Multivariate integration of functions depending explicitly on the extrema of the variables

Calculating orness values of aggregation functions

Jean-Luc Marichal

University of Luxembourg

Introduction

Let $\mathsf{F} : [a, b]^n \to \mathbb{R}$ be an aggregation function.

If F is integrable, its *average value* over the domain $[a, b]^n$ is

$$\overline{\mathsf{F}} = \frac{1}{(b-a)^n} \int_{[a,b]^n} \mathsf{F}(\mathbf{x})\, d\mathbf{x}$$

When F is *internal*, that is,

$$\mathrm{Min}(\mathbf{x}) \leqslant \mathsf{F}(\mathbf{x}) \leqslant \mathrm{Max}(\mathbf{x}),$$

then

$$\overline{\mathrm{Min}} \leqslant \overline{\mathsf{F}} \leqslant \overline{\mathrm{Max}}$$

Example : Geometric mean on $[0, 1]^n$

$$\boxed{\mathrm{GM}(\mathbf{x}) = \prod_{i=1}^{n} x_i^{1/n}}$$

We have

$$\begin{aligned}
\overline{\mathrm{GM}} &= \int_{[0,1]^n} \mathrm{GM}(\mathbf{x})\, d\mathbf{x} = \left(\frac{n}{n+1}\right)^n \\
\overline{\mathrm{Min}} &= \int_{[0,1]^n} \mathrm{Min}(\mathbf{x})\, d\mathbf{x} = \frac{1}{n+1} \\
\overline{\mathrm{Max}} &= \int_{[0,1]^n} \mathrm{Max}(\mathbf{x})\, d\mathbf{x} = \frac{n}{n+1}
\end{aligned}$$

and

$$\frac{1}{n+1} \leqslant \left(\frac{n}{n+1}\right)^n \leqslant \frac{n}{n+1}$$

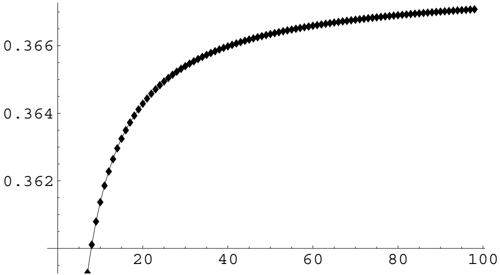
Relative position of $\overline{\mathsf{F}}$ within the interval $[\overline{\mathsf{Min}}, \overline{\mathsf{Max}}]$

Global orness value (Dujmović, 1974)

$$\boxed{\mathrm{orness}_{\mathsf{F}} = \frac{\overline{\mathsf{F}} - \overline{\mathsf{Min}}}{\overline{\mathsf{Max}} - \overline{\mathsf{Min}}} \in [0,1]}$$

Example : Geometric mean on $[0,1]^n$

$$\mathrm{orness}_{\mathsf{GM}} = -\frac{1}{n-1} + \frac{n+1}{n-1}\left(\frac{n}{n+1}\right)^n$$



Alternative definition

Let $\mathsf{F} : [a,b]^n \to \mathbb{R}$ be integrable and internal

Orness distribution function associated with F (Dujmović, 1973)

$$\boxed{\mathrm{odf}_{\mathsf{F}}(\mathbf{x}) = \frac{\mathsf{F}(\mathbf{x}) - \mathsf{Min}(\mathbf{x})}{\mathsf{Max}(\mathbf{x}) - \mathsf{Min}(\mathbf{x})} \in [0,1]}$$

Orness average value of F (Dujmović, 1973)

$$\boxed{\overline{\mathrm{odf}}_{\mathsf{F}} = \frac{1}{(b-a)^n} \int_{[a,b]^n} \mathrm{odf}_{\mathsf{F}}(\mathbf{x})\, d\mathbf{x}}$$

Example : Geometric mean on $[0, 1]^n$

$$\boxed{\mathsf{GM}(\mathbf{x}) = \prod_{i=1}^{n} x_i^{1/n}}$$

$$\overline{\mathrm{odf}}_{\mathsf{GM}} = \int_{[0,1]^n} \frac{\mathsf{GM}(\mathbf{x}) - \mathsf{Min}(\mathbf{x})}{\mathsf{Max}(\mathbf{x}) - \mathsf{Min}(\mathbf{x})}\, d\mathbf{x}$$

Intractable integral ?

