An Optimal Control Approach to Portfolio Optimisation with Conditioning Information

Marc Boissaux Jang Schiltz

Luxembourg School of Finance
Faculty of Law, Economics and Finance
University of Luxembourg

Advanced Mathematical Methods in Finance
Bled, May 2010

Outline

1. Context
   - Portfolio optimisation with conditioning information

Outline

1. Context
   - Portfolio optimisation with conditioning information
2. Main contributions
   - Theoretical results
   - Algorithm testing

Outline

1. Context
   - Portfolio optimisation with conditioning information
2. Main contributions
   - Theoretical results
   - Algorithm testing

Problem context

- Discrete-time optimisation
- Minimise portfolio variance for a given expected portfolio mean
- Postulate that there exists some relationship $\mu(s)$ between a signal s and each asset return r observed at the end of the investment interval:
  $$r = \mu(s) + \epsilon,$$
  with $E[\epsilon|s] = 0$.
- How do we optimally use this information in an otherwise classical portfolio optimisation process?

Problem history

- Hansen and Richard (1983): functional analysis argument suggesting that unconditional moments should enter the optimisation even when conditioning information is known
- Ferson and Siegel (2001): closed-form solution of unconstrained mean-variance problem using unconditional moments
- Chiang (2008): closed-form solutions to the benchmark tracking variant of the Ferson-Siegel problem
- Basu et al. (2006), Luo et al. (2008): empirical studies covering conditioned optima of portfolios of trading strategies

Possible signals

Taken from a continuous scale ranging from purely macroeconomic indices to investor sentiment indicators. Indicators taking into account investor attitude may be based on some model or calculated in an ad-hoc fashion. Examples include

- short-term treasury bill rates (Fama and Schwert 1977);
- CBOE Market Volatility Index (VIX) (Whaley 1993);
- risk aversion indices using averaging and normalisation (UBS Investor Sentiment Index 2003) or PCA reduction (Coudert and Gex 2007) of several macroeconomic indicators;

Possible signals (2)

- global risk aversion indices (GRAI) (Kumar and Persaud 2004) based on a measure of rank correlation between current returns and previous risks;
- option-based risk aversion indices (Tarashev et al. 2003);
- sentiment indicators directly obtained from surveys (e.g. University of Michigan Consumer Sentiment Index)

Outline

1. Context
   - Portfolio optimisation with conditioning information
2. Main contributions
   - Theoretical results
   - Algorithm testing

Aim

- Existing results are useful and instructive but limited to problem variations where a closed-form solution is achievable
- Want to formulate the problem with conditioning information in such a way that more general variations can be tackled (using numerical algorithms if necessary)
- Want to integrate this type of optimisation problem into an existing theoretical framework

Idea

- Formulate the Ferson-Siegel problem in optimal control terms
- Use signal (instead of time) as the independent variable and add a signal density factor so integrals in problem represent expectations
- Realise that the signal support may equal all of $\mathbb{R}$ and so a doubly-infinite version of the Pontryagin Principle must be available

General problem

$$
\begin{aligned}
\text{Minimise} \quad & J_{[\theta,\psi]}(x,u) = \int_{\theta}^{\psi} L(x(s),u(s),s)ds \text{ as } \theta \to -\infty, \psi \to \infty \\
\text{subject to} \quad & \dot{x}(s) = f(x(s),u(s),s)\ \forall s \in [\theta,\psi] \\
& \lim_{s\to-\infty} x(s) = x_-,\ \lim_{s\to\infty} x(s) = x_+, \\
\text{and} \quad & u(s) \in U\ \forall s \in [\theta,\psi]
\end{aligned}
$$

where

- $U \subseteq \mathbb{R}^n$ and convex
- $x(s) \in \mathbb{R}^m$
- L and f continuous and differentiable in both x and u

General problem (2)

- In general, various types of optimality should be defined as the cost function limit need not exist: the above case gives "strong optimality"
- Show that Pontryagin Minimum Principle (PMP) and Mangasarian Theorem are still valid for this type of problem

