Filtrage non linéaire de diffusions faiblement bruitées dans le cas d'un système avec bruits corrélés

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Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ a right-continuous increasing family of sub-σ-algebras of $\mathcal{F}$.

Let w and v be two independent $\mathcal{F}_t$- Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^m$.

If x_t is a semimartingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\circ dx_t$ (respectively dx_t) denotes its Stratonovitch (respectively Itô) differential.

Let us consider the nonlinear filtering problem associated with the system signal/observation pair $(x_t, y_t) \in (\mathbb{R}^m)^2$ solution of the following stochastic differential system:

$$\begin{cases} x_t = x_0 + \int_0^t b(x_s)\,ds + \int_0^t \sigma(x_s)\,dw_s + \int_0^t g(x_s)\,dv_s \\ \\ y_t = \int_0^t h(x_s)\,ds + \varepsilon\, v_t, \end{cases} \tag{1}$$

verifying the following hypotheses:

(H_1) x_0 is an $\mathbb{R}^m$-valued, $\mathcal{F}_0$-measurable random variable independent of w and v with finite moments of all orders.

(H_2) b and h are $\mathcal{C}^2(\mathbb{R}^m, \mathbb{R}^m)$ functions with bounded first and second derivatives.

(H_3) σ and g are bounded $\mathcal{C}^2(\mathbb{R}^m, \mathbb{R}^m \otimes \mathbb{R}^d)$ respectively $\mathcal{C}^2(\mathbb{R}^m, \mathbb{R}^m \otimes \mathbb{R}^m)$ functions with bounded first derivatives.

(H_4) The function $a = \sigma\sigma^\tau + gg^\tau$ is uniformly elliptic.

(H_5) The functions $a^{-\frac{1}{2}}b$ and $h'b$ are Lipschitzian.

Definition 1 *For all t in $[0,T]$ denote by π_t the filter associated with the system* (1), *defined for all functions ψ in $\mathcal{C}_b(\mathbb{R}^m,\mathbb{R})$ by*

$$\pi_t\psi = E\big[\psi(x_t)/\mathcal{Y}_t\big], \tag{2}$$

where $\mathcal{Y}_t = \sigma(y_s\,/\,0 \le s \le t)$.

Moreover we consider the following class of suboptimal filters:

$$m_t = m_0 + \int_0^t b(m_s)\,ds + \frac{1}{\varepsilon}\int_0^t h'^{-1}(m_s)\,K_s\,\big(dy_s - h(m_s)\,ds\big), \tag{3}$$

where $m_0 \in \mathbb{R}^m$ is arbitrary and $\{K_t,\, t \le 0\}$ is a $\mathcal{Y}_t$-progressively measurable bounded process such that for all (t,w) in $[0,T]\times\Omega$, $K_t(w)$ is a uniformly elliptic bounded function.

Remarks:

(i) The definition of the sub-optimal filters implies that the signal and the observation are of the same dimension.

(ii) In the following, if f is a vectorial function of the variable $x \in \mathbb{R}^m$, f' will denote the Jacobian matrix of f; if f is scalar, f' will be a line vector; this notation is extended to functions which may also depend upon other parameters.

Proposition 2 *For each $t_0 > 0$ and $p \geq 1$, we have:*

$$\sup_{t \geq t_0} \|x_t - m_t\|_p = O(\sqrt{\varepsilon}) \tag{4}$$

Proof:

Itô's formula implies that

$$h(x_t) = h(x_0) + \int_0^t Lh(x_s)ds + \int_0^t (h'\sigma)(x_s)dw_s + \int_0^t (h'g)(x_s)dv_s,$$

where L is the second order differential operator defined for any function f in $\mathcal{C}^2(\mathbb{R}^m, \mathbb{R}^m)$ by

$$(Lf)^l(x) = b^i(x)\frac{\partial f^l}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{d} \big(\sigma(x)\big)^i_k \big(\sigma(x)\big)^j_k \frac{\partial^2 f^l}{\partial x_i \partial x_j}(x) + \frac{1}{2}\sum_{k=1}^{m} \big(g(x)\big)^i_k \big(g(x)\big)^j_k \frac{\partial^2 f^l}{\partial x_i \partial x_j}(x).$$

