On Order Invariant Synthesizing Functions

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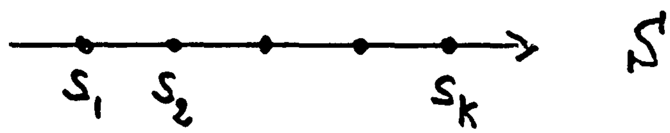
What is a finite ordinal scale ?

Two equivalent approaches :

- **Symbolical approach**

  Finite chain $(S, \preccurlyeq)$

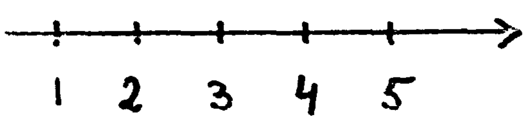
$$S = \{s_1 \prec s_2 \prec \cdots \prec s_k\}$$

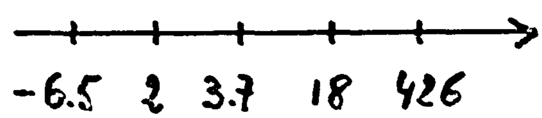


Example : evaluation of a product by a consumer :
$S = \{B \prec RB \prec A \prec MLG \prec G\}$

- **Numerical approach**

  Strictly increasing sequence of real numbers defined up to order





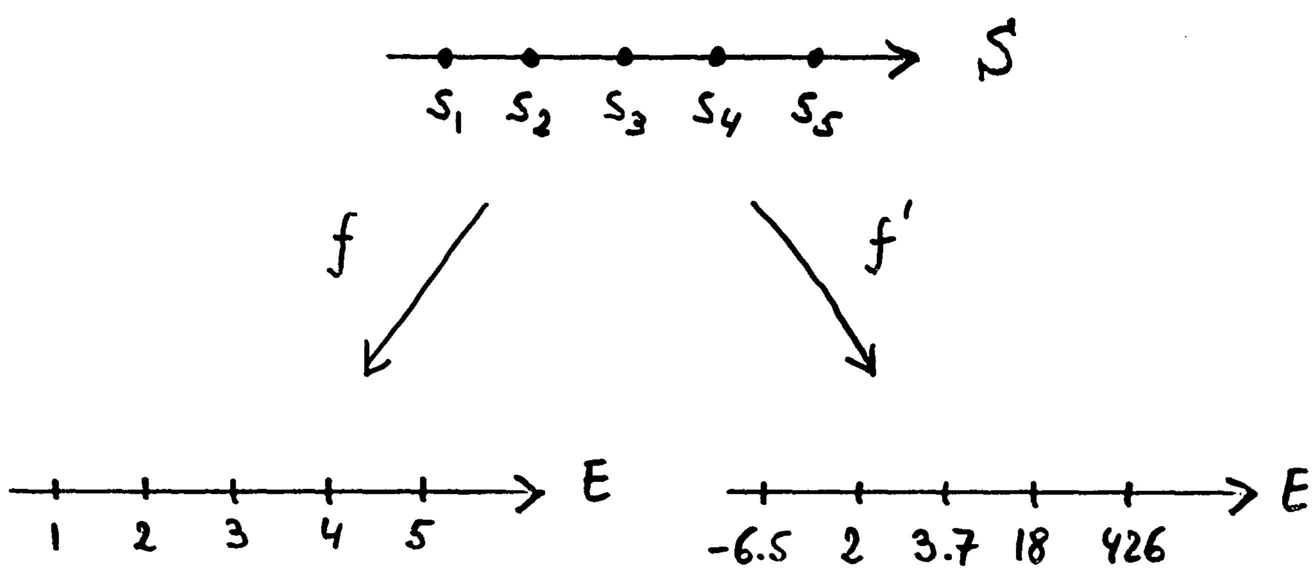
Equivalence $\to$ numerical representation of $\preccurlyeq$

Strictly increasing $f : S \to E \subseteq \mathbb{R}$

$$s_i \preccurlyeq s_j \quad \Leftrightarrow \quad f(s_i) \leqslant f(s_j)$$

f is defined within an automorphism ϕ of E :

$$f' = \phi \circ f$$



$A(E) =$ automorphism group of E

Generally $E = \mathbb{R}$ or $E =]0, 1[$

What if $E = [0, 1]$?

