

Extended higher spin holography and Grassmannian models

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ABSTRACT: We propose holographic dualities between higher spin gravity theories extended with Chan-Paton factor on AdS_3 and a large N limit of two dimensional Grassmannian models with and without supersymmetry. These proposals are natural extensions of the duality without Chan-Paton factor, and the extensions are motivated by a higher dimensional version of the duality, which implies a possible relation to superstring theory via ABJ theory. As evidence for the proposals, we show that the free limit of the Grassmannian models have the higher spin symmetry expected from the dual gravity theory. Furthermore, we construct currents in the 't Hooft limit of the supersymmetric Grassmannian model and compare them with the currents from the bulk theory. One-loop partition function of the supergravity theory is reproduced by the 't Hooft limit of the Grassmannian model after decoupling a gauge sector.

KEYWORDS: Conformal and W Symmetry, String Duality, AdS-CFT correspondence, Supergravity Models

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1 Introduction

Superstring theory includes a lot of massive higher spin states and in the massless limit they are believed to be described by higher spin gauge theory. Since the higher spin states are characteristic to superstring theory, it should be useful to study higher spin gauge theory to improve our understanding of superstring theory. Non-trivial examples of higher spin gauge theories are Vasiliev theories on AdS spacetimes [1], and they have been useful in the construction of simplified versions of the AdS/CFT correspondence. In [2, 3] it was proposed that the Vasiliev theory on AdS₄ is dual to a U(N) or O(N) vector model, and the proposal has been investigated quite actively. A few years ago, the authors of [4] (see [5] for a review) conjectured a lower dimensional version of the duality between the higher spin gauge theory on AdS₃ found in [6] and a large N minimal model. Several generalizations of the duality have been proposed in [7–12].

In [13, 14] one-parameter families of conformal models have been constructed by coupling Chern-Simons gauge fields to three dimensional U(N) or O(N) vector matter. If we take the limit of large Chern-Simons level k , then the theory reduces to that of free vector fields with U(N) or O(N) invariant condition in [3]. It was conjectured in [14] that the dual gravity theory is a parity violating Vasiliev theory, and the parameter characterizing the parity violation is related to the ratio k/N for the large N, k limit of the Chern-Simons vector model. The Vasiliev theory can be extended with supersymmetry and U(M) Chan-Paton factor, and the dual field theory is argued to be the supersymmetric U(N) _{k} × U(M) _{$-k$} Chern-Simons theory with bi-fundamental matter for large N, k but finite M [15]. The most important example of the theory is the $\mathcal{N} = 6$ ABJ theory, which should be dual to type IIA superstring theory on AdS₄ × CP³ with discrete torsion for large N, M, k [16, 17]. In this way, the ABJ theory connects the vector-like model dual to higher spin gauge theory and the matrix-like model dual to superstring theory. Based on these two dualities, the authors of [15] discussed an interesting new relation between higher spin gauge theory and superstring theory.

Motivated by the higher dimensional duality (or triality), we would like to construct a holographic duality involving higher spin theory with Chan-Paton factor on AdS₃.¹ Even in lower dimensions, the higher spin gauge theory can be easily extended with supersymmetry and/or U(M) Chan-Paton factor as discussed in [6]. In the bosonic case, we propose the dual theory is the Grassmannian coset model defined as

$$\frac{\text{su}(N+M)_k}{\text{su}(N)_k \oplus \text{su}(M)_k \oplus \text{u}(1)_{kNM(N+M)}} \quad (1.1)$$

with central charge

$$c = \frac{(k^2 - 1)MN(2k + M + N)}{(N + M + k)(N + k)(M + k)}. \quad (1.2)$$

¹Recently, a holography involving a higher spin theory with U(2) Chan-Paton factor is proposed in an intriguing paper [12]. The dual CFT is based on a Wolf space and has a large $\mathcal{N} = 4$ superconformal symmetry. The $\mathcal{N} = 4$ CFTs on Wolf spaces were suggested to be dual to supersymmetric higher spin theories in [18]. A possible relation to superstring theory on AdS₃ × S³ × S³ × S¹ is also discussed in [12].

In order to compare with the classical gravity theory, we consider the 't Hooft limit with large N, k , but keeping M finite and the ratio

$$\lambda_M = \frac{k}{N + M + k} \quad (1.3)$$

finite. In the limit, the central charge behaves as $c \sim MN\lambda_M(1 - \lambda_M)$, thus the theory is vector-like. On the other hand, if we take the limit with large N, M, k but finite $N - M$ and k/N , then the model is matrix-like as $c \sim \mathcal{O}(N^2)$. Thus the Grassmannian model connects a vector-like model dual to higher spin gauge theory and a matrix-like model dual to string theory like the ABJ theory, even though we do not know what is the string dual of the Grassmannian model currently.

There is a level-rank duality for the coset (1.1) and the dual coset is given by [19, 20]²

$$\frac{\mathrm{su}(k)_N \oplus \mathrm{su}(k)_M}{\mathrm{su}(k)_{N+M}}. \quad (1.4)$$

This form of coset with $M = 1$ is the model dual to the Vasiliev theory without Chan-Paton factor [4]. Therefore, we can say that the Grassmannian model (1.1) is a natural extension for $M \neq 1$ as the dual theory. Moreover, at the large level limit with $k \rightarrow \infty$ in (1.1), the Grassmannian coset reduces to a free system with NM complex bosons (see e.g. [23]). We will show that the free system reproduces the asymptotic symmetry of the dual gravity theory with the corresponding parameter. The one-loop partition functions are also examined.

