Barycentrically associative aggregation functions

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B-associative functions

Let X be a nonempty set and let

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

$F\colon X^* \to X$ is *B-associative* (Bemporad, 1926) if

$$\begin{aligned} & F(x_1, \ldots, x_p, \quad y_1, \ldots, y_q, \quad z_1, \ldots, z_r) \\ = \;& F(x_1, \ldots, x_p, \underbrace{F(y_1, \ldots, y_q) \ldots, F(y_1, \ldots, y_q)}_{q \text{ times}}, z_1, \ldots, z_r) \end{aligned}$$

Example: $F(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$ on $X = \mathbb{R}$

Notation

We regard n-tuples $\mathbf{x}$ in X^n as *n-strings* over X

0-string: ε
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

X^* is endowed with concatenation

Example: $\mathbf{x} \in X^n$, $y \in X$, $\mathbf{z} \in X^m \quad \Rightarrow \quad \mathbf{x}y\mathbf{z} \in X^{n+1+m}$

$$\mathbf{x}^n = \mathbf{x} \cdots \mathbf{x} \qquad (n \text{ times})$$

$$|\mathbf{x}| = \text{length of } \mathbf{x}$$

Functions of multiple arities

Let

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

$$\boxed{F\colon X^* \to X}$$

Components of F :

$$F_n\colon X^n \to X$$

$$\boxed{F_n = F|_{X^n}}$$

F is described by its components $F_1, F_2, F_3, \ldots, F_n, \ldots$

B-associative functions

$F\colon X^* \to X$ is *B-associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Theorem

$F\colon X^* \to X$ is B-associative if and only if

$$F(\mathbf{xy}) \;=\; F(F(\mathbf{x})^{|\mathbf{x}|}F(\mathbf{y})^{|\mathbf{y}|}) \qquad \forall\ \mathbf{xy} \in X^*$$

Interpretation ?

B-associative functions

Theorem (Kolomogorov-Nagumo, 1930)

$F\colon \mathbb{R}^* \to \mathbb{R}$ is B-associative and every F_n is symmetric, continuous, idempotent (i.e., $F_n(x^n) = x$), and strictly increasing in each argument if and only if there exists a continuous and strictly monotone function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that

$$F_n(\mathbf{x}) \;=\; \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right)$$

$\varphi(x)$	Mean
x	arithmetic
x^2	quadratic
x^k	root-power
$1/x$	harmonic
$\log(x)$	geometric
$\exp(x)$	exponential

B-associative functions

Theorem (M., Mathonet, and Tousset, 1999)

$F\colon \mathbb{R}^* \to \mathbb{R}$ is B-associative and every F_n is nondecreasing in each argument and satisfies

$$F_n(rx_1+s,\ldots,rx_n+s) \;=\; r\,F_n(x_1,\ldots,x_n)+s \qquad r,s\in\mathbb{R},\; r>0,$$

if and only if either

(i) $F_n = \min$ for every $n\in\mathbb{N}$, or

(ii) $F_n = \max$ for every $n\in\mathbb{N}$, or

(iii) $F_n(\mathbf{x}) = \sum_i w_i(\theta)\,x_i$ for every $n\in\mathbb{N}$, where

$$w_i(\theta) \;=\; \frac{\theta^{i-1}(1-\theta)^{n-i}}{\sum_j \theta^{j-1}(1-\theta)^{n-j}}$$

B-associative functions

$F\colon X^* \to X$ is *B-associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Theorem

We can assume that $|\mathbf{xz}| \leqslant 1$ in the definition above

That is, $F\colon X^* \to X$ is B-associative if and only if

$$\begin{aligned} F(\mathbf{y}) &= F(F(\mathbf{y})^{|\mathbf{y}|}) \\ F(x\mathbf{y}) &= F(xF(\mathbf{y})^{|\mathbf{y}|}) \\ F(\mathbf{y}z) &= F(F(\mathbf{y})^{|\mathbf{y}|}z) \end{aligned}$$

B-preassociative functions

Let Y be a nonempty set

Definition. We say that $F\colon X^* \to Y$ is *B-preassociative* if

$$|\mathbf{y}| = |\mathbf{y}'| \text{ and } F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Example: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

Proposition

$F\colon X^* \to Y$ is B-preassociative if and only if

$$\left.\begin{array}{l} |\mathbf{x}| = |\mathbf{x}'| \text{ and } F(\mathbf{x}) = F(\mathbf{x}') \\ |\mathbf{y}| = |\mathbf{y}'| \text{ and } F(\mathbf{y}) = F(\mathbf{y}') \end{array}\right\} \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}) = F(\mathbf{x}'\mathbf{y}')$$