Pontryagin Minimum Principle over a doubly-infinite horizon

Theorem

If an admissible pair $(x^*(s), u^*(s))$ *is optimal for the above problem, there exist a constant* $\lambda_0 \in \{0, 1\}$ *and a vector costate function* $\lambda(s) = (\lambda_1(s), \ldots, \lambda_m(s))$ *such that,* $\forall s \in (-\infty, \infty)$,

$$(\lambda_0, \lambda_1(s), \ldots \lambda_m(s)) \neq (0, 0, \ldots 0),$$

Define the Hamiltonian
$\mathcal{H}(x(s), u(s), \lambda_0, \lambda(s), s) = \lambda_0 L(x(s), u(s), s) + \lambda \cdot f(x(s), u(s), s)$.

Pontryagin Minimum Principle over a doubly-infinite horizon(2)

Theorem

$u^*(s)$ minimises the Hamiltonian over all $u \in U$, i.e.
$\forall u \in U, s \in (-\infty, \infty)$, for all admissible pairs $(x(s), u(s))$,

$$\mathcal{H}(x^*(s), u^*(s), \lambda_0, \lambda(s), s) \leq \mathcal{H}(x(s), u(s), \lambda_0, \lambda(s), s)$$

Additionally, the costates λ_i, for $i \in \{1, 2, \dots m\}$, verify

$$\dot{\lambda}_i(s) = -\frac{\partial \mathcal{H}}{\partial x_i}$$

except at any points of discontinuity for $u^*(s)$.

Proof steps for PMP over doubly-infinite horizon

- Start from singly-infinite versions of the PMP (Halkin 1974) and the Bellman Optimality Principle (BOP, easily shown)
- Show that BOP still holds for the doubly-infinite horizon
- Use singly-infinite PMP to establish separate optimality on positive and negative half-axes whatever their origin
- Apply finite and singly-infinite PMP variants on overlapping intervals to establish continuity of all costate components over the whole of $\mathbb{R}$

Mangasarian sufficiency theorem over a doubly-infinite horizon

Theorem

Given the previous problem and notation, suppose additionally that the following conditions are satisfied $\forall s \in \mathbb{R}$ *with* $\lambda_0 = 1$*:*

$$\dot{\lambda}_i(s) = -\frac{\partial \mathcal{H}^*}{\partial x_i} \; \forall i \in \{1, 2, \dots n\}$$

$$\mathcal{H}(x(s), u(s), \lambda_0, \lambda(s), s) \textit{ is jointly convex in } (x(s), u(s))$$

$$\sum_{j=1}^{n} \frac{\partial \mathcal{H}^*}{\partial u_j} \left(u_j^*(s) - u_j(s) \right) \leq 0 \; \forall u(s) \in U.$$

Mangasarian sufficiency theorem over a doubly-infinite horizon(2)

Theorem

Then the admissible pair $(x^*(s), u^*(s))$ *solves the problem if, for all s,*

$$\exists M > 0 : |\lambda(s)| \leq M.$$

If $\mathcal{H}(x(s), u(s), \lambda_0, \lambda(s), s)$ *is strictly convex, the pair* $(x^*(s), u^*(s))$ *constitutes the unique solution to the problem.*

- Proof is immediate for the given optimality definition and boundary conditions

Mean-variance portfolio problem

Minimise

$$J(u) = \int_{s^-}^{s^+} u'(s)\left[(\mu(s) - r_f e)(\mu(s) - r_f e)' + \Sigma_\epsilon^2\right] u(s) p_S(s) ds$$

given the state trajectory

$$\dot{x}_1(s) = u'(s)(\mu(s) - r_f e) p_S(s)$$

with

- μ_p unconditional expected portfolio return
- r_f risk-free rate of return
- Σ_ϵ^2 conditional covariance matrix
- $p_S(s)$ signal density function
- $x_1(s^-) = 0$ and $x_1(s^+) = \mu_p - r_f$
- $u(s) \in U \; \forall s$

Mean-variance portfolio problem(2)

- Assume availability of a risk-free asset with return r_f
- Above expressions are for unconditional variance and expected return in the presence of a signal (see Ferson and Siegel 2001)
- Can apply PMP to this specific case of the general problem
- Formulation corresponds to a variation of a classical LQ minimum energy problem with magnitude constraints (see e.g. Athans and Falb 1966)

