On tolerant or intolerant character of interacting criteria in aggregation by the Choquet integral

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Abstract

In many multi-criteria decision making problems the criteria present some interaction whose nature may vary from one situation to another. When a criterion bounds the global score from above, it is called a veto, due to its rather intolerant character. When it bounds the global score from below, it is then called a favor. In this paper we investigate the tolerance of criteria, or equivalently, the tolerance of the weighted aggregation operator (here the Choquet integral) which is used to aggregate criteria. More specifically, we propose (axiomatically) indices to appraise the extent to which each criterion behaves like a veto or a favor in the aggregation by the Choquet integral.

Keywords: multi-criteria decision-making; interacting criteria; Choquet integral.

1 Introduction

The use of the fuzzy measures in multi-criteria decision-making enables us to model some interaction phenomena existing among criteria; see [4, 6]. For example, when two criteria are positively correlated then the importance of these criteria, taken together, should be strictly less than the sum of the importances of the single criteria.

Of course, there are interaction phenomena whose nature is not linked to correlation. In [6] the author discusses the concepts of substitutiveness and complementarity, which simply represent the opinion of the decision maker on the relative importance of criteria, independently of the partial scores obtained by the alternatives along these criteria.

Another form of dependence among criteria corresponds to the presence of veto or favor criteria [4]. A criterion is a veto (resp. a favor) if the partial score of any alternative along this criterion bounds from above (resp. from below) the global score obtained by aggregation.

For example, consider the problem of evaluating students with respect to various courses (criteria) and suppose that there exists a veto (resp. favor) course. This means that the global grade obtained by any student cannot be greater than (resp. less than) the grade obtained at this course.

In this paper we analyze the tolerance degree of criteria. That is, we propose (axiomatically) veto and favor indices giving the extent to which a given criterion behaves like a veto or a favor. We also propose global indices, called *andness* and *orness* degrees, which measure the overall tolerance of criteria. These latter indices also give the degree to which the aggregation is conjunctive or disjunctive.

We assume here that criteria are all expressed on the same interval scale, and hence we aggregate them by means of the discrete Choquet integral, which has been proved to be the best suitable operator to aggregate interacting criteria defined on such a scale type; see [6]. Thus the main aim of this short paper is to investigate the average tolerance degree of criteria as well as the tolerance degree of each of them when they are aggregated by the Choquet integral.

Throughout this paper, the label set $N = \{1, \ldots, n\}$ represents the set of criteria of a given decision problem.

2 The Choquet integral

The use of the Choquet integral has been proposed by many authors as an adequate substitute to the weighted arithmetic mean to aggregate interacting criteria; see e.g. [3, 6]. In the weighted arithmetic mean model, each criterion $i \in N$ is given a weight $\omega_i \in [0,1]$ representing the importance of this criterion in the decision. In the Choquet integral model, where criteria can be dependent, a fuzzy measure [8] is used to define a weight on each combination of criteria, thus making it possible to model the interaction existing among criteria.

Definition 2.1 *A fuzzy measure on N is a set function $v : 2^N \to [0,1]$, which is nondecreasing with respect to set inclusion and such that $v(\varnothing) = 0$ and $v(N) = 1$.*

Throughout, we will denote by $\mathcal{F}_N$ the set of all fuzzy measures on N.

We now give the definition of the Choquet integral [1, 7].

Definition 2.2 *Let $v \in \mathcal{F}_N$. The (discrete) Choquet integral of $x : N \to \mathbb{R}$ with respect to v is defined by*

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)} \, [v(A_{(i)}) - v(A_{(i+1)})],$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq \ldots \leq x_{(n)}$. Also $A_{(i)} = \{(i), \ldots, (n)\}$, and $A_{(n+1)} = \varnothing$.

Thus defined, the Choquet integral has very good properties for aggregation (see e.g. Grabisch [3]). For instance, it is continuous, non decreasing, comprised between min and max, and coincides with the weighted arithmetic mean (discrete Lebesgue integral) as soon as the fuzzy measure is additive. Since it is also stable under the same transformations of interval scales, we can always restrict its domain to $[0,1]^n$ by means of a suitable affine transformation.

3 Conjunction and disjunction degrees

Consider the cube $[0,1]^n$ as a probability space with uniform distribution. Then the expected value of $\mathcal{C}_v(x)$, that is

$$\mathrm{E}(\mathcal{C}_v) := \int_{[0,1]^n} \mathcal{C}_v(x) \, \mathrm{d}x,$$

represents the *average value* of the Choquet integral $\mathcal{C}_v$ over $[0,1]^n$. This expression gives the average position of $\mathcal{C}_v$ within the interval $[0,1]$.