Likewise,

$$h(m_t) = h(m_0) + \int_0^t \tilde{L}_s h(m_s)ds + \frac{1}{\varepsilon}\int_0^t K_s\big(h(x_s) - h(m_s)\big)ds + \int_0^t K_s dv_s,$$

where $\tilde{L}_t$ is the second order differential operator defined for any function f in $\mathcal{C}^2(\mathbb{R}^m, \mathbb{R}^m)$ by

$$(\tilde{L}_t f)^l(x) = b^i(x)\frac{\partial f^l}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{m} \big(h'^{-1}(x)K_t\big)^i_k \big(h'^{-1}(x)K_t\big)^j_k \frac{\partial^2 f^l}{\partial x_i \partial x_j}(x).$$

Hence,

$$\begin{aligned} h(x_t) - h(m_t) = h(x_0) - h(m_0) &+ \int_0^t \big(Lh(x_s) - \tilde{L}_s h(m_s)\big)\, ds + \int_0^t (h'\sigma)(x_s)\, dw_s \\ &+ \int_0^t (h'g)(x_s)\, dv_s - \frac{1}{\varepsilon}\int_0^t K_s dv_s - \frac{1}{\varepsilon}\int_0^t K_s\big(h(x_s) - h(m_s)\big)ds. \end{aligned}$$

Since K_t is uniformly elliptic, we can prove (4) as in [11].

By applying Itô's formula, we compute $|x_t - m_t|^k$ for even integers k, then, having taken the expectation of the two members of the obtained relation, we use some estimates on ordinary inequations.

□

Let γ be the function defined by $\gamma = \left(h'(\sigma\sigma^\tau + gg^\tau)h'^\tau\right)^{\frac{1}{2}}$.

For any t in $[0,T]$, we set

$$K_t = \gamma(m_t). \tag{5}$$

Theorem 3 *For any $t_0 > 0$ and $p \geq 1$, we have:*

$$\sup_{t \geq t_0} \|m_t - \pi_t I\|_p = O(\varepsilon). \tag{6}$$

Proof:

Consider the process

$$\overline{w}_t = \int_0^t (\gamma^{-1}h'\sigma)(x_s)\, dw_s + \int_0^t (\gamma^{-1}h'g)(x_s)\, dv_s. \tag{7}$$

Levy's theorem then implies that $\overline{w}_t$ is an $\mathbb{R}^m$-valued $(\mathcal{F}_t, P)$- Brownian motion and

$$dx_t = b(x_t)dt + h'^{-1}\gamma(x_t)\, d\overline{w}_t. \tag{8}$$

For any t in $[0,T]$, set

$$Z_t = \exp\Big(\frac{1}{\varepsilon}\int_0^t x_s^\tau \, dy_s - \frac{1}{2\varepsilon^2}\int_0^t |x_s|^2 \, ds\Big) \tag{9}$$

and

$$\Lambda_t = \exp\Big(-\frac{1}{\varepsilon}\int_0^t \big(h(x_s)-h(m_s)\big)^\tau \, d\overline{w}_s + \frac{1}{2\varepsilon^2}\int_0^t |h(x_s)-h(m_s)|^2 \, ds\Big). \tag{10}$$

Novikov's criterion then implies that the processes Z_t^{-1} and Λ_t^{-1} are exponential martingales. So we can apply Girsanov's theorem and define some reference probability measures which allow us to show the desirated estimates via Kallianpur-Striebel's formula.