We can consider the $\mathcal{N} = 2$ supersymmetric higher spin theory with $U(M)$ Chan-Paton factor from [6] in a similar way. We would like to propose that the dual CFT is given by the $\mathcal{N} = (2, 2)$ Grassmannian Kazama-Suzuki model [24, 25]

$$\frac{\mathrm{su}(N + M)_k \oplus \mathrm{so}(2NM)_1}{\mathrm{su}(N)_{k+M} \oplus \mathrm{su}(M)_{k+N} \oplus \mathfrak{u}(1)_\kappa} \quad (1.5)$$

with $\kappa = NM(N + M)(N + M + k)$, whose central charge is

$$c = \frac{3NMk}{N + M + k}. \quad (1.6)$$

We are interested in the 't Hooft limit where $N, k \rightarrow \infty$, but³

$$\lambda = \frac{N}{N + M + k} \quad (1.7)$$

and M finite. From the central charge, we can see that the model again connects a vector-like model in the 't Hooft limit and a matrix-like model with large N, M, k . The coset model with $M = 1$ is dual to the full $\mathcal{N} = 2$ higher spin supergravity without Chan-Paton factor as argued in [9].

²The possibility of holographic duality involving this form of coset was already pointed out in [21]. See also [22].

³We use different notations for the 't Hooft parameter such that (1.3) and (1.7) with $M = 1$ reduce to those of [4] and [9], respectively.

In order to support our duality conjecture, we discuss the free limit and partition functions as in the bosonic case. Moreover, we investigate currents in the Grassmannian model in the 't Hooft limit, but not in the free limit. From the bulk side we calculate several OPEs of these currents and successfully compare these to the CFT results. In the 't Hooft limit with large N, k but finite M, λ and outside the free limit, we see that extra bosonic $\text{su}(M)$ current should be coupled. Thus the cosets we are dealing with are effectively of the form as

$$\frac{\text{su}(N+M)_k}{\text{su}(N)_k \oplus \text{su}(M)_k \oplus \mathfrak{u}(1)_{kNM(N+M)}} \oplus \text{su}(M)_k \simeq \frac{\text{su}(N+M)_k}{\text{su}(N)_k \oplus \mathfrak{u}(1)_{kNM(N+M)}} \quad (1.8)$$

for the bosonic case and

$$\frac{\text{su}(N+M)_k \oplus \text{so}(2NM)_1}{\text{su}(N)_{k+M} \oplus \text{su}(M)_{k+N} \oplus \mathfrak{u}(1)_\kappa} \oplus \text{su}(M)_{k+N} \simeq \frac{\text{su}(N+M)_k \oplus \text{so}(2NM)_1}{\text{su}(N)_{k+M} \oplus \mathfrak{u}(1)_\kappa} \quad (1.9)$$

for the supersymmetric case. In (1.9) there is a subtlety since we add only bosonic $\text{su}(M)$. In other words, if we write the coset in terms of $\mathcal{N} = 1$ currents, then fermionic components of $\mathcal{N} = 1$ $\text{su}(M)$ currents should be decoupled.⁴ With this form of coset (1.9) we show that the gravity partition function can be reproduced from the CFT partition function in the 't Hooft limit.

This paper is organized as follows: In section 2 we study the higher spin gravity theories with $U(M)$ Chan-Paton factor and derive the partition functions of the gravity theories. In section 3 we propose that MN complex free bosons (and MN complex free fermions) are dual to the higher spin theories with a particular mass parameter based on symmetry arguments. Here we assign the $U(N)$ invariant condition, but the $U(M)$ symmetry is treated as a global symmetry. Furthermore, we relate the free systems to the Grassmannian cosets (1.1) and (1.5). We study the Grassmannian models in some details for the bosonic case in section 4 and for the supersymmetric case in section 5. In particular we discuss the roles of the $\text{su}(M)$ symmetry. We examine the characters of the coset models and reproduce the gravity partition functions in the free limits. In section 6 we construct currents in the Grassmannian model in the 't Hooft limit and compare them with those from the bulk theory. Conclusion and discussions are given in section 7. In appendix A some properties of the $\mathfrak{gl}(M)$ extended higher spin algebra are given, and we construct $w_\infty[\lambda]$ algebras extended with a matrix algebra from ghost systems in appendix B. In appendix C we find agreement between the supergravity partition function and that of (1.9) in the 't Hooft limit but outside the free limit.

2 Matrix valued higher spin theory

In this section, we introduce the $\mathcal{N} = 2$ higher spin gauge theory with $U(M)$ Chan-Paton factor on AdS_3 found in [6]. The bosonic truncation of the higher spin theory with $M = 1$ was used to construct a duality with a large N minimal model (1.4) in [4]. Moreover, the

⁴After the paper [12] appeared in the arXiv, we have noticed that the model (1.9) with $M = 2$ coincides with their coset.

total $\mathcal{N} = 2$ higher spin supergravity with $M = 1$ was argued to be dual to the \mathbb{CP}^N Kazama-Suzuki model (1.5) in [9]. Since the equations of motion are written in terms of a $*$ -product, we can easily replace the fields by $M \times M$ matrix valued ones. Using the reality condition of the fields discussed in [6], we can construct a higher spin gauge theory with $U(M)$ Chan-Paton factor. In the next section we introduce a bosonic truncation of the higher spin supergravity with $U(M)$ Chan-Paton factor in [6], and in section 2.2 we examine the full $\mathcal{N} = 2$ supergravity theory. In particular, we obtain the one-loop partition functions of these higher spin theories with $U(M)$ Chan-Paton factor, even though they easily follow from those without Chan-Paton factor.

