An Optimal Control Solution to the Constrained Weight Portfolio Optimisation Problem with Conditioning Information

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Outline

1. Context
   - Portfolio optimisation with conditioning information

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2. Main contributions
   - Theoretical results
   - Algorithm testing

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Problem context

- Discrete-time optimisation
- Minimise portfolio variance for a given expected portfolio mean
- Postulate that there exists some relationship $\mu(s)$ between a signal s and each asset return r observed at the end of the investment interval:
$$r = \mu(s) + \epsilon,$$
with $E[\epsilon|s] = 0$.
- How do we optimally use this information in an otherwise classical portfolio optimisation process?

Problem history

- Hansen and Richard (1983): functional analysis argument suggesting that unconditional moments should enter the optimisation even when conditioning information is known
- Ferson and Siegel (2001): closed-form solution of unconstrained mean-variance problem using unconditional moments
- Chiang (2008): closed-form solutions to the benchmark tracking variant of the Ferson-Siegel problem
- Basu et al. (2006), Luo et al. (2008): empirical studies covering conditioned optima of portfolios of trading strategies

Possible signals

Taken from a continuous scale ranging from purely macroeconomic indices to investor sentiment indicators. Indicators taking into account investor attitude may be based on some model or calculated in an ad-hoc fashion. Examples include

- short-term treasury bill rates (Fama and Schwert 1977);
- CBOE Market Volatility Index (VIX) (Whaley 1993);
- risk aversion indices using averaging and normalisation (UBS Investor Sentiment Index 2003) or PCA reduction (Coudert and Gex 2007) of several macroeconomic indicators;

Possible signals (2)

- global risk aversion indices (GRAI) (Kumar and Persaud 2004) based on a measure of rank correlation between current returns and previous risks;
- option-based risk aversion indices (Tarashev et al. 2003);
- sentiment indicators directly obtained from surveys (e.g. University of Michigan Consumer Sentiment Index)

Outline

Aim

- Existing results are useful and instructive but limited to problem variations where a closed-form solution is achievable
- Want to formulate the problem with conditioning information in such a way that more general variations can be tackled (using numerical algorithms if necessary)
- Want to integrate this type of optimisation problem into an existing theoretical framework

Idea

- Formulate the Ferson-Siegel problem in optimal control terms
- Use signal (instead of time) as the independent variable and add a signal density factor so integrals in problem represent expectations
- Realise that the signal support may equal all of $\mathbb{R}$ and so a doubly-infinite version of the Pontryagin Principle must be available: this is shown in Boissaux and Schiltz (upcoming) along with a doubly-infinite version of the Mangasarian sufficiency theorem

The portfolio problem in optimal control terms

Minimise

$$J(u) = \int_{s^-}^{s^+} u'(s)\Big[(\mu(s) - r_f e)(\mu(s) - r_f e)' + \Sigma_\epsilon^2\Big]u(s)p_S(s)ds$$

given the state trajectory

$$\dot{x}_1(s) = u'(s)(\mu(s) - r_f e)p_S(s)$$

with

- μ_p unconditional expected portfolio return
- r_f risk-free rate of return
- Σ_ϵ^2 conditional covariance matrix
- $p_S(s)$ signal density function
- $x_1(s^-) = 0$ and $x_1(s^+) = \mu_p - r_f$
- $u(s) \in U \; \forall s$ (asset weights in convex set)

The portfolio problem in optimal control terms(2)

- Assume availability of a risk-free asset with return r_f
- Above expressions are for unconditional variance and expected return in the presence of a signal (modulo constants; see Ferson and Siegel 2001)
- Given a version of the PMP valid on the whole of $\mathbb{R}$, signals with infinite support are covered provided we adapt notation for the boundary conditions
- Formulation corresponds to a variation of a classical LQ minimum energy problem with magnitude constraints (see e.g. Athans and Falb 1966)

Pontryagin Minimum Principle over a doubly-infinite horizon

Theorem

If an admissible pair $(x^*(s), u^*(s))$ *is optimal for the above problem, there exist a constant* $\lambda_0 \in \{0, 1\}$ *and a vector costate function* $\lambda(s) = (\lambda_1(s), \ldots, \lambda_m(s))$ *such that,* $\forall s \in (-\infty, \infty)$,

$$(\lambda_0, \lambda_1(s), \ldots \lambda_m(s)) \neq (0, 0, \ldots 0),$$

Define the Hamiltonian
$\mathcal{H}(x(s), u(s), \lambda_0, \lambda(s), s) = \lambda_0 L(x(s), u(s), s) + \lambda \cdot f(x(s), u(s), s).$
$u^*(s)$ *minimises the Hamiltonian over all* $u \in U$, *i.e.*
$\forall u \in U, s \in (-\infty, \infty)$, *for all admissible pairs* $(x(s), u(s))$,

$$\mathcal{H}(x^*(s), u^*(s), \lambda_0, \lambda(s), s) \leq \mathcal{H}(x(s), u(s), \lambda_0, \lambda(s), s)$$