B-preassociative functions

Remark. If $F\colon X^* \to X$ is B-associative, then it is B-preassociative

Proof. Suppose $|\mathbf{y}| = |\mathbf{y}'|$ and $F(\mathbf{y}) = F(\mathbf{y}')$
Then $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) = F(\mathbf{x}F(\mathbf{y}')^{|\mathbf{y}'|}\mathbf{z}) = F(\mathbf{xy}'\mathbf{z})$ □

Proposition

$F\colon X^* \to X$ is B-associative if and only if it is B-preassociative and $F(F(\mathbf{x})^{|\mathbf{x}|}) = F(\mathbf{x})$

Proof. (Necessity) OK.
(Sufficiency) We have $F(\mathbf{y}) = F(F(\mathbf{y})^{|\mathbf{y}|})$
Hence, by B-preassociativity, $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$ □

B-preassociative functions

Proposition

If $F\colon X^* \to Y$ is B-preassociative, then so is $F \circ (g, \ldots, g)$ for every function $g\colon X \to X$, where

$$F \circ (g, \ldots, g) \quad : \qquad x_1 \cdots x_n \ \mapsto F_n(g(x_1) \cdots g(x_n))$$

Example: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \qquad (X = Y = \mathbb{R})$

Proposition

If $F\colon X^* \to Y$ is B-preassociative, then so is $g \circ F$ for every function $g\colon Y \to Y$ such that $g|_{\mathrm{ran}(F)}$ is constant or one-to-one

Example: $F_n(\mathbf{x}) = \exp(x_1^2 + \cdots + x_n^2) \qquad (X = Y = \mathbb{R})$

B-preassociative functions

Proposition

Assume $F\colon X^* \to Y$ is B-preassociative
If F_n is constant, then so is F_{n+1}

Proof. If $F_n(\mathbf{y}) = F_n(\mathbf{y}')$ for all $\mathbf{y}, \mathbf{y}' \in X^n$, then $F_{n+1}(x\mathbf{y}) = F_{n+1}(x\mathbf{y}')$ and hence F_{n+1} depends only on its first argument... □

Open question:
Find necessary and sufficient conditions on a B-preassociative function F for F_{n+1} to be completely determined by F_n

B-preassociative functions

We have seen that $F\colon X^* \to X$ is B-associative if and only if it is B-preassociative and $F(F(\mathbf{x})^{|\mathbf{x}|}) = F(\mathbf{x})$

Remark. The latter condition can be rewritten as

$$F_n(F_n(\mathbf{x})^n) \;=\; F_n(\mathbf{x}) \qquad \forall\, n \in \mathbb{N}$$

or

$$\delta_{F_n}(F_n(\mathbf{x})) \;=\; F_n(\mathbf{x}) \qquad \forall\, n \in \mathbb{N}$$

Relaxation of $\quad \delta_{F_n} \circ F_n = F_n$:

$$\boxed{\mathrm{ran}(\delta_{F_n}) \;=\; \mathrm{ran}(F_n) \quad \forall\, n \in \mathbb{N}}$$

B-preassociative functions

Theorem

Let $F\colon X^* \to Y$ be a function. The following assertions are equivalent:

(i) F is B-preassociative and satisfies $\mathrm{ran}(\delta_{F_n}) = \mathrm{ran}(F_n)$

(ii) F_n can be factorized into $F_n = f_n \circ H_n$,
where $H\colon X^* \to X$ is B-associative
and $f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one.
In this case, we have $f_n = \delta_{F_n}|_{\mathrm{ran}(H_n)}$ and $F_n = \delta_{F_n} \circ H_n$

Open problems

(1) Suppress the condition $\mathrm{ran}(\delta_{F_n}) = \mathrm{ran}(F_n)$ in this theorem

(2) Find necessary and sufficient conditions on δ_{F_n} for a function F of the form $F_n = \delta_{F_n} \circ H_n$, where H is B-associative, to be B-preassociative.

Axiomatizations of function classes

Theorem

$F\colon \mathbb{R}^* \to \mathbb{R}$ is B-preassociative and
every F_n is symmetric, continuous, and strictly increasing in each argument
if and only if there exist continuous and strictly increasing function $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi_n\colon \mathbb{R} \to \mathbb{R}$ ($n \in \mathbb{N}$) such that

$$F_n(\mathbf{x}) \;=\; \psi_n\left(\sum_{i=1}^{n} \varphi(x_i)\right)$$

Open problem

Find new axiomatizations of classes of B-preassociative functions from existing axiomatizations of classes of B-associative functions

Thank you for your attention !