2.1 A bosonic truncation

Before including the Chan-Paton factor, let us start from a bosonic truncation of the higher spin supergravity in [6]. The gravity theory includes gauge fields with spin $s = 2, 3, 4, \dots$ and massive complex scalar fields. The gauge fields can be described by Chern-Simons gauge theory with the action [26]

$$S = S_{\text{CS}}[A] - S_{\text{CS}}[\tilde{A}], \quad (2.1)$$

where

$$S_{\text{CS}}[A] = \frac{k_{\text{CS}}}{4\pi} \int \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.2)$$

The gauge fields A, \tilde{A} take values in an infinite dimensional higher spin algebra $\text{hs}[\lambda]$, where $\text{hs}[\lambda]$ can be truncated to $\text{sl}(N)$ at $\lambda = \pm N$.⁵ The generators of $\text{hs}[\lambda]$ are given by

$$V_m^s \quad (s = 2, 3, 4, \dots, m = -s + 1, -s + 2, \dots, s - 1), \quad (2.3)$$

and the commutation relations are (see, e.g., [27])

$$[V_m^s, V_n^t] = \sum_{u=2,4,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda) V_{m+n}^{s+t-u}. \quad (2.4)$$

We do not need the explicit form of the structure constants, but they can be found in [28] (see also [27]). The products of gauge fields can be described by the ‘‘lone star product’’ [28] as

$$V_m^s * V_n^t = \frac{1}{2} \sum_{u=1,2,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda) V_{m+n}^{s+t-u}. \quad (2.5)$$

The one-loop partition function for spin s gauge field was computed in [29] as

$$Z_B^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2}. \quad (2.6)$$

⁵To be more precise, the generators of spin higher than N form an ideal, \mathcal{I}_N , at these integer values $\lambda = \pm N$. Factoring $\text{hs}[\lambda]$ by this ideal gives $\text{sl}(N)$, $\text{hs}[\lambda]/\mathcal{I}_N \cong \text{sl}(N)$.

In order to compute thermal partition function, the space-time is set to be Euclidean AdS₃. The modulus of the boundary torus is $q = \exp \tau$. The total partition function for the gauge sector is

$$Z_0 = \prod_{s=2}^{\infty} Z_B^{(s)} \quad (2.7)$$

by taking products of those for spin s gauge fields.

Along with the higher spin gauge fields, the gravity theory includes two complex scalars with mass

$$M_B^2 = -1 + \lambda^2. \quad (2.8)$$

We may decouple one complex scalar depending on the situation [30].⁶ Now we set $0 \leq \lambda \leq 1$ and then two types of boundary condition for the scalar fields are allowed. We choose to assign the standard boundary condition for one scalar and the alternative boundary condition for the other scalar, then the conformal weights for operators dual to the scalar fields are (h_{\pm}, h_{\pm}) with

$$h_{\pm} = \frac{1 \pm \lambda}{2}. \quad (2.9)$$

The partition function for a complex scalar with dual conformal weight (h, h) is [32, 33]

$$Z_{\text{scalar}}^h = \prod_{n,m=0}^{\infty} \frac{1}{(1 - q^{h+n} \bar{q}^{h+m})^2}. \quad (2.10)$$

In order to compare the result with the CFT computation, it is useful to rewrite the expression in terms of characters of Young diagrams as [34, 35]

$$Z_{\text{scalar}}^h = \sum_{\Lambda_l, \Lambda_r} |\text{ch}_{\Lambda_l}(U(h)) \text{ch}_{\Lambda_r}(U(h))|^2. \quad (2.11)$$

The character of Λ is defined as

$$\text{ch}_{\Lambda}(U(h)) = \sum_{T \in \text{Tab}_{\Lambda}} \prod_{j \in T} q^{h+j}, \quad U(h)_{jj} = q^{h+j}, \quad (2.12)$$

where Tab_{Λ} denotes a Young tableau of a shape Λ . In a Young tableau we assign a non-negative number $c_{i,j}$ to the box at the i -th row and the j -th column of Λ with the rule that $c_{i,j} \leq c_{i,j+1}$ and $c_{i,j} < c_{i+1,j}$. Combined with the gauge sector, the total one-loop partition function of the gravity theory with two complex scalars is

$$\begin{aligned} Z^{\text{Bulk}} &= Z_0 Z_{\text{scalar}}^{h_+} Z_{\text{scalar}}^{h_-} \\ &= Z_0 \sum_{\Lambda_l, \Lambda_r, \Xi_l, \Xi_r} |\text{ch}_{\Lambda_l}(U(h_+)) \text{ch}_{\Lambda_r}(U(h_+)) \text{ch}_{\Xi_l}(U(h_-)) \text{ch}_{\Xi_r}(U(h_-))|^2. \end{aligned} \quad (2.13)$$

⁶ In the refined version of the conjecture, we should include only one complex scalar [5, 31]. However, at the strict 't Hooft limit of the minimal model, the original proposal with two complex scalars still works. It is argued that non-perturbative objects of gravity theory play roles of one complex scalar outside the 't Hooft limit.

