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Tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral

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Aggregation in multicriteria decision aid

- Alternatives $A = \{a, b, c, \ldots\}$
- Criteria $N = \{1, 2, \ldots, n\}$
- Profile $a \in A \quad \longrightarrow \quad \underbrace{(x_1^a, \ldots, x_n^a)}_{\text{same interval scale}} \in \mathbb{R}^n$ or $[0,1]^n$
- Aggregation function

$$M : \mathbb{R}^n \to \mathbb{R} \qquad\qquad M : [0,1]^n \to [0,1]$$

$$(x_1, \ldots, x_n) \mapsto M(x_1, \ldots, x_n)$$

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Choquet integral

Capacity on N:

$$v : 2^N \to [0,1],\ \text{monotone},\ v(\varnothing) = 0,\ \text{and}\ v(N) = 1$$
$$\mathcal{F}_n := \{\text{capacities on } N\}$$

Choquet integral of $x \in \mathbb{R}^n$ w.r.t. v:

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)} \Big[v\big[(i), \ldots, (n)\big] - v\big[(i+1), \ldots, (n)\big] \Big]$$

with the convention that $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$.

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Examples

If $x_3 \leqslant x_1 \leqslant x_2$, we have

$$\begin{aligned} \mathcal{C}_v(x_1, x_2, x_3) &= x_3[v(3,1,2) - v(1,2)] \\ &\quad + x_1[v(1,2) - v(2)] \\ &\quad + x_2 v(2) \end{aligned}$$

If $x_2 \leqslant x_3 \leqslant x_1$, we have

$$\begin{aligned} \mathcal{C}_v(x_1, x_2, x_3) &= x_2[v(2,3,1) - v(3,1)] \\ &\quad + x_3[v(3,1) - v(1)] \\ &\quad + x_1 v(1) \end{aligned}$$

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Properties of the Choquet integral

- **Linearity w.r.t. the capacity**
  $\exists\ f_T : \mathbb{R}^n \to \mathbb{R}$ $(T \subseteq N)$ s.t.
  $$\mathcal{C}_v = \sum_{T\subseteq N} v(T) f_T \qquad (v \in \mathcal{F}_n)$$

- **Monotonicity**
  For any $x, x' \in \mathbb{R}^n$ and any $v \in \mathcal{F}_n$
  $$x \leqslant x' \quad \Rightarrow \quad \mathcal{C}_v(x) \leqslant \mathcal{C}_v(x')$$

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- **Affine invariance**
  For any $x \in \mathbb{R}^n$, $v \in \mathcal{F}_n$, $r > 0$ and $s \in \mathbb{R}$,
  $$\mathcal{C}_v(rx_1 + s, \ldots, rx_n + s) = r\, \mathcal{C}_v(x_1, \ldots, x_n) + s$$
  $\Rightarrow$ The partial scores may be embedded in $[0, 1]$

- **Meaningfulness of weights**
  $$v(S) = \mathcal{C}_v(\mathbf{1}_S) \qquad (S \subseteq N,\ v \in \mathcal{F}_n)$$
  Example : $N = \{1, 2, 3, 4\}$
  $$v(2, 3, 4) = \mathcal{C}_v(0, 1, 1, 1)$$

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Axiomatization

Consider a family of n-variable aggregation functions

$$\{M_v : \mathbb{R}^n \to \mathbb{R} \mid v \in \mathcal{F}_n\}$$

This family satisfies the four properties above if and only if

$$M_v = \mathcal{C}_v \qquad (v \in \mathcal{F}_n)$$

(Marichal, 2000)

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Evaluation of students

Student	St	Pr	Al
a	19	15	18
b	19	18	15
c	11	15	18
d	11	18	15

on $[0, 20]$

$\longleftrightarrow$

St	Pr	Al
0.95	0.75	0.90
0.95	0.90	0.75
0.55	0.75	0.90
0.55	0.90	0.75

on $[0, 1]$

$$x_i' = r x_i + s$$

$$r = 1/20,\ s = 0$$

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Assumptions

- St and Pr more important than Al
- St and Pr are somewhat substitutive
- $a \succ b$ and $d \succ c$ ($\Rightarrow$ nonadditive Choquet integral)

$$
\begin{array}{ll}
v(\varnothing) = 0 & v(\text{St, Pr}) = 0.50 \\
v(\text{St}) = 0.35 & v(\text{St, Al}) = 0.80 \\
v(\text{Pr}) = 0.35 & v(\text{Pr, Al}) = 0.80 \\
v(\text{Al}) = 0.30 & v(\text{St, Pr, Al}) = 1
\end{array}
$$