Let us define the probability measures $\mathring{P}$ and $\tilde{P}$ by the Radon-Nicodym derivatives

$$\frac{d\mathring{P}}{dP}\bigg|_{\mathcal{F}_t} = Z_t^{-1}, \tag{11}$$

and

$$\frac{d\tilde{P}}{d\mathring{P}}\bigg|_{\mathcal{F}_t} = \Lambda_t^{-1}. \tag{12}$$

Hence, by Girsanov's theorem, under $\tilde{P}$, $\tilde{w}_t = \overline{w}_t - \frac{1}{\varepsilon}\int\limits_0^t \big(h(x_s)-h(m_s)\big)\,ds$ and $y_{\frac{t}{\varepsilon}}$ are two independent standard Brownian motions.

$$dx_t = \frac{1}{\varepsilon}\,(h'^{-1}\gamma)(x_t)\big(h(x_t)-h(m_t)\big)\,dt + b(x_t)\,dt + (h'^{-1}\gamma)(x_t)\,d\tilde{w}_t. \tag{13}$$

Let F be the function defined for each x, m in $\mathbb{R}^m$, by

$$F(x,m) = \big(h(x) - h(m)\big)^\tau \gamma^{-1}(m)\big(h(x) - h(m)\big). \tag{14}$$

Then, Itô's formula implies that

$$\begin{aligned} F(x_t, m_t) = F(x_0, m_0) &+ 2\int_0^t \big(h(x_s) - h(m_s)\big)^\tau \gamma^{-1}(m_s) h'(x_s) dx_s \\ &+ \int_0^t \big(h(x_s) - h(m_s)\big)^\tau \frac{\partial \gamma^{-1}}{\partial m_i}(m_s)\big(h(x_s) - h(m_s)\big)\, dm_s^i \\ &- 2\int_0^t \big(h(x_s) - h(m_s)\big)^\tau (\gamma^{-1}h')(m_s)\, dm_s + \int_0^t (A_x F + A_{x,m}(F) + A_m F)(x_s, m_s)\, ds, \end{aligned}$$

where A_x, $A_{x,m}$ and A_m are the second order differential operators defined for any function f in $\mathcal{C}^2\big((\mathbb{R}^m)^2\big)$ by

$$A_x f(x,m) = \frac{1}{2}\sum_{k=1}^{d} \big(\sigma(x)\big)_k^i \big(\sigma(x)\big)_k^j \frac{\partial^2 f}{\partial x_i \partial x_j}(x,m) + \frac{1}{2}\sum_{k=1}^{m} \big(g(x)\big)_k^i \big(g(x)\big)_k^j \frac{\partial^2 f}{\partial x_i \partial x_j}(x,m),$$

$$A_{x,m} f(x,m) = \frac{1}{2}\sum_{k=1}^{d} \big(\gamma(m)\big)_k^i \big(\sigma(x)\big)_k^j \frac{\partial^2 f}{\partial m_i \partial x_j}(x,m)$$

and

$$A_m f(x,m) = \frac{1}{2}\sum_{k=1}^{d} \big(\gamma(x)\big)_k^i \big(\gamma(x)\big)_k^j \frac{\partial^2 f}{\partial m_i \partial m_j}(x,m).$$

$$
\begin{aligned}
F(x_t,m_t) = F(x_0,m_0) &- 2\int_0^t \big(h(x_s)-h(m_s)\big)^\tau \big(\gamma^{-1}(x_s)-\gamma^{-1}(m_s)\big)h'(x_s)dx_s \\
&+ 2\int_0^t \big(h(x_s)-h(m_s)\big)^\tau\, d\overline{w}_s - \frac{2}{\varepsilon}\int_0^t \big(h(x_s)-h(m_s)\big)^\tau \big[dy_s - h(m_s)ds\big] \\
&+ 2\int_0^t \big(h(x_s)-h(m_s)\big)^\tau \big((\gamma^{-1}h'b)(x_s)-(\gamma^{-1}h'b)(m_s)\big)\, ds \\
&+ \int_0^t (A_xF + A_{x,m}F + A_mF)(x_s,m_s)\, ds \\
&+ \int_0^t \big(h(x_s)-h(m_s)\big)^\tau \frac{\partial\gamma^{-1}}{\partial m_i}(m_s)\big(h(x_s)-h(m_s)\big)\, dm_s^i.
\end{aligned}
$$