In [5], the author defined the conjunction degree or the degree of andness of $\mathcal{C}_v$ as the relative position of $\mathrm{E}(\mathcal{C}_v)$ with respect to the lower bound of the interval $[\mathrm{E}(\min), \mathrm{E}(\max)]$:

$$\mathrm{andness}(\mathcal{C}_v) := \frac{\mathrm{E}(\max) - \mathrm{E}(\mathcal{C}_v)}{\mathrm{E}(\max) - \mathrm{E}(\min)}$$

This value represents the degree to which the average value of $\mathcal{C}_v$ is close to that of "min". In some sense, it also reflects the extent to which $\mathcal{C}_v$ behaves like a minimum or has a conjunctive behavior.

Similarly, the relative position of $\mathrm{E}(\mathcal{C}_v)$ with respect to E(max) is called the disjunction degree or the degree of orness of $\mathcal{C}_v$:

$$\mathrm{orness}(\mathcal{C}_v) := \frac{\mathrm{E}(\mathcal{C}_v) - \mathrm{E}(\min)}{\mathrm{E}(\max) - \mathrm{E}(\min)}$$

It measures the degree to which $\mathcal{C}_v$ behaves like a maximum or has a disjunctive behavior.

Historically, these two concepts have been introduced as early as 1974 by Dujmović [2] in the particular case of the root-mean-power. Here we have simply applied his definitions to the Choquet integral. Actually, the concept of orness was also defined independently by Yager [9] in the particular case of OWA operators.

By definition, both andness($\mathcal{C}_v$) and orness($\mathcal{C}_v$) lie in the unit interval $[0, 1]$. Furthermore they fulfill the following property:

$$\mathrm{andness}(\mathcal{C}_v) + \mathrm{orness}(\mathcal{C}_v) = 1$$

The degree of orness is actually a measure of global tolerance of criteria or, equivalently, a measure of the tolerance of the decision maker. Indeed, tolerant decision makers can accept that only *some* criteria are satisfied. This corresponds to a disjunctive behavior ($\mathrm{orness}(\mathcal{C}_v) > 0.5$), whose extreme example is max. On the other hand, intolerant decision makers demand that *most* criteria are satisfied. This corresponds to a conjunctive behavior ($\mathrm{orness}(\mathcal{C}_v) < 0.5$), whose extreme example is min. When $\mathrm{orness}(\mathcal{C}_v) = 0.5$ the decision maker is medium (neither tolerant nor intolerant).

4 Veto and favor effects

Interesting behavioral phenomena in aggregation of criteria are the veto and favor effects [4].

A criterion $k \in N$ is said to be a *veto* for $\mathcal{C}_v$ if

$$\mathcal{C}_v(x) \leq x_k \qquad (x \in [0,1]^n).$$

This definition is motivated by the fact that the non satisfaction of a veto criterion should entail a low global score.

Similarly, the criterion k is a *favor* for $\mathcal{C}_v$ if

$$\mathcal{C}_v(x) \geq x_k \qquad (x \in [0,1]^n).$$

In this case, the satisfaction of criterion k entails necessarily a high global score.

Note that if the decision maker considers that a given criterion must absolutely be satisfied (veto criterion), then (s)he is conjunctive oriented. Similarly, if the decision maker considers that the satisfaction of a given criterion is sufficient (favor criteria) then (s)he is disjunctive oriented.

The following result provides equivalent conditions for k to be a veto for $\mathcal{C}_v$.

Proposition 4.1 *Let $k \in N$ and $v \in \mathcal{F}_N$. Then the following three assertions are equivalent:*

i) *k is a veto for $\mathcal{C}_v$*

ii) $v(N \setminus \{k\}) = 0$

iii) $\exists \lambda \in [0,1)$ *such that* $\forall x \in [0,1]^n$ *we have* $x_k \leq \lambda \Rightarrow \mathcal{C}_v(x) \leq \lambda$

The equivalence between *i)* and *iii)* in Proposition 4.1 is surprising. By imposing that $\mathcal{C}_v(x) \leq \lambda$ whenever $x_k \leq \lambda$ for a given threshold $\lambda \in [0,1)$, we necessarily consider k as a veto for $\mathcal{C}_v$. For instance, consider again the problem of evaluating students with respect to different subjects (courses) and suppose that the teacher of course k decides that if a student gets a mark less than 18/20 for course k then the global mark over all courses must be less than 18/20. In this case, this teacher has a veto behavior with respect to his/her colleagues.