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Results

Student	St	Pr	Al	Global evaluation
a	19	15	18	17.75
b	19	18	15	16.85
c	11	15	18	15.10
d	11	18	15	15.25

Ranking

$$\boxed{a \succ b \succ d \succ c}$$

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Special types of interaction: veto and favor effects

A criterion $k \in N$ is

- a *veto* for $\mathcal{C}_v$ if

$$\mathcal{C}_v(x) \leqslant x_k \qquad (x \in [0,1]^n)$$

- a *favor* for $\mathcal{C}_v$ if

$$\mathcal{C}_v(x) \geqslant x_k \qquad (x \in [0,1]^n)$$

(Grabisch, 1997)

If k is both a veto and a favor then k is a *dictator*

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Surprising behavior of $\mathcal{C}_v$

Example of rule

$$\text{“ } x_k \leqslant 18/20 \quad \Rightarrow \quad \mathcal{C}_v(x) \leqslant 18/20 \text{ ”}$$
$$\Downarrow$$
$$k \text{ is a veto}$$

Proposition

1) k is a veto for $\mathcal{C}_v$ if and only if $\exists\, \lambda \in [0,1[$ s.t. $\forall x \in [0,1]^n$

$$x_k \leqslant \lambda \quad \Rightarrow \quad \mathcal{C}_v(x) \leqslant \lambda$$

2) k is a favor for $\mathcal{C}_v$ iff $\exists\, \lambda \in]0,1]$ s.t. $\forall x \in [0,1]^n$

$$x_k \geqslant \lambda \quad \Rightarrow \quad \mathcal{C}_v(x) \geqslant \lambda$$

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More surprising...

One can show that

$$k \text{ is a veto} \quad \Leftrightarrow \quad \exists\, x < y \ : \ \mathcal{C}_v(y,\ldots,y,\overset{(k)}{x},y\ldots,y) = x$$

$$k \text{ is a favor} \quad \Leftrightarrow \quad \exists\, x > y \ : \ \mathcal{C}_v(y,\ldots,y,\overset{(k)}{x},y\ldots,y) = x$$

Example

$$\mathcal{C}_v(10,11,11) = 10 \quad \Rightarrow \quad k = 1 \text{ is a veto}$$

$$\Rightarrow \ \mathcal{C}_v(10,20,20) = 10 \text{ !!}$$

$$\mathcal{C}_v(19,18,18) = 19 \quad \Rightarrow \quad k = 1 \text{ is a favor}$$

$$\Rightarrow \ \mathcal{C}_v(19,0,0) = 19 \text{ !!}$$

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Analysis

- Veto and favor criteria are rarely encountered
- Some criteria might behave nearly like a veto or a favor
- How can we measure the degree of veto (or favor) of criteria ?

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Average value of $\mathcal{C}_v$ over $[0,1]^n$:

$$E(\mathcal{C}_v) := \int_{[0,1]^n} \mathcal{C}_v(x)\, dx \qquad \in [0,1]$$

Examples

- $E(\min) = \frac{1}{n+1}$ (most intolerant)
- $E(\max) = \frac{n}{n+1}$ (most tolerant)
- $E(\mathrm{OS}_k) = \frac{k}{n+1}$
- $E(\mathrm{WAM}_\omega) = E(\mathrm{median}) = \frac{1}{2}$

$$E(\min) \leqslant E(\mathcal{C}_v) \leqslant E(\max)$$

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Conjunction and disjunction degrees

Position of $E(\mathcal{C}_v)$ within the interval $[E(\min), E(\max)]$:

$$\begin{aligned} \mathrm{orness}(\mathcal{C}_v) &:= \frac{E(\mathcal{C}_v) - E(\min)}{E(\max) - E(\min)} \\ \mathrm{andness}(\mathcal{C}_v) &:= \frac{E(\max) - E(\mathcal{C}_v)}{E(\max) - E(\min)} \end{aligned}$$

(Dujmović, 1974)

$$\mathrm{andness}(\mathcal{C}_v) + \mathrm{orness}(\mathcal{C}_v) = 1$$

Examples

- orness(min) $= 0$ (most intolerant)
- orness(max) $= 1$ (most tolerant)
- orness(WAM_ω) $=$ orness(median) $= \frac{1}{2}$

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$$\boxed{k \text{ is a veto for } \mathcal{C}_v \quad \Rightarrow \quad \text{orness}(\mathcal{C}_v) \leqslant 1/2 \quad \text{(not tolerant)}}$$

Proof:

$$\mathcal{C}_v(x) \leqslant x_k$$
$$\Downarrow$$
$$E(\mathcal{C}_v) = \int_{[0,1]^n} \mathcal{C}_v(x)\,dx \leqslant \int_{[0,1]^n} x_k\,dx = 1/2$$

