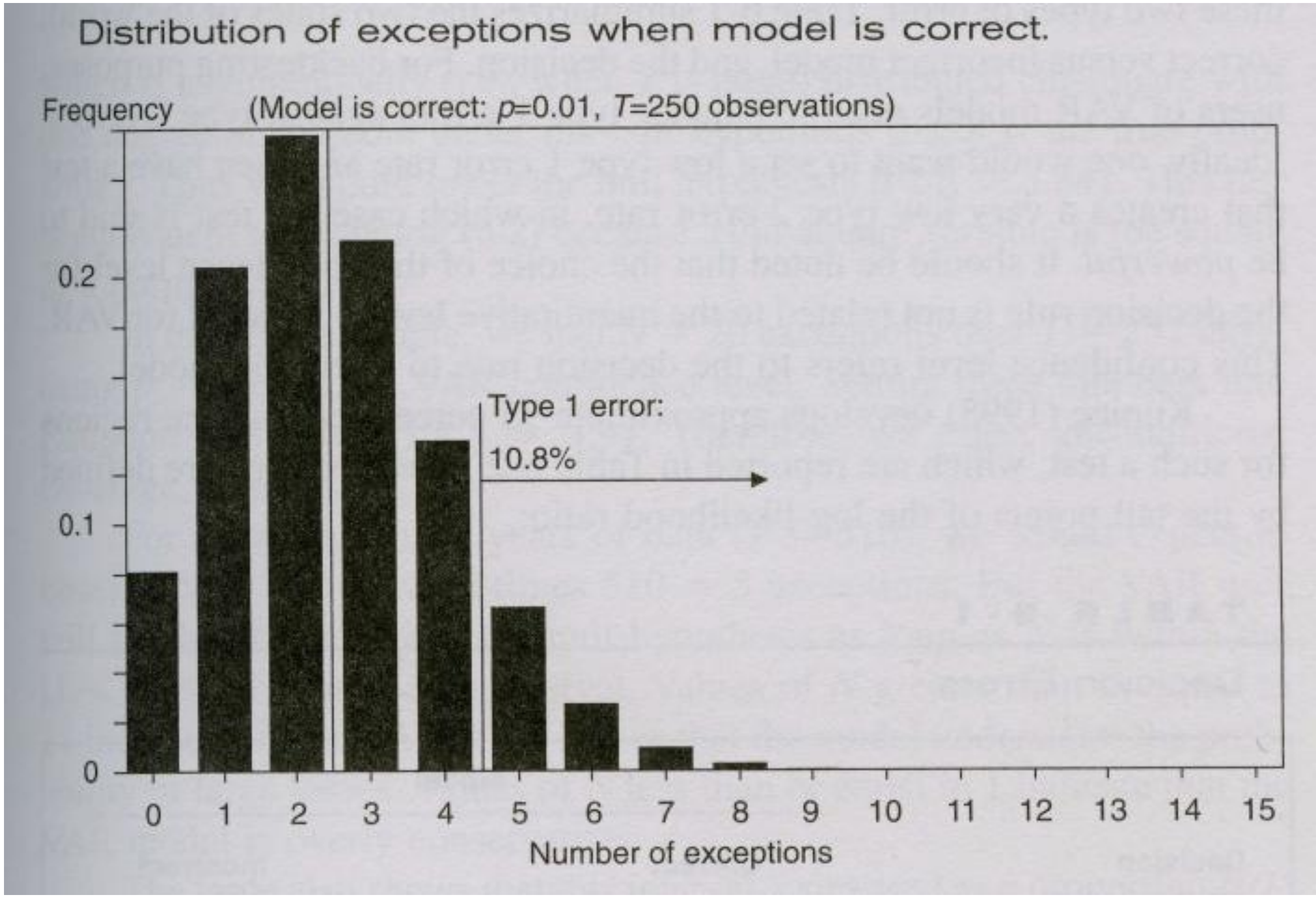
If the model is correctly calibrated, the number of exceptions N follows a binomial probability distribution of parameters T and p. This means