Open loop optimal portfolio weights

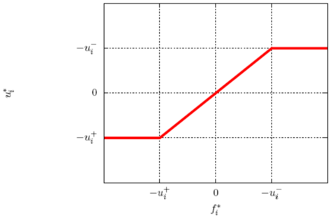
Define the *saturation function*

$$\mathrm{sat}_{u_i^-,u_i^+}(f_i^*) = \begin{cases} -u_i^+ \text{ if } f_i^* < -u_i^+ \\ f_i^* \text{ if } -u_i^+ < f_i^* < -u_i^- \\ -u_i^- \text{ if } f_i^* > -u_i^- . \end{cases}$$

with

$$f^*(s) = ((\mu(s) - r_f e)(\mu(s) - r_f e)' + \Sigma_\epsilon^2)^{-1}(\mu(s) - r_f e)\lambda^*(s)$$

Saturation function example with $u_i^- < 0$ and $u_i^+ > 0$



Open loop optimal portfolio weights (2)

We get a per-asset expression for the optimal weight:

$$u_i^*(s) = -\mathrm{sat}_{u_i^-, u_i^+}(f_i^*(s)),$$

- From the PMP costate equation, the $\lambda_i^*(s) = \lambda_i^*$ are constant
- The optimal weights are piecewise linear in f^* but not, of course, in s

Outline

1. Context
   - Portfolio optimisation with conditioning information
2. Main contributions
   - Theoretical results
   - Algorithm testing

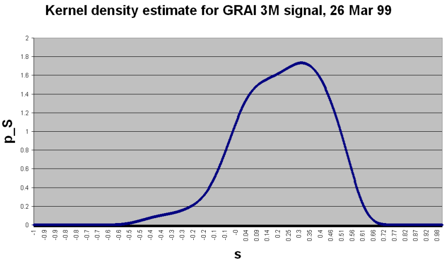
Data set

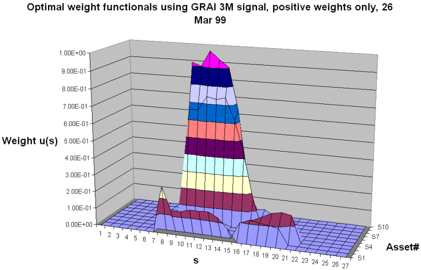
- 11 years of daily data, from January 1999 to February 2010 (2891 samples)
- Risky assets: 10 different EUR-based funds commercialised in Luxembourg chosen across asset categories (equity, fixed income) and across Morningstar style criteria
- Risk-free proxy: EURIBOR with 1 week tenor
- Signal: Kumar and Persaud currency-based GRAI obtained using 3 monthly forward rates

Experiment

- Rebalance Markowitz-optimal portfolio alongside portfolio optimal with conditioning information over the 11-year period
- Assume lagged relationship $\mu(s)$ between signal and return can be represented by a linear regression
- Use kernel density estimates for signal densities
- Use direct (Gaussian collocation) method for numerical problem solutions (Benson 2005)
- Obtain efficient frontier for every date and choose portfolio based on quadratic utility functions with risk aversion coefficients between 0 and 10
- Compare Sharpe ratios (ex ante), ongoing returns (ex post) of both strategies

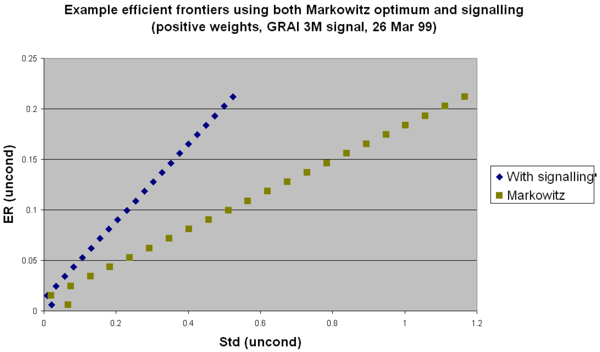
Typical kernel density estimate for signal and resulting optimal weight functionals