So, we finally have,

$$
\begin{aligned}
Z_t\Lambda_t = \exp\Big(&-\frac{1}{2\varepsilon}\big(F(x_t,m_t)-F(x_0,m_0)\big) + \frac{1}{\varepsilon}\int_0^t \chi_1(x_s,m_s)\, ds \\
&+\frac{1}{\varepsilon}\int_0^t \chi_2^\tau(x_s,m_s)\, dm_s + \frac{1}{\varepsilon}\int_0^t \chi_3^\tau(x_s,m_s)\, dx_s \\
&+\frac{1}{\varepsilon^2}\int_0^t h^\tau(m_s)\, dy_s - \frac{1}{2\varepsilon^2}\int_0^t |h(m_s)|^2\, ds\Big),
\end{aligned} \tag{15}
$$

with

$$
\begin{aligned}
\chi_1(x,m) &= \big(h(x)-h(m)\big)^\tau \big((\gamma^{-1}h'b)(x)-(\gamma^{-1}h'b)(m)\big) \\
&\qquad\qquad + \frac{1}{2}(A_xF + A_{x,m}F + A_mF)(x,m), \\
\chi_2^i(x,m) &= \frac{1}{2}\big(h(x)-h(m)\big)^\tau \frac{\partial\gamma^{-1}}{\partial m_i}(m)\big(h(x)-h(m)\big), \quad i=1,...,m \\
\chi_3(x,m) &= -(h')^\tau(x)\big(\gamma^{-1}(x)-\gamma^{-1}(m)\big)^\tau\big(h(x)-h(m)\big).
\end{aligned}
$$

In the following, if f denotes a function of (x,m), we denote $\frac{\partial f}{\partial x}$ by f'.

By the rules of the Malliavin Calculus, if $\overline{D}$ (respectively $\tilde{D}$) denotes the derivation operator in the direction of $\overline{w}$ (respectively $\tilde{w}$), (13) implies that for all $0 \le s \le t$,

$$\tilde{D}_s x_t = \zeta_{st}\,(h'^{-1}\gamma)(x_s), \tag{16}$$

where $\{\zeta_{st},\, t \ge s\}$ is the solution of the stochastic differential equation

$$\begin{aligned}\zeta_{st} = 1 &+ \frac{1}{\varepsilon}\int_s^t \zeta_{sr}\,(h'^{-1}\gamma)'(x_r)\big(h(x_r)-h(m_r)\big)\,dr + \frac{1}{\varepsilon}\int_s^t \zeta_{sr}(h'^{-1}\gamma h')(x_r)\,dr \\ &+ \int_s^t \zeta_{sr}\,b'(x_r)\,dr + \int_s^t \zeta_{sr}\,(h'^{-1}\gamma)'(x_r)\,d\tilde{w}_r + \int_s^t \zeta_{sr}\,g'(x_r)\,dv_r.\end{aligned} \tag{17}$$

Since

$$\tilde{D}_s y_t = \tilde{D}_s m_t = 0 \tag{18}$$

it follows from (15) and the chain rule that

$$\begin{aligned}\tilde{D}_s\,\log(Z_t\Lambda_t) = &-\frac{1}{2\varepsilon}F'(x_t,m_t)\tilde{D}_s x_t + \frac{1}{\varepsilon}\int_s^t \chi_1'(x_r,m_r)\tilde{D}_s x_r\,dr \\ &+\frac{1}{\varepsilon}\int_s^t dm_r^\tau \chi_2'(x_r,m_r)\tilde{D}_s x_r + \frac{1}{\varepsilon}\int_s^t dx_r^\tau \chi_3'(x_r,m_r)\tilde{D}_s x_r \\ &+\frac{1}{\varepsilon}\int_s^t \chi_3(x_r,m_r)\tilde{D}_s x_r\,dr.\end{aligned}$$