Pontryagin Minimum Principle over a doubly-infinite horizon(2)

Theorem

Additionally, the costates λ_i, for $i \in \{1, 2, \ldots m\}$, verify

$$\dot{\lambda}_i(s) = -\frac{\partial \mathcal{H}}{\partial x_i}$$

except at any points of discontinuity for $u^*(s)$.

- Similarly, the (standard) Mangasarian sufficiency condition is easily extended to the doubly-infinite case.

Open loop optimal portfolio weights

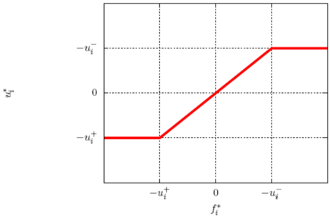
Define the *saturation function*

$$\text{sat}_{u_i^-, u_i^+}(f_i^*) = \begin{cases} -u_i^+ \text{ if } f_i^* < -u_i^+ \\ f_i^* \text{ if } -u_i^+ < f_i^* < -u_i^- \\ -u_i^- \text{ if } f_i^* > -u_i^- . \end{cases}$$

with

$$f^*(s) = ((\mu(s) - r_f e)(\mu(s) - r_f e)' + \Sigma_\epsilon^2)^{-1}(\mu(s) - r_f e)\lambda^*(s)$$

Saturation function example with $u_i^- < 0$ and $u_i^+ > 0$



Open loop optimal portfolio weights (2)

We get a per-asset expression for the optimal weight:

$$u_i^*(s) = -\mathrm{sat}_{u_i^-, u_i^+}(f_i^*(s)),$$

- From the PMP costate equation, the $\lambda_i^*(s) = \lambda_i^*$ are constant
- The optimal weights are piecewise linear in f^* but not, of course, in s

Classical special cases of problem

- weight limits $u_i^*(s) \rightarrow \infty$: saturation function becomes identity. We obtain per-asset equations directly yielding costate value and Ferson-Siegel result
- weight limits $u_i^*(s) \rightarrow \infty$, $\mu(s) = \mu$: no informative relationship between signal and return. Formulation yields classical Markowitz solution in closed form.

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Data set

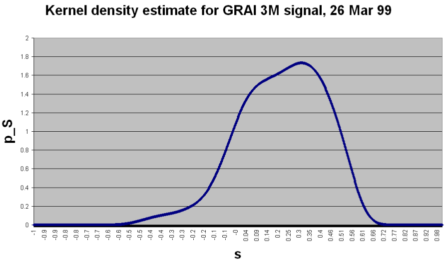
- 11 years of daily data, from January 1999 to February 2010 (2891 samples)
- Risky assets: 10 different EUR-based funds commercialised in Luxembourg chosen across asset categories (equity, fixed income) and across Morningstar style criteria
- Risk-free proxy: EURIBOR with 1 week tenor
- Signal: Kumar and Persaud currency-based GRAI obtained using 3 monthly forward rates

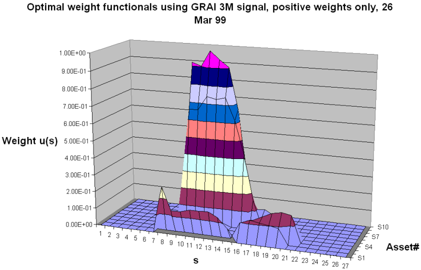
Experiment

- Rebalance Markowitz-optimal portfolio alongside portfolio optimal with conditioning information over the 11-year period
- Assume lagged relationship $\mu(s)$ between signal and return can be represented by a linear regression
- Use kernel density estimates for signal densities
- Use direct (Gaussian collocation) method for numerical problem solutions (Benson 2005)
- Obtain efficient frontier for every date and choose portfolio based on quadratic utility functions with risk aversion coefficients between 0 and 10
- Compare Sharpe ratios (ex ante), ongoing returns (ex post) of both strategies