It is easy to extend the above analysis to the case with $U(M)$ Chan-Paton factor. The fields are now of $M \times M$ matrix form instead of the functional form. Here the reality condition is assigned as in [6]. The field equations for the master fields with $M = 1$ are written in terms of the $*$ -product in [6]. Since $*$ -product has already non-abelian nature, the field equations for the fields with $M \neq 1$ can be obtained by using the product of the $*$ -product and the usual matrix multiplication. For the physical gauge fields in the master fields, the $*$ -product reduces to the lone star product in (2.5). Thus now the gauge fields take values in algebra generated by

$$V_m^s \otimes t_a \quad (2.14)$$

with t_a being a generator of $\mathfrak{gl}(M)$. Notice that we should include $V_0^1 \otimes t_a$ since they cannot be decoupled for $M \neq 1$ in general, but we can still decouple $V_0^1 \otimes 1_M$ where 1_M is the identity in $\mathfrak{gl}(M)$. For $\lambda = 1$ we can show that generators $V_0^1 \otimes t_a$ can be decoupled as well. The scalar fields are also replaced by $M \times M$ matrix valued fields. In the free limit, the only effect of introducing the $U(M)$ Chan-Paton factor is that the number of fields simply is multiplied by M^2 . Therefore, the one-loop partition function is simply given by the M^2 -th power of the one of $M = 1$; that is

$$Z_M^{\text{Bulk}} = (Z^{\text{Bulk}})^{M^2} (Z_B^{(1)})^{M^2-1}. \quad (2.15)$$

The last factor comes from the fact that elements generated by $V_0^1 \otimes t_a$ with t_a as a generator of $\mathfrak{sl}(M)$ do not decouple for $\lambda \neq 1$.

2.2 The $\mathcal{N} = 2$ supergravity

Since the inclusion of the Chan-Paton factor can be made in the same way as in the bosonic case, we first review the $\mathcal{N} = 2$ higher spin supergravity in [6] without Chan-Paton factor. The massless sector can be described by $\text{shs}[\lambda] \oplus \text{shs}[\lambda]$ Chern-Simons theory, where $\text{shs}[\lambda]$ can be truncated to the Lie superalgebra $\mathfrak{sl}(N+1|N)$ at $\lambda = N+1$. See, e.g., [9, 36–38] for the details of $\text{shs}[\lambda]$ Lie superalgebra. The gauge sector includes bosonic gauge fields with spin $s = 2, 3, \dots$ generated by $V_m^{(s)+}$ ($m < |s|, m \in \mathbb{Z}$) and $s = 1, 2, \dots$ generated by $V_m^{(s)-}$ ($m < |s|, m \in \mathbb{Z}$). Moreover there are two sets of fermionic gauge fields with spin $s = 3/2, 5/2, \dots$ generated by $V_m^{(s)\pm}$ ($m < |s|, m \in \mathbb{Z} + 1/2$). The one-loop partition functions of spin $s - 1/2$ gauge fields are given by [9]

$$Z_F^{(s-1/2)} = \prod_{n=s}^{\infty} |1 + q^{n-1/2}|^2, \quad (2.16)$$

thus the one-loop partition function of the massless sector is

$$\mathcal{Z}_0 = \prod_{s=2}^{\infty} Z_B^{(s)} (Z_F^{(s-1/2)})^2 Z_B^{(s-1)}. \quad (2.17)$$

We can include the $U(M)$ Chan-Paton factor as in the bosonic case by replacing the scalar coefficients in front of generators of $\text{shs}[\lambda] \oplus \mathbb{C}$ in gauge fields by $M \times M$ matrix valued

ones. The gauge fields thus take values in algebra generated by $V_m^{(s)\pm} \otimes t_a$ with $\mathfrak{gl}(M)$ generator t_a . Here $V_0^{(1)+} \otimes t_a$ are included, but $V_0^{(1)+} \otimes 1_M$ with the identity 1_M in $\mathfrak{gl}(M)$ is decoupled. We can show that $V_0^{(1)+} \otimes t_a$ can be excluded for $\lambda = 0$.

The massive sector includes four massive complex scalars and four massive Dirac fermions. The masses are parameterized by a parameter λ as [6]

$$(M_+^B)^2 = -1 + (\lambda - 1)^2, \quad (M_-^B)^2 = -1 + \lambda^2, \quad (M_{\pm}^F)^2 = (\lambda - \frac{1}{2})^2, \quad (2.18)$$

where two of the four bosons/fermions have the + index and the others have the - index. We choose the boundary conditions such that the dual conformal dimensions are [9, 35]

$$(\Delta_+^B, \Delta_{\pm}^F, \Delta_-^B) = (2 - \lambda, \frac{3}{2} - \lambda, 1 - \lambda), \quad (\lambda, \frac{1}{2} + \lambda, 1 + \lambda). \quad (2.19)$$

With this choice, the supersymmetry is enhanced to be $\mathcal{N} = (2, 2)$ at the boundary. The one-loop partition function of a Dirac spin 1/2 fermion with $(h, h - 1/2)$ and $(h - 1/2, h)$ is [9]

$$Z_{\text{spinor}}^h = \prod_{n,m=0}^{\infty} (1 + q^{h+n} \bar{q}^{h-1/2+m})^2 (1 + q^{h-1/2+n} \bar{q}^{h+m})^2. \quad (2.20)$$