We will see how to evaluate exactly such an integral...

Questions

- Which definition is the best ? $\mathrm{orness}_{\mathsf{F}}$ or $\overline{\mathrm{odf}}_{\mathsf{F}}$?
  Should we integrate first and then normalize or normalize first and then integrate ?

  Answer : It depends on the interpretation

- Which are the functions F such that $\mathrm{orness}_{\mathsf{F}} = \overline{\mathrm{odf}}_{\mathsf{F}}$?

  (Open question)

- How to calculate exactly $\overline{\mathrm{odf}}_{\mathsf{F}}$?

  Answer : See next result...

Main results

Let $\mathrm{G} : [a,b]^{n+1} \to \mathbb{R}$ be integrable

$$\int_{[a,b]^n} \mathrm{G}(x_1,\ldots,x_n,\mathrm{Min}(\mathbf{x}))\,d\mathbf{x}$$
$$= \sum_{j=1}^{n} \int_a^b du \int_{[u,b]^{n-1}} \mathrm{G}(x_1,\ldots,\overset{(j)}{u},\ldots,x_n,u)\,dx_1\cdots[dx_j]\cdots dx_n$$

$$\int_{[a,b]^n} \mathrm{G}(x_1,\ldots,x_n,\mathrm{Max}(\mathbf{x}))\,d\mathbf{x}$$
$$= \sum_{j=1}^{n} \int_a^b dv \int_{[a,v]^{n-1}} \mathrm{G}(x_1,\ldots,\overset{(j)}{v},\ldots,x_n,v)\,dx_1\cdots[dx_j]\cdots dx_n$$

Main results

Let $\mathrm{G} : [a, b]^{n+2} \to \mathbb{R}$ be integrable

$$\int_{[a,b]^n} \mathrm{G}(x_1, \ldots, x_n, \mathrm{Min}(\mathbf{x}), \mathrm{Max}(\mathbf{x}))\, d\mathbf{x}$$

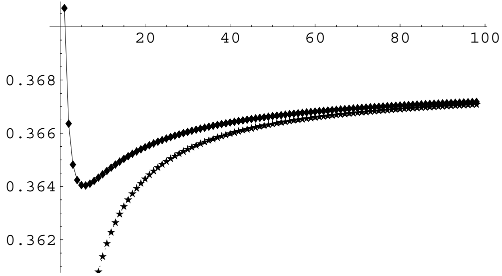
$$= \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \int_a^b dv \int_a^v du \int_{[u,v]^{n-2}} \mathrm{G}(x_1, \ldots, \overset{(j)}{u}, \ldots, \overset{(k)}{v}, \ldots, x_n, u, v)$$

$$\times\ dx_1 \cdots [dx_j] \cdots [dx_k] \cdots dx_n$$

Example : Geometric mean on $[0, 1]^n$

$$\boxed{\mathrm{GM}(\mathbf{x}) = \prod_{i=1}^{n} x_i^{1/n}}$$

$$\overline{\mathrm{odf}}_{\mathrm{GM}} = \begin{cases} -1 + \ln 4, & \text{if } n = 2 \\ n(n-1)\left(\dfrac{n}{n+1}\right)^{n-2} \displaystyle\int_0^1 \frac{x^n(1-x^{n+1})^{n-2}}{1-x^n}\,dx - \frac{1}{n-2}, & \text{if } n > 2 \end{cases}$$



Example : Discrete Choquet integral on $[0,1]^n$

$$\boxed{\mathsf{C}_a(\mathbf{x}) = \sum_{S \subseteq \{1,\ldots,n\}} a(S) \min_{i \in S} x_i}$$

The set function $a : 2^{\{1,\ldots,n\}} \to \mathbb{R}$ is chosen such that $\mathsf{C}_a(\mathbf{x})$ is nondecreasing and $\mathsf{C}_a(\mathbf{0}) = 0$ and $\mathsf{C}_a(\mathbf{1}) = 1$

