Weighted Lattice Polynomials

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Lattice polynomials

Let L be a lattice with lattice operations $\wedge$ and $\vee$

We assume that L is

- bounded (with bottom 0 and top 1)
- distributive

Definition (Birkhoff 1967)

An n-ary *lattice polynomial* is a well-formed expression involving n variables $x_1, \ldots, x_n \in L$ linked by the lattice operations $\wedge$ and $\vee$ in an arbitrary combination of parentheses

Example.

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

Lattice polynomial functions

Any lattice polynomial naturally defines a *lattice polynomial function* (l.p.f.) $p : L^n \to L$.

Example.

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

If p and q represent the same function, we say that p and q are equivalent and we write $p = q$

Example.

$$x_1 \vee (x_1 \wedge x_2) = x_1$$

Disjunctive and conjunctive forms of l.p.f.'s

Notation. $[n] := \{1, \ldots, n\}$.

Proposition (Birkhoff 1967)

Let $p : L^n \to L$ be any l.p.f.
Then there are nonconstant set functions $v, w : 2^{[n]} \to \{0, 1\}$, with $v(\varnothing) = 0$ and $w(\varnothing) = 1$, such that

$$p(x) = \bigvee_{\substack{S \subseteq [n] \\ v(S)=1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ w(S)=0}} \bigvee_{i \in S} x_i .$$

Example.

$$(x_1 \wedge x_2) \vee x_3 = (x_1 \vee x_3) \wedge (x_2 \vee x_3)$$

$$v(\{3\}) = v(\{1, 2\}) = 1$$
$$w(\{1, 3\}) = w(\{2, 3\}) = 0$$

The set functions v and w, which generate p, are not unique :

$$x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2)$$

Notation. $\mathbf{1}_S :=$ characteristic vector of $S \subseteq [n]$ in $\{0,1\}^n$.

Proposition (Marichal 2002)

From among all the set functions v that disjunctively generate the l.p.f. p, only one is isotone :

$$v(S) = p(\mathbf{1}_S)$$

From among all the set functions w that conjunctively generate the l.p.f. p, only one is antitone :

$$w(S) = p(\mathbf{1}_{[n]\setminus S})$$

Consequently, any n-ary l.p.f. can always be written as

$$p(x) = \bigvee_{\substack{S \subseteq [n] \\ p(\mathbf{1}_S)=1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ p(\mathbf{1}_{[n]\setminus S})=0}} \bigvee_{i \in S} x_i$$

Example. $p(x) = (x_1 \wedge x_2) \vee x_3$

S	$p(\mathbf{1}_S)$	$p(\mathbf{1}_{[n]\setminus S})$
$\varnothing$	0	1
$\{1\}$	0	1
$\{2\}$	0	1
$\{3\}$	1	1
$\{1,2\}$	1	1
$\{1,3\}$	1	0
$\{2,3\}$	1	0
$\{1,2,3\}$	1	0

$$\begin{aligned} p(x) &= x_3 \vee (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3) \\ p(x) &= (x_1 \vee x_3) \wedge (x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3) \end{aligned}$$

Particular cases : order statistics

Denote by $x_{(1)}, \ldots, x_{(n)}$ the *order statistics* resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order : $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$.

Proposition (Ovchinnikov 1996, Marichal 2002)

p is a symmetric l.p.f. $\iff$ p is an order statistic

Notation. Denote by $os_k : L^n \to L$ the kth order statistic function.

$$os_k(x) := x_{(k)}$$

Then we have

$$\begin{aligned} os_k(\mathbf{1}_S) = 1 \quad &\iff \quad |S| \geqslant n-k+1 \\ os_k(\mathbf{1}_{[n]\setminus S}) = 0 \quad &\iff \quad |S| \geqslant k \end{aligned}$$

Weighted lattice polynomials

We can generalize the concept of l.p.f. by regarding some variables as parameters.

Example. For $c \in L$, we consider

$$p(x_1, x_2) = (c \vee x_1) \wedge x_2$$

Definition

$p : L^n \to L$ is an n-ary *weighted lattice polynomial* function (w.l.p.f.) if there exist parameters $c_1, \ldots, c_m \in L$ and a l.p.f. $q : L^{n+m} \to L$ such that

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n, c_1, \ldots, c_m)$$

Disjunctive and conjunctive forms of w.l.p.f.'s

Proposition (Lausch & Nöbauer 1973)

Let $p : L^n \to L$ be any w.l.p.f.
Then there are set functions $v, w : 2^{[n]} \to L$ such that

$$p(x) = \bigvee_{S \subseteq [n]} \Big[v(S) \wedge \bigwedge_{i \in S} x_i \Big] = \bigwedge_{S \subseteq [n]} \Big[w(S) \vee \bigvee_{i \in S} x_i \Big].$$

Remarks.

- p is a l.p.f. if v and w range in $\{0, 1\}$, with $v(\varnothing) = 0$ and $w(\varnothing) = 1$.
- Any w.l.p.f. is entirely determined by 2^n parameters, even if more parameters have been considered to construct it.