For an $\mathcal{N} = 2$ multiplet we have

$$\mathcal{Z}_{\text{matter}}^h = Z_{\text{scalar}}^h (Z_{\text{spinor}}^{h+1/2})^2 Z_{\text{scalar}}^{h+1/2}. \quad (2.21)$$

As in the bosonic case, we rewrite the partition function in terms of supercharacter of the Young diagram Λ as [35]

$$\mathcal{Z}_{\text{matter}}^h = \sum_{\Lambda_l, \Lambda_r} |\text{sch}_{\Lambda_l}(\mathcal{U}(h)) \text{sch}_{\Lambda_r}(\mathcal{U}(h))|^2, \quad (2.22)$$

where the supercharacter of Λ is

$$\text{sch}_{\Lambda}(\mathcal{U}(h)) = \sum_{T \in \text{STab}_{\Lambda}} \prod_{i \in T} q^{h+i/2}, \quad \mathcal{U}(h)_{ii} = (-1)^i q^{h+i/2}. \quad (2.23)$$

The Young supertableau STab_{Λ} is defined with a non-negative integer $c_{i,j}$ in each box at the i -th row and the j -th column of a Young diagram Λ . The numbers should satisfy $c_{i,j} \leq c_{i,j+1}$ and $c_{i,j} \leq c_{i+1,j}$, moreover $c_{i,j} < c_{i,j+1}$ if both $c_{i,j}$ and $c_{i,j+1}$ are odd and $c_{i,j} < c_{i+1,j}$ if both $c_{i,j}$ and $c_{i+1,j}$ are even (see, for instance, [35] for more details).

In total, the one-loop partition function of the gravity theory is

$$\begin{aligned} \mathcal{Z}^{\text{Bulk}} &= \mathcal{Z}_0 \mathcal{Z}_{\text{matter}}^{\frac{\lambda}{2}} \mathcal{Z}_{\text{matter}}^{\frac{1-\lambda}{2}} \\ &= \mathcal{Z}_0 \sum_{\Lambda_l, \Lambda_r, \Xi_l, \Xi_r} |\text{sch}_{\Lambda_l}(\mathcal{U}(\frac{\lambda}{2})) \text{sch}_{\Lambda_r}(\mathcal{U}(\frac{\lambda}{2})) \text{sch}_{\Xi_l}(\mathcal{U}(\frac{1-\lambda}{2})) \text{sch}_{\Xi_r}(\mathcal{U}(\frac{1-\lambda}{2}))|^2. \end{aligned} \quad (2.24)$$

With the $U(M)$ Chan-Paton factor, all fields are now of $M \times M$ matrix form including the fermionic fields. Therefore, the one-loop partition function is given by roughly the M^2 -th power as

$$\mathcal{Z}_M^{\text{Bulk}} = (\mathcal{Z}^{\text{Bulk}})^{M^2} (Z_B^{(1)})^{M^2-1}. \quad (2.25)$$

The last factor is the contribution from the spin 1 fields generated by $V_0^{(1)+} \otimes t_a$ with $\mathfrak{sl}(M)$ generator t_a , which do not decouple in general.

3 Dual free CFTs

In order to find CFTs dual to the gravity theories, it is useful to utilize their asymptotic symmetry. For the gravity theory used in [4], the gauge sector can be described by $\text{hs}[\lambda] \oplus \text{hs}[\lambda]$ Chern-Simons theory. Assigning the asymptotically AdS boundary condition, the classical symmetry around the AdS boundary was found to be $W_\infty[\lambda]$ symmetry [27, 39–41]. The $W_\infty[\lambda]$ symmetry algebra is non-linear and generated by

$$W_m^s \quad (s = 2, 3, \dots, m \in \mathbb{Z}). \quad (3.1)$$

In the large central charge limit, the wedge subalgebra generated by W_m^s with $m = -s + 1, -s + 2, \dots, s - 1$ reduces to the $\text{hs}[\lambda]$ Lie algebra (see, e.g., [27]). It was shown in [27, 31] that the $W_\infty[\lambda]$ symmetry algebra is realized by the coset (1.4) with $M = 1$ as

$$\frac{\text{su}(N)_k \oplus \text{su}(N)_1}{\text{su}(N)_{k+1}}. \quad (3.2)$$

We need to take the 't Hooft limit where $N, k \rightarrow \infty$ with

$$\lambda_{\text{GG}} = \frac{N}{N + k} \quad (3.3)$$

finite and we identify $\lambda_{\text{GG}} = \lambda$.

It is not easy to extend the analysis for the case with $M \neq 1$ since the symmetry algebra of the coset (1.4) would be quite complicated. However, the $W_\infty[\lambda]$ algebra becomes simple for $\lambda = 0, 1$, and the algebra is realized by a system of N complex free fermions for $\lambda = 0$ or k complex free bosons for $\lambda = 1$. In this section we mainly focus on the limit where coset models considered reduce to free theories. The coset (1.4) is mapped to the Grassmannian coset (1.1) with the level-rank duality [19, 20] where k and N are exchanged. After reviewing the known results with $M = 1$ in the next subsection, we argue in section 3.2 that the system reduces to MN complex free bosons in the $\lambda_M = 1$ limit of the 't Hooft parameter (1.3) in the Grassmannian coset (1.1). Moreover, we can see that the free system generates an algebra with correct wedge subalgebra for the $M \neq 1$ cases. In section 3.3 we examine the supersymmetric case. For that case, we just need to consider MN complex free fermions in addition to the MN complex free bosons. The free system arises in the $\lambda = 0$ limit of the 't Hooft parameter (1.7) in the supersymmetric Grassmannian coset (1.5). In section 3.4, we analyze the symmetry of the coset (1.5) in the 't Hooft limit, but with $\lambda \neq 0$.