Algorithm testing

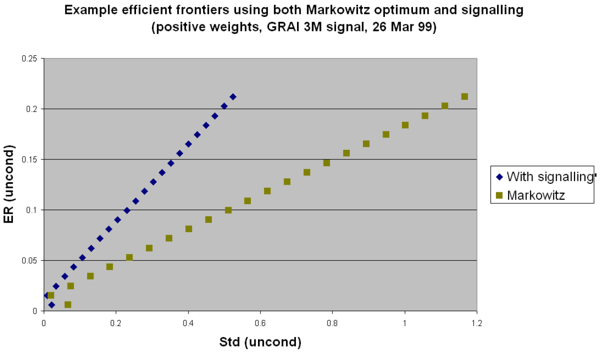
Typical kernel density estimate for signal and resulting optimal weight functionals





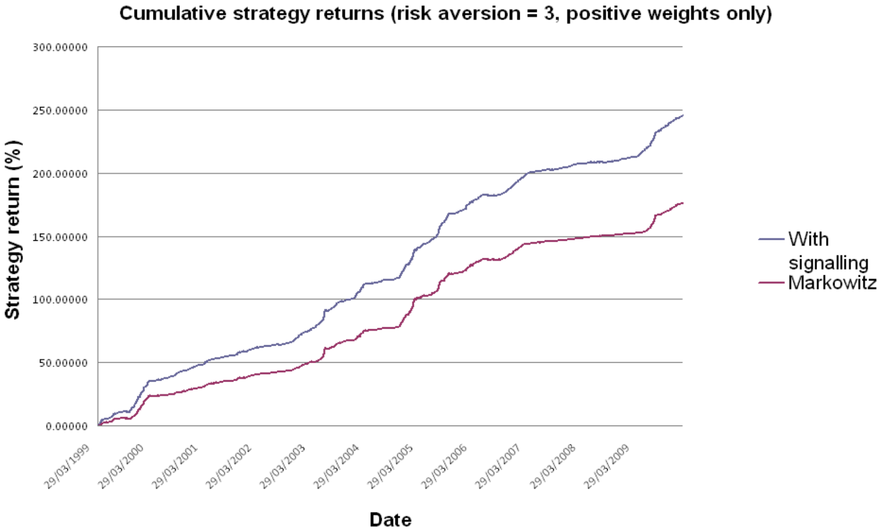
- As would be expected, the constrained optimal weights are not simply a truncated version of the unconstrained optimal (Ferson-Siegel) weights

Typical efficient frontiers using Markowitz portfolios and portfolios using signal information

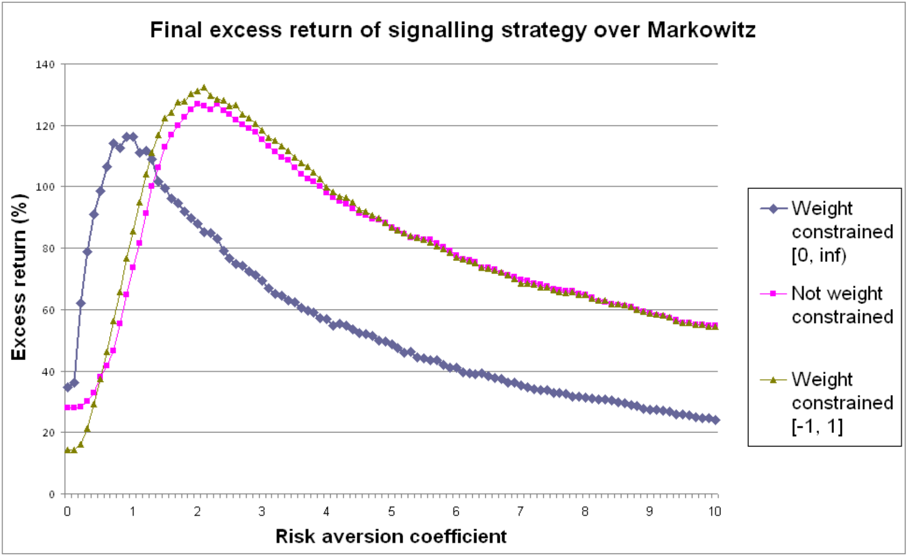


- Average Sharpe ratios over 2891 samples: Markowitz - 0.289, using signal - 0.373 (using business daily returns and volatilities)

Evolution of strategy returns



Comparison of strategy excess returns over Markowitz



Summary

- By using the *signal* as the independent variable, we can express portfolio optimisation problems with conditioning information in an optimal control format
- Necessity and sufficiency results in optimal control theory can easily be generalised to the required doubly-infinite horizon context
- In this way, it becomes possible to solve more general types of optimisation problem through applying any of the numerous numerical optimal control solution approaches available
- The presented problem yields significant outperformance of a pure Markowitz strategy in a more realistic setting than the unconstrained optimum with signalling can afford