Proposition 4.1 can be easily rewritten for favor criteria as follows:

Proposition 4.2 *Let $k \in N$ and $v \in \mathcal{F}_N$. Then the following three assertions are equivalent:*

i) k *is a favor for* $\mathcal{C}_v$

ii) $v(\{k\}) = 1$

iii) $\exists \lambda \in (0,1]$ *such that* $\forall x \in [0,1]^n$ *we have* $x_k \geq \lambda \Rightarrow \mathcal{C}_v(x) \geq \lambda$

Now, it seems sensible to define indices that measure the degree of veto or favor of a given criterion $j \in N$. In [5] the author introduced the following indices, based on an axiomatic characterization:

$$\mathrm{veto}(\mathcal{C}_v, j) := 1 - \frac{1}{n-1} \sum_{T \subseteq N \setminus \{j\}} \frac{(n-t-1)!\, t!}{(n-1)!} v(T)$$

$$\mathrm{favor}(\mathcal{C}_v, j) := \frac{1}{n-1} \sum_{T \subseteq N \setminus \{j\}} \frac{(n-t-1)!\, t!}{(n-1)!} v(T \cup \{j\}) - \frac{1}{n-1}$$

Although the form of these indices seems not very informative, the axiomatic that supports them is rather natural. We present it in the following theorem.

Theorem 4.1 *Consider a family of real numbers $\psi(\mathcal{C}_v, j)$ $(j \in N, v \in \mathcal{F}_N)$. These numbers*

- *are linear w.r.t. the fuzzy measures, that is, there exist real constants p_T^j $(T \subseteq N)$ such that*

$$\psi(\mathcal{C}_v, j) = \sum_{T \subseteq N} p_T^j\, v(T) \qquad (j \in N, v \in \mathcal{F}_N)$$

- *are symmetric, that is, for any permutation π on N, we have*

$$\psi(\mathcal{C}_v, j) = \psi(\mathcal{C}_{\pi v}, \pi(j)) \qquad (j \in N, v \in \mathcal{F}_N),$$

  where πv is the fuzzy measure of $\mathcal{F}_N$ defined by $\pi v(\pi(S)) = v(S)$ for all $S \subseteq N$.

- *fulfill the "boundary" axiom, that is, for any $T \subseteq N$, $T \neq \varnothing$, and any $j \in T$, we have*

$$\psi(\min_T, j) = 1, \quad (\textit{resp. } \psi(\max_T, j) = 1)$$

- *fulfill the "normalization" axiom, that is, for any $v \in \mathcal{F}_N$,*

$$\psi(\mathcal{C}_v, i) = \psi(\mathcal{C}_v, j) \quad \forall i, j \in N$$
$$\Downarrow$$
$$\psi(\mathcal{C}_v, j) = \mathrm{andness}(\mathcal{C}_v) \quad (\textit{resp. } \mathrm{orness}(\mathcal{C}_v)) \quad \forall j \in N$$

if and only if $\psi(\mathcal{C}_v, j) = \mathrm{veto}(\mathcal{C}_v, j)$ (resp. $\mathrm{favor}(\mathcal{C}_v, j)$) for all $j \in N$ and all $v \in \mathcal{F}_N$.

Let us comment on the axioms presented in this characterization. First, we ask the veto and favor indices to be linear with respect to the fuzzy measures. We also require that these indices be symmetric, that is, independent of the numbering of criteria. Next, the boundary axiom is motivated by the observation that any $j \in T$ is a veto (resp. favor) criterion for $\min_T$ (resp. $\max_T$), which is a particular Choquet integral defined by

$$\min_T(x) = \min_{i \in T} x_i \quad (\text{resp. } \max_T(x) = \max_{i \in T} x_i).$$

Finally, the normalization axiom says that if the degree of veto (resp. favor) does not depend on criteria, then it identifies with the degree of intolerance (resp. tolerance) of the decision maker.

Thus defined, we see that $\text{veto}(\mathcal{C}_v, j)$ is more or less the degree to which the decision maker demands that criterion j is satisfied. Similarly, $\text{favor}(\mathcal{C}_v, j)$ is the degree to which the decision maker considers that a good score along criterion j is sufficient to be satisfied.

It is easy to observe that $\text{veto}(\mathcal{C}_v, j) \in [0,1]$ and $\text{favor}(\mathcal{C}_v, j) \in [0,1]$. Furthermore, we have, for any $v \in \mathcal{F}_N$,

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n} \text{veto}(\mathcal{C}_v, j) &= \text{andness}(\mathcal{C}_v), \\ \frac{1}{n} \sum_{j=1}^{n} \text{favor}(\mathcal{C}_v, j) &= \text{orness}(\mathcal{C}_v). \end{aligned}$$

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