3.1 Free limits of the W_N minimal model

Let us start from the large level k limit of the coset (3.2), which means $\lambda_{\text{GG}} = 0$. Introducing N free complex fermions ψ_i ($i = 1, 2, \dots, N$) satisfying

$$\psi_i(z)\psi_j^\dagger(0) \sim \frac{\delta_{ij}}{z}, \quad (3.4)$$

we can define currents

$$J^{(s)}(z) = \frac{2^{-s+1}(s-1)!}{(2s-3)!!} \sum_{i=1}^N \sum_{l=0}^{s-1} (-1)^l \binom{s-1}{l}^2 \partial^{s-l-1} \psi_i^\dagger \partial^l \psi_i, \quad (3.5)$$

where

$$J^{(s)}(z) = \sum_{m=-\infty}^{\infty} \frac{W_m^s}{z^{m+s}}. \quad (3.6)$$

Here the spin of current is denoted by $s (= 1, 2, 3, \dots)$. They generate a linear algebra $w_{1+\infty}$ with $c = 2N$ [42]. The wedge subalgebra is $\text{hs}[0] \oplus \mathfrak{u}(1)$, and decoupling the $\mathfrak{u}(1)$ current we obtain the non-linear algebra $W_\infty[0]$ from the linear algebra $w_{1+\infty}$ [43]. The dual operator with $h = 1/2$ is given by

$$\mathcal{O}(z, \bar{z}) = \sum_{i=1}^N \psi_i(z) \otimes \bar{\psi}_i(\bar{z}) \quad (3.7)$$

and its complex conjugate. Here $\bar{\psi}_i$ are free complex fermions in the anti-holomorphic sector. At $\lambda = 0$, the two operators dual to scalar fields with different boundary conditions have the same conformal weight $h = 1/2$ as in (2.9). Thus they are indistinguishable at $\lambda = 0$ and we can pick up only one of them as discussed in [44]. By bosonizing the fermions we have N free real bosons. The summation over the i -index corresponds to the $U(N)$ invariant condition discussed in [44]. In order to construct a modular invariant theory, we need to add sectors twisted by the $U(N)$ action. In this paper we neglect the sectors since they decouple from the $U(N)$ invariant sector in the 't Hooft limit.

Next we consider the symmetry algebra for the $\lambda_{\text{GG}} = 1$ limit of the coset (3.2). The $W_\infty[1]$ algebra reduces to the linear algebra W_∞^{PRS} of [28], and its wedge subalgebra is $\text{hs}[1]$. The central charge of the coset (3.2) is $c = 2k + \mathcal{O}(1/N)$, and the algebra with central charge $c = 2k$ can be realized by k free complex bosons as shown in [45]. We choose the normalization of the free bosons ϕ_i ($i = 1, 2, \dots, k$) and their complex conjugates ϕ_i^\dagger such as to have OPEs

$$\phi_i(z) \phi_j^\dagger(0) \sim -\delta_{ij} \ln z. \quad (3.8)$$

Then, the generators of $W_\infty[1]$ with $c = 2k$ are written as

$$J^{(s)}(z) = -\frac{2^{-s+1}s!}{(2s-3)!!} \sum_{i=1}^k \sum_{l=0}^{s-2} (-1)^l \frac{1}{s-1} \binom{s-1}{l} \binom{s-1}{s-l} \partial^{s-l-1} \phi_i^\dagger \partial^{l+1} \phi_i \quad (3.9)$$

with $s = 2, 3, \dots$. In terms of free bosons, the operator dual to the scalar field can be given by

$$\mathcal{O}(z, \bar{z}) = \sum_{i=1}^k \partial \phi_i(z) \otimes \bar{\partial} \phi_i(\bar{z}) \quad (3.10)$$

and its complex conjugate. The conformal dimension of the operator is $h = 1$. The other operator has dimension $h = 0$ according to (2.9), and this value is the same as that of the vacuum. Thus it is natural to pick up one complex scalar field whose dual operator has conformal dimension $h = 1$.

The free boson system in the dual coset description (1.1) with $M = 1$ can be understood as

$$\frac{\text{su}(N+1)_k}{\text{su}(N)_k \oplus \text{u}(1)_{kN(N+1)}}. \quad (3.11)$$

According to [23], the coset in the classical limit with $k \rightarrow \infty$ has the same algebra $W_\infty[1]$ generated by N free complex bosons. It was claimed that the $\text{su}(N)_k$ current algebra is flattened out to be the $\text{u}(1)^{N^2-1}$ current algebra in large k limit. This implies that the limit of the Grassmannian model has $\text{u}(1)^N \otimes \text{u}(1)^N$ currents generated by N complex bosons. From the complex bosons we can construct operators generating the $W_\infty[1]$ algebra as in (3.9) with k replaced by N .