By integrating that last equality between 0 and t, we get

$$\begin{aligned}\frac{1}{2}F'(x_t,m_t) = & -\frac{\varepsilon}{t}\int_0^t \tilde{D}_s \,\log(Z_t\Lambda_t)(\tilde{D}_s x_t)^{-1}\,ds \\ & +\frac{1}{t}\int_0^t\int_s^t \chi_1'(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\,dr\,ds \\ & +\frac{1}{t}\int_0^t\int_s^t dm_r^\tau\,\chi_2'(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\,ds \\ & +\frac{1}{t}\int_0^t\int_s^t dx_r^\tau\,\chi_3'(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\,ds \\ & +\frac{1}{t}\int_0^t\int_s^t \chi_3(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\,dr\,ds\end{aligned} \tag{19}$$

On the other hand, by the definition of the function F,

$$\frac{1}{2}F'(x_t,m_t) = \big(h(x_t)-h(m_t)\big)^\tau\,\gamma^{-1}(m_t)h'(x_t), \tag{20}$$

so

$$\begin{aligned}E\Big[\frac{1}{2}F'(x_t,m_t)/\mathcal{Y}_t\Big] = & \,E[\big(h(x_t)-h(m_t)\big)^\tau/\mathcal{Y}_t]\,(\gamma^{-1}h')(m_t) \\ & +E[\big(h(x_t)-h(m_t)\big)^\tau\gamma^{-1}(m_t)\big(h'(x_t)-h'(m_t)\big)/\mathcal{Y}_t].\end{aligned}$$

Consequently, proposition 1.2 implies that

$$E\Big[\frac{1}{2}F'(x_t,m_t)/\mathcal{Y}_t\Big] = E[\big(h(x_t)-h(m_t)\big)^\tau/\mathcal{Y}_t]\,(\gamma^{-1}h')(m_t)+O(\varepsilon).$$

Hence,it follows from the assumptions (H_3) and (H_4) that the theorem will be proved if we show that for any $t_0>0$, $p\geq 1$,

$$\sup_{t\geq t_0}\big\|E\big[\frac{1}{2}F'(x_t,m_t)/\mathcal{Y}_t\big]\big\|_p = O(\varepsilon). \tag{21}$$

So, by equality (19) it suffices to show

$$(i) \quad \sup_{t\geq t_0} \frac{1}{t} \big\| E\Big[\int_0^t \tilde{D}_s \, \log(Z_t\Lambda_t)(\tilde{D}_s x_t)^{-1}\, ds/\mathcal{Y}_t\Big]\big\|_p \leq c_p,$$

$$(ii) \quad \sup_{t\geq t_0} \frac{1}{t} \big\| E\Big[\int_0^t\int_s^t \chi_1'(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\, dr\, ds/\mathcal{Y}_t\Big]\big\|_p = O(\varepsilon),$$

$$(iii) \quad \sup_{t\geq t_0} \frac{1}{t} \big\| E\Big[\int_0^t\int_s^t dm_r^\tau\, \chi_2'(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\, ds/\mathcal{Y}_t\Big]\big\|_p = O(\varepsilon),$$

$$(iv) \quad \sup_{t\geq t_0} \frac{1}{t} \big\| E\Big[\int_0^t\int_s^t dx_r^\tau\, \chi_3'(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\, ds/\mathcal{Y}_t\Big]\big\|_p = O(\varepsilon),$$

$$(v) \quad \sup_{t\geq t_0} \frac{1}{t} \big\| E\Big[\int_0^t\int_s^t \chi_3(x_r,m_r)\tilde{D}_s x_r(\tilde{D}_s x_t)^{-1}\, dr\, ds/\mathcal{Y}_t\Big]\big\|_p = O(\varepsilon).$$

Lemma 4 *For each $\varepsilon > 0$, $p \geq 1$, there exist strictly positive constants $a(p)$ and $\tilde{a}(p)$ such that*

$$\|\zeta_{ts}\|_p \leq a(p)\, \exp\Big[-\frac{\tilde{a}(p)}{\varepsilon}(t-s)\Big], \quad 0 \leq s \leq t. \tag{22}$$

Proof:

By (17) and the definition of $\tilde{w}_t$, $\{\zeta_{st},\, t \geq s\}$ is the unique solution of the stochastic differential equation

$$\zeta_{st} = 1 + \frac{1}{\varepsilon}\int_s^t \zeta_{sr}(h'^{-1}\gamma h')(x_r)\, dr + \int_s^t \zeta_{sr}\, b'(x_r)\, dr + \sum_{i=1}^m \int_s^t \zeta_{sr}\, \frac{\partial}{\partial x_i}(h'^{-1}\gamma)(x_r)\, d\overline{w}_r^i. \tag{23}$$

Hence,

$$\begin{aligned} E \sup_{t\varepsilon[0,T]} |\zeta_{st}|^p &\leq E\Big\{1 + \sup_{t\varepsilon[0,T]}\Big(\big|\frac{1}{\varepsilon}\int_s^t \zeta_{sr}(h'^{-1}\gamma h')(x_r)\, dr\big|^p + \big|\int_s^t \zeta_{sr}\, b'(x_r)\, dr\big|^p \\ &\qquad + \big|\sum_{i=1}^m \int_s^t \zeta_{sr}\, \frac{\partial}{\partial x_i}(h'^{-1}\gamma)(x_r)\, d\overline{w}_r^i\big|^p \\ &\leq c\Big\{E\Big[\frac{1}{\varepsilon}\int_s^T \big|\zeta_{sr}(h'^{-1}\gamma h')(x_r)\big|^p dr\Big] + E\Big[\int_s^T \big|\zeta_{sr}\, b'(x_r)\big|^p dr\Big] \\ &\qquad + E\Big(\sum_{i=1}^m \int_s^T \big|\zeta_{sr}\, \frac{\partial}{\partial x_i}(h'^{-1}\gamma)(x_r)\big|^2 dr\Big)^{\frac{p}{2}}\Big\}, \end{aligned}$$

by Burkhölder's inequatlity. Since γ is hypoelliptic, the assumptions $(H_2) - (H_4)$ imply that there exist some strictly positive constants c et c' independant of ε such that

$$\|\zeta_{st}\|_p \leq \frac{c}{\varepsilon}\int_s^T \|\zeta_{sr}\|_p\, dr + c'\int_s^T \|\zeta_{sr}\|_p\, dr. \tag{24}$$

The lemma then follows from Gronwall's theorem and the fact that $\zeta_{ts} = \zeta_{st}^{-1}$. □

Lemma 5 *For any $p \geq 1$, there exists a constant $c(p)$ such that*

$$\|\overline{D}_s \zeta_{t0}\|_p \leq c(p), \tag{25}$$

and

$$\|\tilde{D}_s \zeta_{t0}\|_p \leq c(p). \tag{26}$$

Proof:

By means of relation (25), we get for any t in $[0, T]$,

$$\zeta_{0t} = 1 + \frac{1}{\varepsilon} \int_0^t \zeta_{0r} (h'^{-1} \gamma h')(x_r)\, dr + \int_0^t \zeta_{0r}\, b'(x_r)\, dr + \sum_{i=1}^m \int_0^t \zeta_{0r} \frac{\partial}{\partial x_i} (h'^{-1}\gamma)(x_r)\, d\overline{w}_r^i. \tag{27}$$