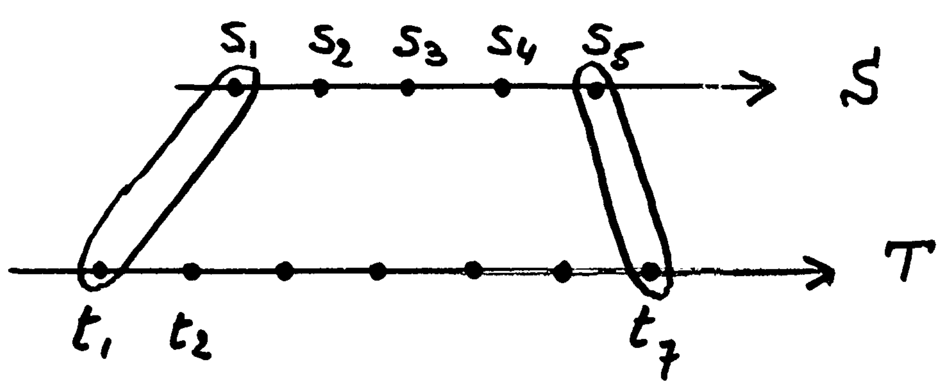
Let $(S, \preccurlyeq)$ be a finite ordinal scale. Then

$$f(s_i) = 0 \quad \Rightarrow \quad i = 1$$
$$f(s_i) = 1 \quad \Rightarrow \quad i = |S|$$

$\Rightarrow$ We assume that $f : S \to E$ is endpoint-preserving

Consider two independent ordinal scales $(S, \preccurlyeq_S)$ and $(T, \preccurlyeq_T)$

- If $E =]0, 1[$, those scales have nothing in common
- If $E = [0, 1]$, those scales have fixed end-points



Aggregation on finite ordinal scales

$$(a_1, \ldots, a_n) \in S^n \quad \mapsto \quad a \in S$$

First approach :

Define $G : S^n \to S$

$$(a_1, \ldots, a_n) \in S^n \quad \mapsto \quad G(a_1, \ldots, a_n) \in S$$

Example : $n = 2$ and $|S| = 3$

$a_2 \backslash a_1$	s_1	s_2	s_3
s_1	s_2	s_1	s_3
s_2	s_1	s_1	s_3
s_3	s_3	s_2	s_2

There are $|S|^{|S|^n} = 19\,683$ such functions G !

Second approach :

Scale independent function $M : E^n \to E$

$$(x_1, \ldots, x_n) \in E^n \quad \mapsto \quad M(x_1, \ldots, x_n) \in E$$

Example: $M =$ median

$$\text{median}(0.1, 0.3, 0.6) = 0.3$$

Definition (Marichal and Roubens 1993)
$M : E^n \to E$ *is an invariant function if, for any* $\phi \in A(E)$, *we have*

$$M[\phi(x_1), \ldots, \phi(x_n)] = \phi[M(x_1, \ldots, x_n)]$$

Examples:

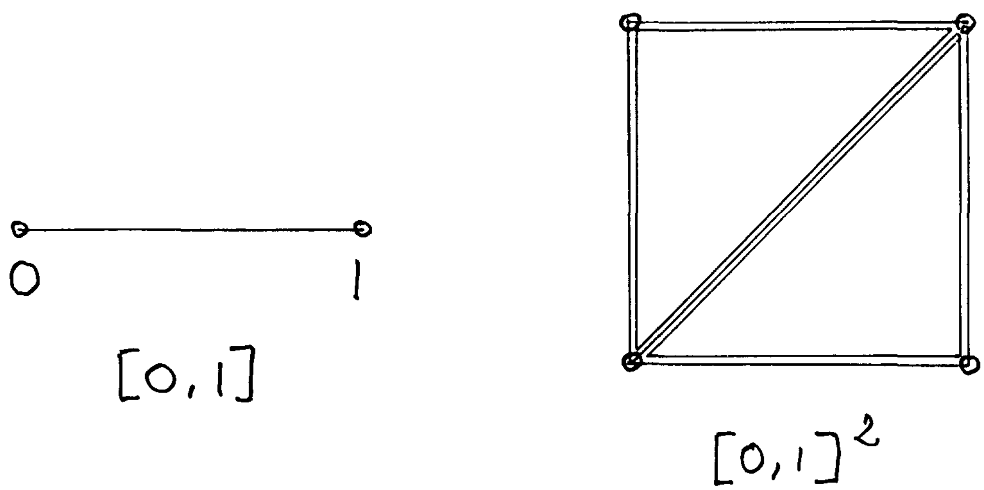
median : OK
arithmetic mean : No !

Description (Ovchinnikov 1998, Mesiar 2002)

1. $I \subseteq E^n$ is *invariant* if

$$x \in I \quad \Rightarrow \quad \phi(x) \in I \qquad \forall \phi \in A(E)$$

   I is *minimal invariant* if $\nexists J$ invariant and $J \subsetneq I$.