3.2 Adding the Chan-Paton factor

From the above consideration, we can guess that the CFT dual to the higher spin theory with $U(M)$ Chan-Paton factor at $\lambda = 1$ is given by the free system of NM complex bosons ϕ_{iA} with $i = 1, 2, \dots, N$ and $A = 1, 2, \dots, M$. We assign the following OPEs to the bosons

$$\phi_{iA}(z)\phi_{jB}^\dagger(0) \sim -\delta_{ij}\delta_{AB} \ln z. \quad (3.12)$$

We extend the currents (3.9) for $M = 1$ in a natural way as

$$[J^{(s)}(z)]_{AB} = -\frac{2^{-s+1}s!}{(2s-3)!!} \sum_{i=1}^N \sum_{l=0}^{s-2} (-1)^l \frac{1}{s-1} \binom{s-1}{l} \binom{s-1}{s-l} \partial^{s-l-1} \phi_{iA}^\dagger \partial^{l+1} \phi_{iB} \quad (3.13)$$

with $s = 2, 3, \dots$. Notice that spin $s = 1$ currents do not appear and the wedge subalgebra is given by $\text{hs}[1] \otimes \mathcal{M}$ with the matrix algebra \mathcal{M} as desired. The operators dual to $M \times M$ matrix valued scalar fields are

$$[\mathcal{O}(z, \bar{z})]_{AB} = \sum_{i=1}^N \partial \phi_{iA}(z) \otimes \bar{\partial} \phi_{iB}(\bar{z}) \quad (3.14)$$

and their complex conjugates.

A point here is that we have contracted the i index, but not the A index, and because of this we have $M \times M$ matrix valued currents invariant under the $U(N)$ symmetry. In the case of ABJ theory, the theory includes $U(N)_k \times U(M)_{-k}$ Chern-Simons gauge fields coupled with bi-fundamental fields $A_{\alpha i}^{(a)}$ and $B_{j\beta}^{(b)}$ with $a, b = 1, 2$, $i, j = 1, 2, \dots, N$ and $\alpha, \beta = 1, 2, \dots, M$ [16, 17]. When we consider the duality with a higher spin gauge theory, we should take the large N, k limit, but keep M finite [15]. The operators dual to the bulk higher spin fields are proposed to be of the form $\sum_{i=1}^N A_{\alpha i}^{(a)} B_{i\beta}^{(b)}$, since gluing for i -index is much stronger than that for α -index when $N \gg M$. Therefore, the $U(N)$ and

$U(M)$ gauge groups play very different roles, and this was interpreted in [15] as a confinement/deconfinement phase transition of the ABJ theory in the bulk 't Hooft parameter $\lambda^{\text{Bulk}} = M/N$.

We would like to claim that the free boson system arises in the $\lambda_M = 1$ limit of the Grassmannian model (1.1) as for the $M = 1$ case. An element of $SU(M + N)$ in the numerator of the coset (1.1) may be expressed by an $(M + N) \times (M + N)$ matrix. Here and in the following argument we neglect the trace part just for simplicity. We embed $SU(N)$ and $SU(M)$ gauge groups in the denominator as

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in SU(M + N) \quad (3.15)$$

for $A \in SU(M)$ and $D \in SU(N)$. Since A, D are gauged out, there are off-diagonal components left. It is natural to expect that these NM complex elements reduce to NM complex free boson in large k limit. Indeed, we can also obtain the same conclusion by making use of the argument in [23] as in the $M = 1$ case.

3.3 Supersymmetric extension

We can discuss the supersymmetric case in the same way. Without Chan-Paton factor, the gauge fields of the $\mathcal{N} = 2$ higher spin theory in [6] take values in the $\text{shs}[\lambda]$ algebra, and the asymptotic symmetry of the gravity theory is found to be the $\mathcal{N} = 2$ $W_\infty[\lambda]$ algebra [9, 18, 37, 46]. If we consider the large central charge limit, then the wedge subalgebra of $\mathcal{N} = 2$ $W_\infty[\lambda]$ should reduce to $\text{shs}[\lambda]$. Moreover, at $\lambda = 0$, it is known that $\mathcal{N} = 2$ $W_\infty[0]$ can be generated by complex bosons and fermions [42]. In other words, the $\mathcal{N} = 2$ higher spin theory with $\lambda = 0$ should be dual to N complex free bosons and N complex free fermions.

As in the bosonic case, we can easily extend the above argument to the case with $U(M)$ Chan-Paton factor, since we just need to introduce NM free fermions ψ_{iA} with $i = 1, 2, \dots, N$ and $A = 1, 2, \dots, M$ satisfying

$$\psi_{iA}(z)\psi_{jB}^\dagger(0) \sim \frac{\delta_{ij}\delta_{AB}}{z} \quad (3.16)$$

in addition to the NM free bosons satisfying (3.12). The bosonic currents are

$$[J^{(s)-}(z)]_{AB} = \frac{2^{-s+1}(s-1)!}{(2s-3)!!} \sum_{i=1}^N \sum_{l=0}^{s-1} (-1)^l \binom{s-1}{l}^2 \partial^{s-l-1} \psi_{iA}^\dagger \partial^l \psi_{iB} \quad (3.17)$$

with $s = 1, 2, \dots$ as a natural extension of (3.5) and $[J^{(s)+}(z)]_{AB} = [J^{(s)}(z)]_{AB}$ in (3.13) with $s = 2, 3, \dots$. The fermionic ones are