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Veto and favor degrees

$$\begin{aligned}
\text{veto}(\mathcal{C}_v, k) \quad &:= \quad 1 - \tfrac{1}{n-1} \sum_{T \subseteq N \setminus \{k\}} \tfrac{(n-t-1)!\,t!}{(n-1)!}\, v(T) \\
\text{favor}(\mathcal{C}_v, k) \quad &:= \quad \tfrac{1}{n-1} \sum_{T \subseteq N \setminus \{k\}} \tfrac{(n-t-1)!\,t!}{(n-1)!}\, v(T \cup \{k\}) - \tfrac{1}{n-1}
\end{aligned}$$

Why these ugly definitions ?

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Properties of veto($\mathcal{C}_v, k$)

- **Linearity w.r.t. the capacity**
  $\exists\ p_T^k \in \mathbb{R}$ ($T \subseteq N$) s.t.

$$\mathrm{veto}(\mathcal{C}_v, k) = \sum_{T \subseteq N} v(T) p_T^k \qquad (k \in N,\ v \in \mathcal{F}_n)$$

- **Symmetry**
  For any permutation π on N

$$\mathrm{veto}(\mathcal{C}_v, k) = \mathrm{veto}(\mathcal{C}_{\pi v}, \pi(k)) \qquad (k \in N,\ v \in \mathcal{F}_n)$$

where $\pi v(\pi(S)) = v(S)$ for all $S \subseteq N$.

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- **Boundary conditions**
  For any $\varnothing \neq T \subseteq N$ we have

$$\mathrm{veto}(\min_T, k) = 1 \qquad \forall k \in T$$

$$(\min_{i \in T} x_i \leqslant x_k \qquad \forall k \in T)$$

- **Normalization**
  Given $v \in \mathcal{F}_n$

$$\mathrm{veto}(\mathcal{C}_v, 1) = \mathrm{veto}(\mathcal{C}_v, 2) = \cdots = \mathrm{veto}(\mathcal{C}_v, n)$$
$$\Downarrow$$
$$\mathrm{veto}(\mathcal{C}_v, i) = \mathrm{andness}(\mathcal{C}_v)$$

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Axiomatization

Consider a family of real numbers

$$\{\psi(\mathcal{C}_v, k) \in \mathbb{R} \mid k \in N,\ v \in \mathcal{F}_n\}$$

This family satisfies the four properties above if and only if

$$\psi(\mathcal{C}_v, k) = \mathrm{veto}(\mathcal{C}_v, k) \qquad (k \in N,\ v \in \mathcal{F}_n)$$

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Evaluation of students

$$\boxed{\mathrm{orness}(\mathcal{C}_v) = 0.517}$$

	St	Pr	Al
$\mathrm{veto}(\mathcal{C}_v, k)$	0.437	0.437	0.575
$\mathrm{favor}(\mathcal{C}_v, k)$	0.500	0.500	0.550
$\phi_k(v)$	0.292	0.292	0.417

Shapley coefficients:

$$\phi_k(v) := \sum_{T \subseteq N \setminus \{k\}} \frac{(n-t-1)!\, t!}{n!} \left[v(T \cup \{k\}) - v(T)\right]$$

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$$\frac{1}{n}\sum_{k=1}^{n}\text{veto}(\mathcal{C}_v,k) \quad = \quad \text{andness}(\mathcal{C}_v)$$

$$\frac{1}{n}\sum_{k=1}^{n}\text{favor}(\mathcal{C}_v,k) \quad = \quad \text{orness}(\mathcal{C}_v)$$

$$\frac{1}{n}\sum_{k=1}^{n}\phi_k(v) \quad = \quad \frac{1}{n}$$

	St	Pr	Al
$\text{veto}(\mathcal{C}_v,k)/\text{andness}(\mathcal{C}_v)$	0.905	0.905	1.190
$\text{favor}(\mathcal{C}_v,k)/\text{orness}(\mathcal{C}_v)$	0.968	0.968	1.064
$n\,\phi_k(v)$	0.875	0.875	1.250

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Conclusion

- We have introduced
  - global tolerance indices: *orness*, *andness*
  - local tolerance indices: *veto*, *favor*
- Behavioral parameters: importance, interaction, dispersion, *tolerance*...
- Identification of capacities: by optimization, learning data, constraints on behavioral parameters...

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Paper presented

J.-L. Marichal, Tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral, *European Journal of Operational Research*, to appear.

Preprint available at: `math.byu.edu/~marichal`