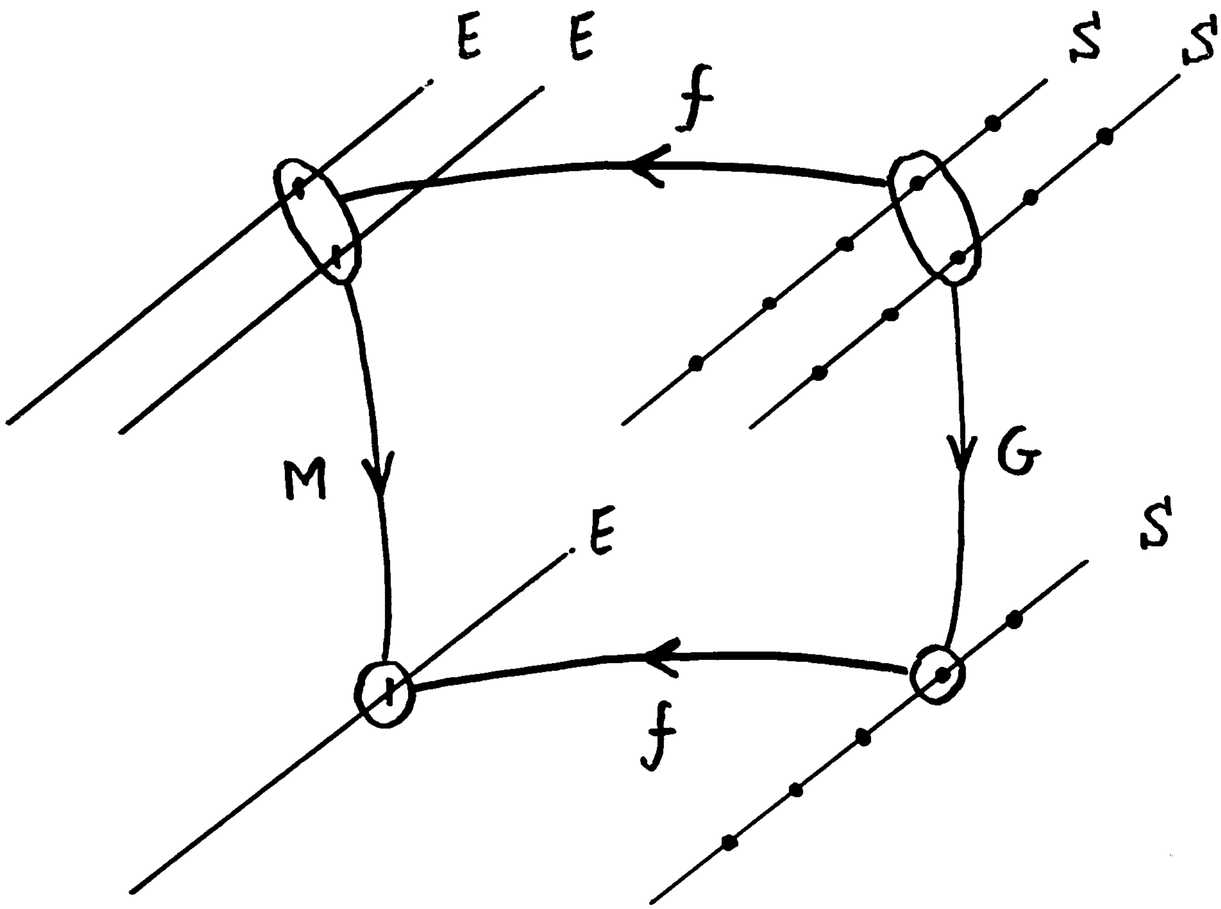


2. $M : E^n \to E$ is an invariant function iff, for any invariant $I \subseteq E^n$,

   - either $M|_I = \inf E$ (if $\inf E \in E$)
   - or $M|_I = \sup E$ (if $\sup E \in E$)
   - or $\exists\, i \in \{1, \ldots, n\}$ s.t. $M|_I = P_i$

Proposition (Marichal and Mesiar 2002)

$$M : E^n \to E \textit{ is invariant}$$
$$\Updownarrow$$
$$\forall\,(S,\preccurlyeq),\ \exists\, G : S^n \to S,\ \forall f : S \to E,$$
$$M[f(a_1),\ldots,f(a_n)] = f[G(a_1,\ldots,a_n)]$$

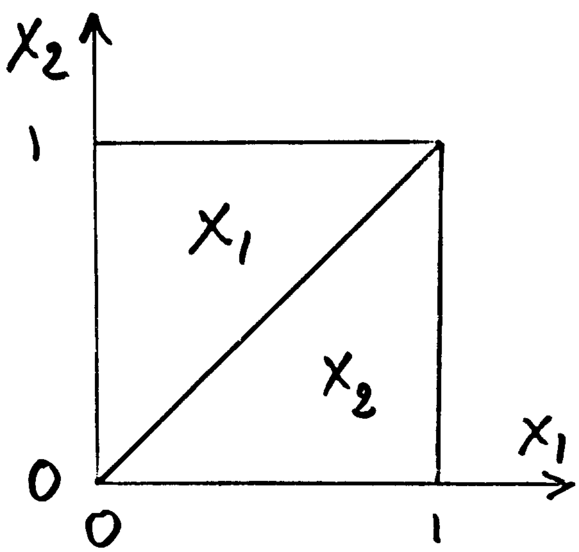


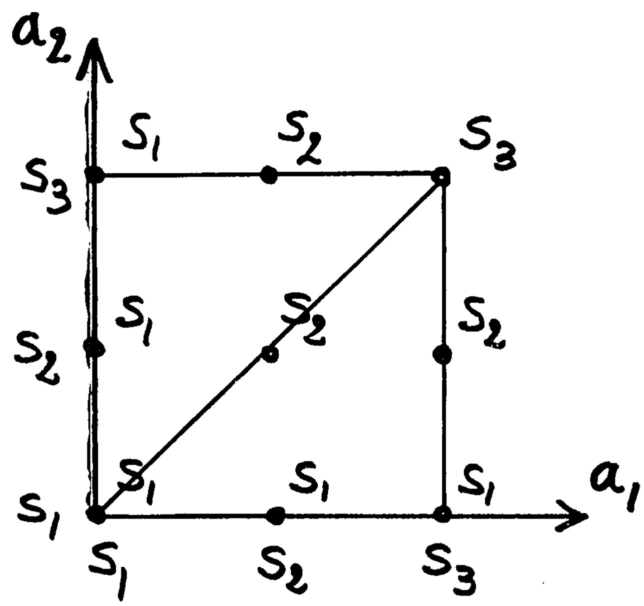
G represents M in $(S,\preccurlyeq)$

G is uniquely determined

Example:

$$M(x_1, x_2) = x_1 \wedge x_2 \quad \Rightarrow \quad G(a_1, a_2) = a_1 \curlywedge a_2$$





Remark:
Considering $G : S^n \to S$ is not equivalent to considering $M : E^n \to E$ invariant

Example: $n = 2$ and $E = \mathbb{R}$

- 4 invariant functions $M : \mathbb{R}^2 \to \mathbb{R}$
- $|S|^{|S|^2}$ discrete functions $G : S^2 \to S$

Continuous invariant functions

Description (Ovchinnikov 1998, Marichal 2002)
$M : E^n \to E$ is a continuous invariant function iff

- either $M \equiv \inf E \quad (\text{if } \inf E \in E)$
- or $M \equiv \sup E \quad (\text{if } \sup E \in E)$
- or $\exists$ a lattice polynomial $L : E^n \to E$ such that $M = L$

$$\left.\begin{array}{l}\text{continuity}\\ \text{order invariance}\end{array}\right\}\ \text{meaning ??}$$

Definition (Godo and Sierra 1988)
$G : S^n \to S$ is smooth if,
$\forall\, a, b \in S^n$, with $a = b$ except on ith coordinate where $|\text{ind}(a_i) - \text{ind}(b_i)| = 1$

$$\Rightarrow \quad \Big|\text{ind}[G(a)] - \text{ind}[G(b)]\Big| \leqslant 1$$

Example: $n = 2$ and $|S| = 3$

$a_2 \backslash a_1$	s_1	s_2	s_3
s_1	s_2	s_1	s_3
s_2	s_1	s_1	s_3
s_3	s_3	s_2	s_2

Not smooth !

Proposition (Marichal and Mesiar 2002)
An invariant function is continuous if and only if it is represented only by smooth discrete functions

A more general case

Two finite ordinal scales $(S, \preccurlyeq_S)$ and $(T, \preccurlyeq_T)$

$$(a_1, \ldots, a_n) \in S^n \quad \mapsto \quad a \in T$$

First approach :

Define $G : S^n \to T$

$$(a_1, \ldots, a_n) \in S^n \quad \mapsto \quad G(a_1, \ldots, a_n) \in T$$

Example: $n = 2$, $|S| = 3$, and $|T| = 5$

$a_2 \backslash a_1$	s_1	s_2	s_3
s_1	t_4	t_1	t_2
s_2	t_4	t_3	t_5
s_3	t_2	t_1	t_1

Second approach :
Scale independent function $M : E^n \to \mathbb{R}$

Definition (Orlov 1981)
$M : E^n \to \mathbb{R}$ is a comparison meaningful function if, for any $\phi \in A(E)$, there is a strictly increasing $\psi_\phi : \mathbb{R} \to \mathbb{R}$ such that

$$M[\phi(x_1), \ldots, \phi(x_n)] = \psi_\phi[M(x_1, \ldots, x_n)]$$

Description (Mesiar 2002)
$M : E^n \to \mathbb{R}$ is a comparison meaningful function if and only if, for any invariant $I \subseteq E^n$, there exist

$i)$ $i \in \{1, \ldots, n\}$
$ii)$ $g : \mathbb{R} \to \mathbb{R}$, *constant or str. monotonic*