Disjunctive and conjunctive forms of w.l.p.f.'s

Proposition (Marichal 2006)

From among all the set functions v that disjunctively generate the w.l.p.f. p, only one is isotone :

$$v(S) = p(\mathbf{1}_S)$$

From among all the set functions w that conjunctively generate the w.l.p.f. p, only one is antitone :

$$w(S) = p(\mathbf{1}_{[n]\setminus S})$$

Disjunctive and conjunctive forms of w.l.p.f.'s

Consequently, any n-ary w.l.p.f. can always be written as

$$p(x) = \bigvee_{S \subseteq [n]} \Big[p(\mathbf{1}_S) \wedge \bigwedge_{i \in S} x_i \Big] = \bigwedge_{S \subseteq [n]} \Big[p(\mathbf{1}_{[n] \setminus S}) \vee \bigvee_{i \in S} x_i \Big]$$

Example. $p(x) = (c \vee x_1) \wedge x_2$

S	$p(\mathbf{1}_S)$	$p(\mathbf{1}_{[n]\setminus S})$
$\varnothing$	0	1
$\{1\}$	0	c
$\{2\}$	c	0
$\{1,2\}$	1	0

$$\begin{aligned} p(x) &= (0 \wedge 1) \vee (0 \wedge x_1) \vee (c \wedge x_2) \vee (1 \wedge x_1 \wedge x_2) \\ &= (c \wedge x_2) \vee (x_1 \wedge x_2) \\ p(x) &= (1 \vee 0) \wedge (c \vee x_1) \wedge (0 \vee x_2) \wedge (0 \vee x_1 \vee x_2) \\ &= (c \vee x_1) \wedge x_2 \end{aligned}$$

Particular case : the Sugeno integral

Let us generalize the concept of discrete Sugeno integral in the framework of bounded distributive lattices.

Definition (Sugeno 1974)

An L-valued *fuzzy measure* on $[n]$ is an isotone set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = 0$ and $\mu([n]) = 1$.

The *Sugeno integral* of a function $x : [n] \to L$ with respect to μ is defined by

$$\mathcal{S}_\mu(x) := \bigvee_{S \subseteq [n]} \Big[\mu(S) \wedge \bigwedge_{i \in S} x_i\Big]$$

Remark. A function $f : L^n \to L$ is an n-ary Sugeno integral if and only if f is a w.l.p.f. fulfilling $f(\mathbf{1}_\varnothing) = 0$ and $f(\mathbf{1}_{[n]}) = 1$.

Particular case : the Sugeno integral

Notation. The median function is the function $os_2 : L^3 \to L$.

Proposition (Marichal 2006)

For any w.l.p.f. $p : L^n \to L$, there is a fuzzy measure $\mu : 2^{[n]} \to L$ such that

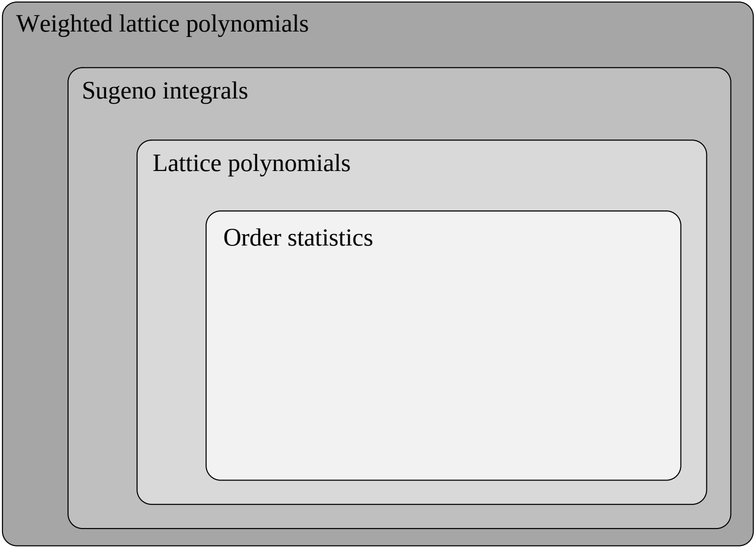
$$p(x) = \text{median}\left[p(\mathbf{1}_\varnothing), \mathcal{S}_\mu(x), p(\mathbf{1}_{[n]})\right]$$

Corollary (Marichal 2006)

Consider a function $f : L^n \to L$.
The following assertions are equivalent :

- f is a Sugeno integral
- f is an idempotent w.l.p.f., that is such that $f(x, \ldots, x) = x$
- f is a w.l.p.f. fulfilling $f(\mathbf{1}_\varnothing) = 0$ and $f(\mathbf{1}_{[n]}) = 1$.

Inclusion properties



The median based decomposition formula

Let $f : L^n \to L$ and $k \in [n]$ and define $f_k^0, f_k^1 : L^n \to L$ as

$$\begin{aligned} f_k^0(x) &:= f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \\ f_k^1(x) &:= f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) \end{aligned}$$

Remark. If f is a w.l.p.f., so are f_k^0 and f_k^1

Consider the following system of n functional equations, called the *median based decomposition formula*

$$\boxed{f(x) = \text{median}\big[f_k^0(x), x_k, f_k^1(x)\big] \qquad (k = 1, \ldots, n)}$$

The median based decomposition formula

Any solution of the median based decomposition formula

$$f(x) = \text{median}\big[f_k^0(x), x_k, f_k^1(x)\big] \qquad (k = 1, \ldots, n)$$

is an n-ary w.l.p.f.

Example. For $n = 2$ we have

$$f(x_1, x_2) = \text{median}\big[f(x_1, 0), x_2, f(x_1, 1)\big]$$

with

$$\begin{aligned} f(x_1, 0) &= \text{median}\big[f(0,0), x_1, f(1,0)\big] \quad \text{(w.l.p.f.)} \\ f(x_1, 1) &= \text{median}\big[f(0,1), x_1, f(1,1)\big] \quad \text{(w.l.p.f.)} \end{aligned}$$

The median based decomposition formula

The median based decomposition formula characterizes the w.l.p.f.'s

Theorem (Marichal 2006)

The solutions of the median based decomposition formula are exactly the n-ary w.l.p.f.'s

Thanks for your attention !