$$P(N=K)=C_T^k p^k (1-p)^{T-k}.$$

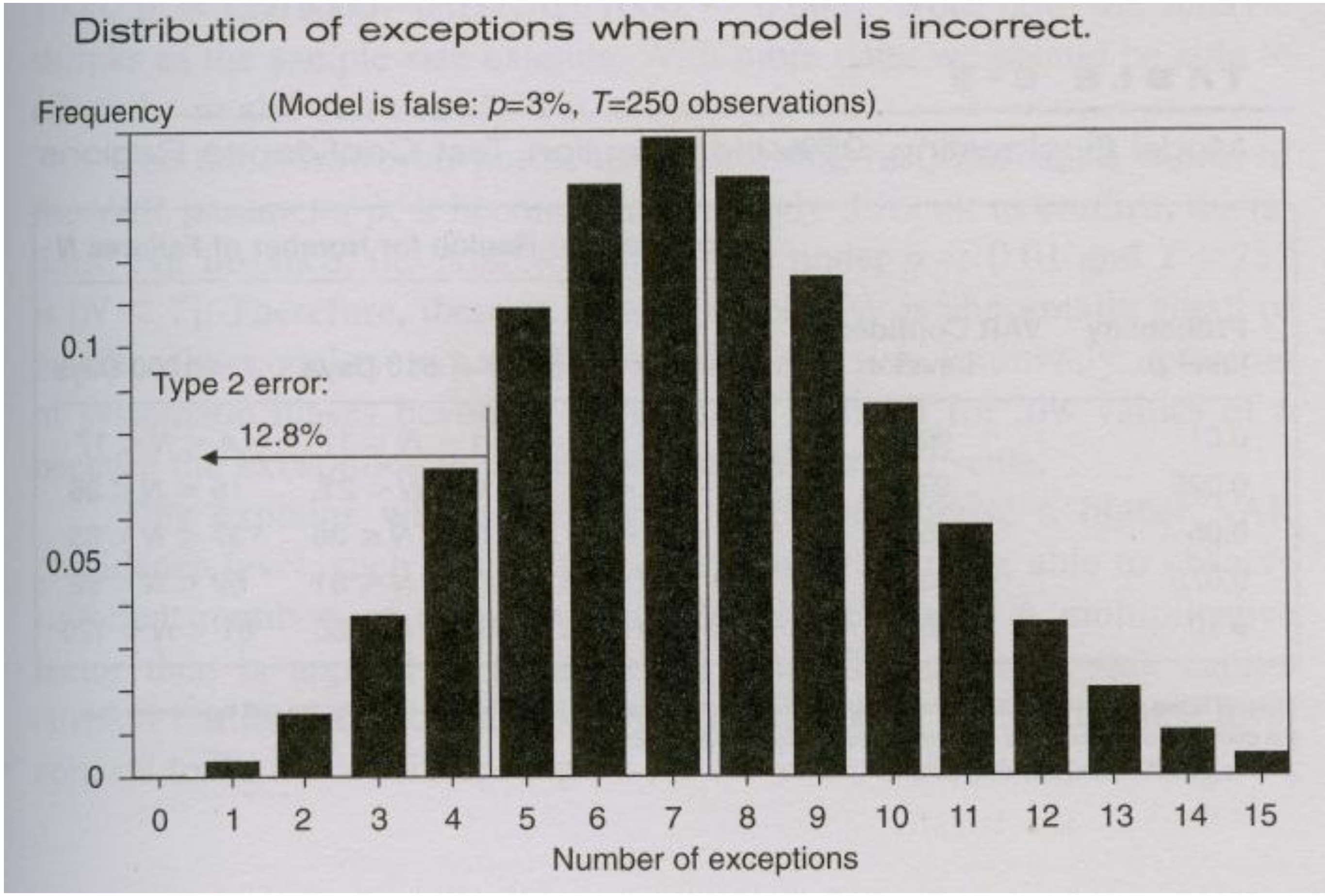
When T is large, we can approximate the binomial distribution by a normal distribution of mean pT and variance p(1-p)T.