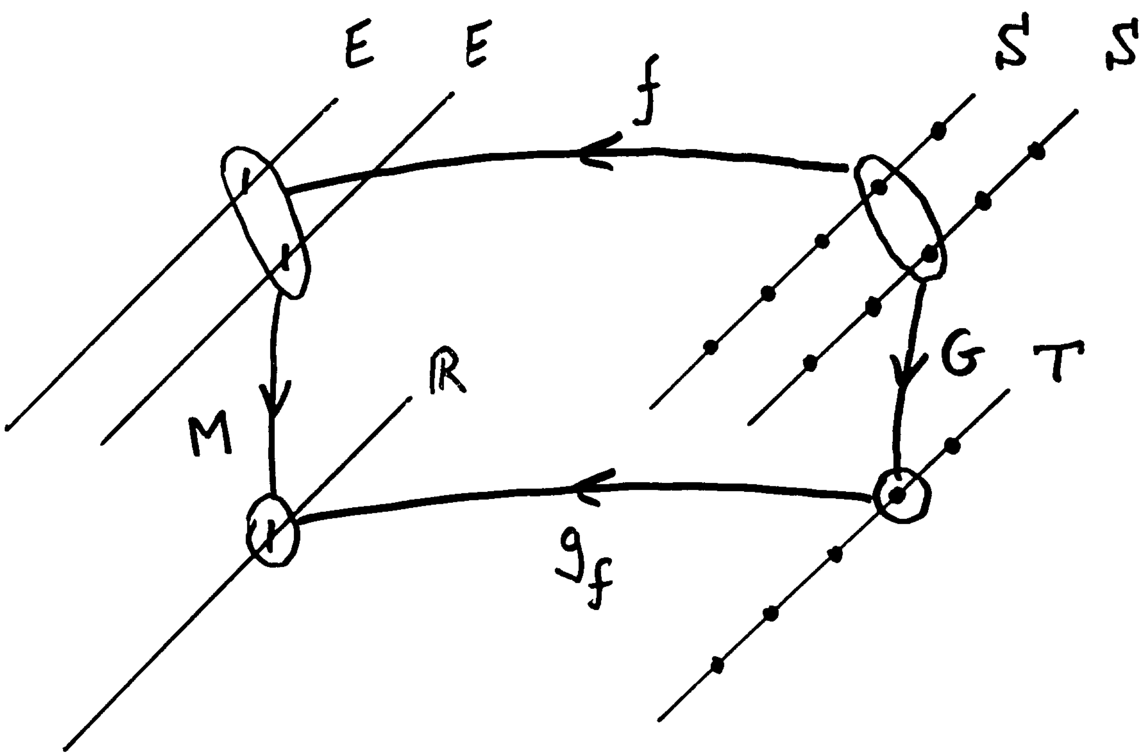
such that

$$M|_I = g \circ P_i$$

(+ some extra conditions)

Proposition (Marichal and Mesiar 2002)

$$M : E^n \to \mathbb{R} \text{ is comparison meaningful}$$
$$\Updownarrow$$
$$\forall\,(S,\preccurlyeq_S),\ \exists\,(T,\preccurlyeq_T)\ \ \&\ \ G : S^n \to T,$$
$$\forall\, f : S \to E,\ \exists\, g_f : T \to \mathbb{R},$$
$$M[f(a_1),\ldots,f(a_n)] = g_f[G(a_1,\ldots,a_n)]$$



G represents M in $(S,\preccurlyeq_S)$

G and g_f are uniquely determined

Continuous comparison meaningful functions

Description (Yanovskaya 1989, Marichal 2002)
$M : E^n \to \mathbb{R}$ is a continuous comparison meaningful function if and only if there exist

$i)$ $L : E^n \to E$, lattice polynomial
$ii)$ $g : E \to \mathbb{R}$, constant
or continuous and str. monotonic

such that

$$M = g \circ L$$

Proposition (Marichal and Mesiar 2002)
A continuous comparison meaningful function is represented only by smooth discrete functions

The converse is false !!!