Itô's formula then implies

$$\begin{aligned}
\zeta_{t0} = 1 &- \frac{1}{\varepsilon} \int_0^t (h'^{-1}\gamma h')(x_r)\, \zeta_{r0} dr - \int_0^t b'(x_r)\, \zeta_{r0}\, dr \\
&- \sum_{i=1}^m \int_0^t \frac{\partial}{\partial x_i} (h'^{-1}\gamma)(x_r)\, \zeta_{r0}\, d\overline{w}_r^i + \sum_{i=1}^m \int_0^t \Big(\frac{\partial}{\partial x_i} (h'^{-1}\gamma h') \big(\frac{\partial}{\partial x_i} (h'^{-1}\gamma h') \big)^\tau (x_r) \Big) \zeta_{r0}\, dr.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\overline{D}_s \zeta_{ts} = 1 &- \frac{1}{\varepsilon} \int_0^t (h'^{-1}\gamma h')'(x_r)\, \zeta_{r0}\, dr - \frac{1}{\varepsilon} \int_0^t (h'^{-1}\gamma h')(x_r)\, \overline{D}_s \zeta_{r0}\, dr \\
&- \int_0^t b''(x_r)\, \zeta_{r0}\, dr - \int_0^t b'(x_r)\, \overline{D}_s \zeta_{r0}\, dr \\
&+ \sum_{i=1}^m \int_0^t \Big(\frac{\partial}{\partial x_i} (h'^{-1}\gamma h') \big(\frac{\partial}{\partial x_i} (h'^{-1}\gamma h') \big)^\tau (x_r) \Big)' \zeta_{r0}\, dr \\
&+ \sum_{i=1}^m \int_0^t \Big(\frac{\partial}{\partial x_i} (h'^{-1}\gamma h') \big(\frac{\partial}{\partial x_i} (h'^{-1}\gamma h') \big)^\tau (x_r) \Big) \overline{D}_s \zeta_{r0}\, dr.
\end{aligned}$$

By using lemma 1.4., the assumptions $(H_2) - (H_4)$ and Burkhölder's inequality, one then can show like in the proof of the previous lemma that there exist some strictly positive constants c and c' such that

$$\|\overline{D}_s \zeta_{t0}\|_p \leq c' + c \int_0^t \|\overline{D}_s \zeta_{r0}\|_p\, dr \tag{28}$$

Hence relation (30). Relation (31) can be obtained by similar computations.

□

Let us now pass to the proof of the expressions (i) to (vii).

Some computations imply

$$\frac{1}{t}E\Big[\int_0^t \tilde{D}_s \log(Z_t\Lambda_t)(\tilde{D}_s x_t)^{-1}\,ds/\mathcal{Y}_t\Big] = \frac{1}{t}E\Big[\int_0^t (\gamma^{-1}h')(x_s)\zeta_{0s}\big(\overline{D}_s\zeta_{t0} - \tilde{D}_s\zeta_{t0}\big)\,ds/\mathcal{Y}_t\Big]$$
$$-\frac{1}{\varepsilon t}E\Big[\int_0^t \big(h(x_s) - h(m_s)\big)\zeta_{ts}(\gamma^{-1}h')\,ds/\mathcal{Y}_t\Big].$$

and (i) then follows from the lemmas 4 and 5.

Moreover, the boundedness of the functions $\|\chi_1'\|_p$ and γ respectively lemma 1.5 imply that

$$\begin{aligned}\int_0^t\int_s^t \chi_1'(x_r, m_r)\,\tilde{D}_s x_r\,(\tilde{D}_s x_t)^{-1}\,dr\,ds &\le c(p)\int_0^t\int_s^t \zeta_{sr}\zeta_{ts}\,dr\,ds \\ &\le c(p)\int_0^t\int_s^t \exp\big[\frac{\tilde{a}(p)}{\varepsilon}(r-s)\big]\exp\big[-\frac{\tilde{a}(p)}{\varepsilon}(t-s)\big]\,dr\,ds \\ &\le \varepsilon\,c(p)\int_0^t\big(\exp\big[\frac{\tilde{a}(p)}{\varepsilon}(t-s)\big]-1\big)\exp\big[\frac{-\tilde{a}(p)}{\varepsilon}(t-s)\big]\,ds \\ &= \varepsilon\,t\,c(p).\end{aligned} \tag{29}$$

hence we have (ii).

Similar calculations imply the expressions (iii)-(v). (Let us notice that in expression (iii) there appears an integral with respect to m_t, which is of order $O(\frac{1}{\sqrt{\varepsilon}})$. But, $\|\chi_2'\|_p$ is of order $O(\sqrt{\varepsilon})$, so there is no problem to conclude).

This completes the proof of the theorem.

□

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