Standard test theory then gives adecision rule for any given confidence level.

Model verification based on failure rates

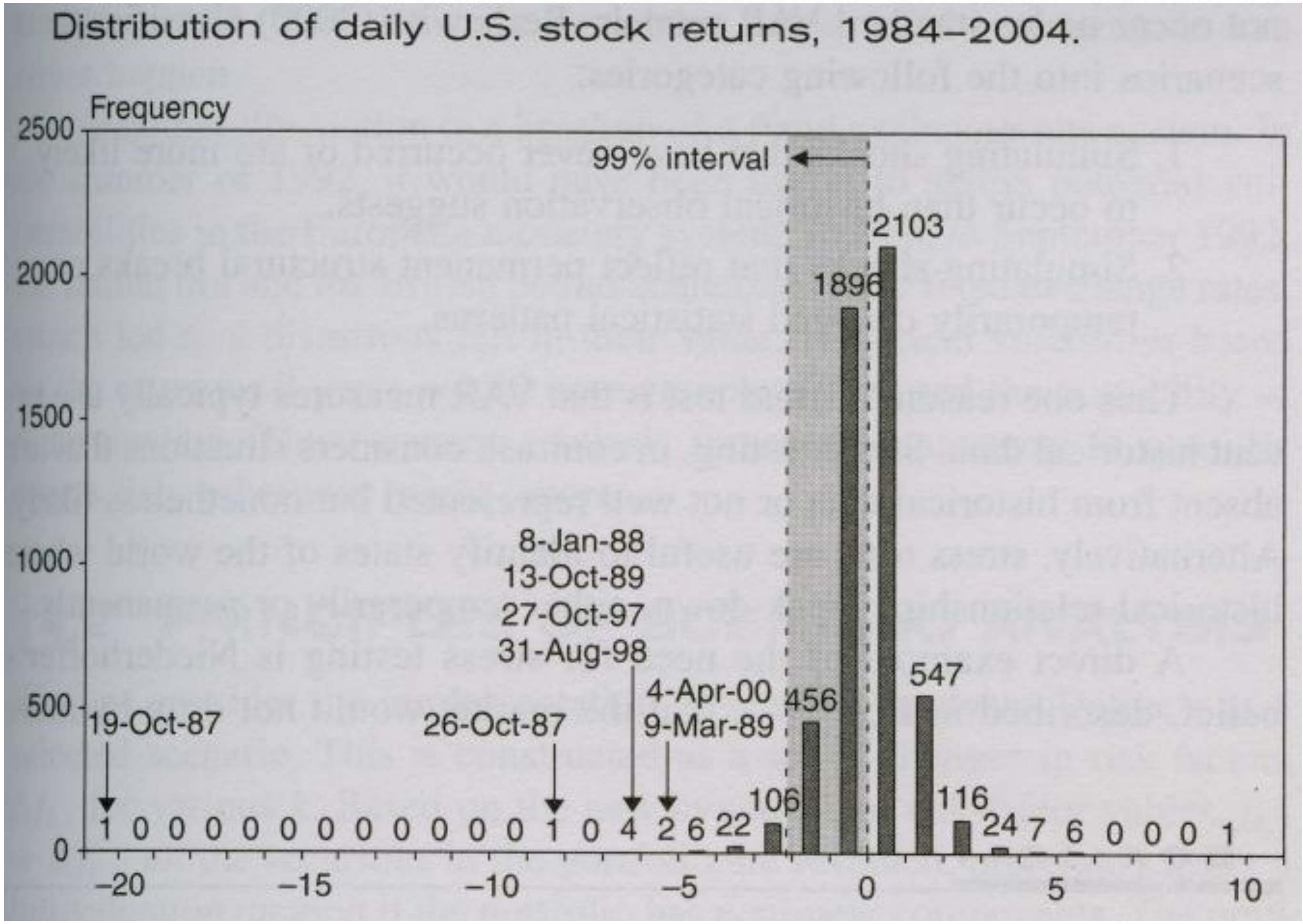


Model verification based on failure rates



Stresstesting

Why stress testing?



Scenario analysis