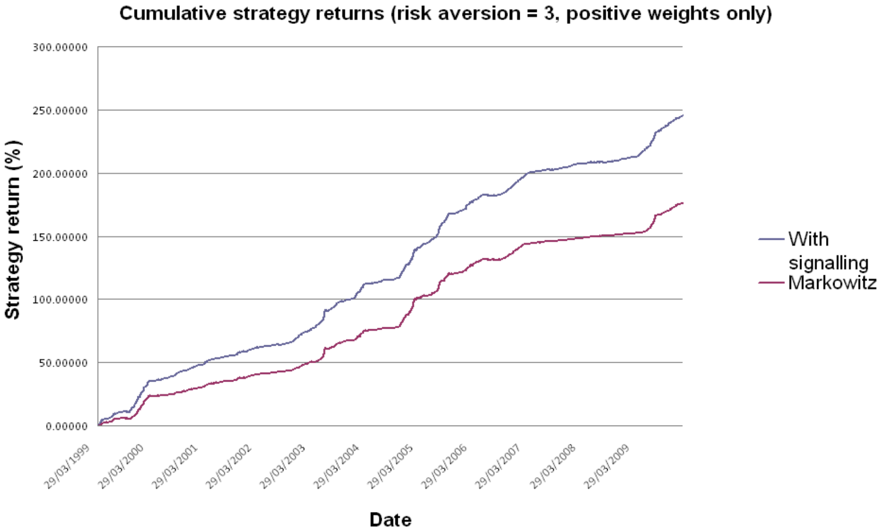
- As would be expected, the constrained optimal weights are not simply a truncated version of the unconstrained optimal (Ferson-Siegel) weights

Typical efficient frontiers using Markowitz portfolios and portfolios using signal information

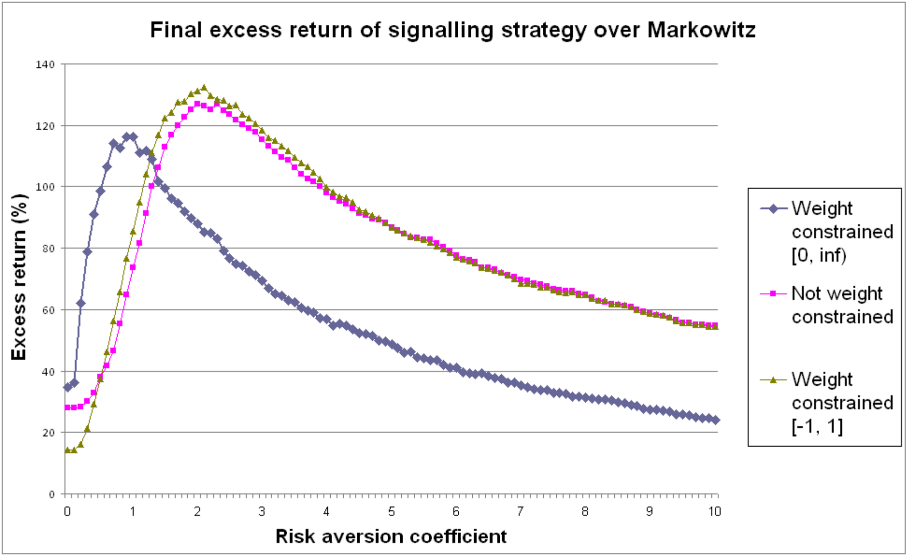


- Average Sharpe ratios over 2891 samples: Markowitz - 0.289, using signal - 0.373 (using business daily returns and volatilities)

Evolution of strategy returns



Comparison of strategy excess returns over Markowitz



Summary

- By using the *signal* as the independent variable, we can express portfolio optimisation problems with conditioning information in an optimal control format
- Necessity and sufficiency results in optimal control theory can easily be generalised to the required doubly-infinite horizon context
- In this way, it becomes possible to solve more general types of optimisation problem through applying any of the numerous numerical optimal control solution approaches available
- The presented problem yields significant outperformance of a pure Markowitz strategy in a more realistic setting than the unconstrained optimum with signalling can afford