Theorem

$$\mathrm{orness}_{\mathsf{C}_a} = \overline{\mathrm{odf}}_{\mathsf{C}_a} = \frac{1}{n-1} \sum_{S \subseteq \{1,\ldots,n\}} a(S) \, \frac{n - |S|}{|S| + 1}$$

Conjunctive functions

A function $\mathsf{F} : [a, b]^n \to \mathbb{R}$ is *conjunctive* if

$$a \leqslant \mathsf{F}(\mathbf{x}) \leqslant \mathrm{Min}(\mathbf{x})$$

If F is integrable then

$$a \leqslant \overline{\mathsf{F}} \leqslant \overline{\mathrm{Min}}$$

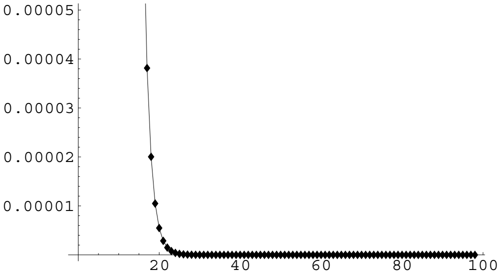
Global idempotency value (Kolesárová, 2006)

$$\boxed{\mathrm{idemp}_{\mathsf{F}} = \frac{\overline{\mathsf{F}} - a}{\overline{\mathrm{Min}} - a} \in [0, 1]}$$

Example : Product on $[0,1]^n$

$$\boxed{\mathsf{P}(\mathbf{x}) = \prod_{i=1}^{n} x_i}$$

$$\mathrm{idemp}_{\mathsf{P}} = (n+1)\int_{[0,1]^n} \mathsf{P}(\mathbf{x})\, d\mathbf{x} \;=\; \frac{n+1}{2^n}$$



Alternative definition

Let $\mathsf{F} : [a,b]^n \to \mathbb{R}$ be integrable and conjunctive

Idempotency distribution function associated with F

$$\boxed{\mathrm{idf}_{\mathsf{F}}(\mathbf{x}) = \frac{\mathsf{F}(\mathbf{x}) - a}{\mathsf{Min}(\mathbf{x}) - a} \in [0,1]}$$

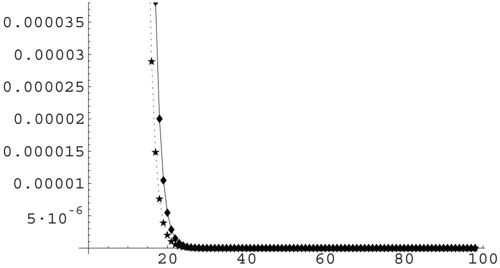
Idempotency average value of F

$$\boxed{\overline{\mathrm{idf}}_{\mathsf{F}} = \frac{1}{(b-a)^n} \int_{[a,b]^n} \mathrm{idf}_{\mathsf{F}}(\mathbf{x})\, d\mathbf{x}}$$

Example : Product on $[0,1]^n$

$$\boxed{\mathsf{P}(\mathbf{x}) = \prod_{i=1}^{n} x_i}$$

$$\overline{\mathrm{idf}}_{\mathsf{P}} = \int_{[0,1]^n} \frac{\mathsf{P}(\mathbf{x})}{\mathsf{Min}(\mathbf{x})}\, d\mathbf{x} \;=\; \frac{2^{n-1}}{\binom{2n-1}{n}}$$



Conclusion

We have provided an alternative expression of

$$\int_{[a,b]^n} \mathrm{G}(x_1, \ldots, x_n, \mathrm{Min}(\mathbf{x}), \mathrm{Max}(\mathbf{x}))\, d\mathbf{x}$$

Applications :

- Aggregation functions

  Example : *Orness average values* and *idempotency average values*

- Integration in probability theory

  Example : Average value of the *variance-to-range* ratio function

$$\frac{1}{(b-a)^n} \int_{[a,b]^n} \frac{s^2(\mathbf{x})}{\mathrm{Max}(\mathbf{x}) - \mathrm{Min}(\mathbf{x})}\, d\mathbf{x} \;=\; \frac{n+2}{12n}(b-a)$$

where $s^2(\mathbf{x}) = \frac{1}{n-1} \sum_i (x_i - \frac{1}{n} \sum_i x_i)^2$