$$[F^{(s)}(z)]_{AB} = \frac{2^{-s+2}(s-1)!}{(2s-1)!!} \sum_{l=0}^{s-2} (-1)^l \binom{s-2}{l} \binom{s-1}{l} \partial^{s-l-1} \phi_{iA}^\dagger \partial^l \psi_{iB} \quad (3.18)$$

and $[F^{(s)\dagger}(z)]_{AB}$ is simply obtained by replacing ϕ_{iA}^\dagger and ψ_{iB} with ϕ_{iA} and ψ_{iB}^\dagger , respectively. Here the label s for $[F^{(s)}(z)]_{AB}$ and $[F^{(s)\dagger}(z)]_{AB}$ take values $s = 2, 3, \dots$. At $\lambda = 0$ we see that $[J^{(1)+}(z)]_{AB}$ do not appear.

From the free fields, we can construct operators as

$$[\mathcal{O}(z, \bar{z})]_{AB}^{(\frac{1}{2}, \frac{1}{2})} = \sum_{i=1}^N \psi_{iA}(z) \otimes \bar{\psi}_{iB}(\bar{z}), \quad [\mathcal{O}(z, \bar{z})]_{AB}^{(1,1)} = \sum_{i=1}^N \partial\phi_{iA}(z) \otimes \bar{\partial}\phi_{iB}(\bar{z}), \quad (3.19)$$

$$[\mathcal{O}(z, \bar{z})]_{AB}^{(1, \frac{1}{2})} = \sum_{i=1}^N \partial\phi_{iA}(z) \otimes \bar{\psi}_{iB}(\bar{z}), \quad [\mathcal{O}(z, \bar{z})]_{AB}^{(\frac{1}{2}, 1)} = \sum_{i=1}^N \psi_{iA}(z) \otimes \bar{\partial}\phi_{iB}(\bar{z}), \quad (3.20)$$

and their complex conjugates. They form one of two $\mathcal{N} = 2$ multiplets with

$$(\Delta_+^B, \Delta_\pm^F, \Delta_-^B) = (2, \frac{3}{2}, 1), \quad (3.21)$$

and this implies that the gravity theory dual to the free theory includes only half of the sets of scalars and fermions.

We can see that the free system arises in the $\lambda = 0$ limit of the 't Hooft parameter (1.7) in the supersymmetric Grassmannian model (1.5). There is an affine $\mathfrak{so}(2NM)_1$ algebra in the numerator of the coset (1.5), which yields NM complex fermions. The other parts are the same as the bosonic coset in (1.1) in the large k limit, and thus lead to NM complex bosons. Totally we have the free system with NM complex bosons and NM complex fermions.

3.4 Beyond the free 't Hooft limit

In this subsection, we will see how a $U(M)$ -valued $\mathcal{N} = 2$ super W -algebra appears in the 't Hooft limit of the coset for arbitrary λ . The construction depends crucially on normalization of fields. Recall that we want to have fields that have norm scaling with N . For example the Virasoro field has to have norm $c/2$. But the coset algebra is usually constructed using fields having a well-defined norm, that is a finite norm. Our strategy will thus be to find the coset fields having finite norm, and then rescale them to have norm scaling with N . It will turn out that the coset fields already provide a $U(M)$ -valued algebra, and the rescaled fields will then generate the desired $\mathcal{N} = 2$ super W -algebra with independent strong generators of dimension $1, 1, 3/2, 3/2, 2, 2, \dots$

Let us first consider the coset (1.5) at finite N . Normally, an efficient way to determine the spectrum of the coset algebra is to take the large k limit and invoke invariance under deformation to finite k . In this limit, the coset can be described by $2NM$ free bosons $\partial\phi_{iA}, \partial\phi_{Ai}^\dagger$ and $2NM$ free fermions $\psi_{iA}, \psi_{Ai}^\dagger$ in the tensor product of the fundamental representation of $SU(N)$ and $SU(M)$ and its conjugate, with OPEs

$$\partial\phi_{iA}(z, \bar{z})\partial\phi_{Bj}^\dagger(w, \bar{w}) = \frac{\delta_{AB}\delta_{ij}}{(z-w)^2}, \quad \psi_{iA}(z)\psi_{Bj}^\dagger(w) = \frac{\delta_{AB}\delta_{ij}}{(z-w)}, \quad (3.22)$$

as mentioned above. The coset algebra is then the invariant subalgebra under the action of $SU(N) \times SU(M) \times U(1)$. Using Weyl's first theorem of invariant theory for the unitary group [47] one obtains that the coset algebra is generated by normal ordered products of

type

$$\begin{aligned}
\sum_{i=1}^N \sum_{A=1}^M : \partial^a \phi_{iA}(z) \partial^b \phi_{Ai}^\dagger(z) : , & \quad \sum_{i=1}^N \sum_{A=1}^M : \partial^a \psi_{iA}(z) \partial^b \phi_{Ai}^\dagger(z) : , \\
\sum_{i=1}^N \sum_{A=1}^M : \partial^a \phi_{iA}(z) \partial^b \psi_{Ai}^\dagger(z) : , & \quad \sum_{i=1}^N \sum_{A=1}^M : \partial^a \psi_{iA}(z) \partial^b \psi_{Ai}^\dagger(z) : .
\end{aligned} \tag{3.23}$$

This gives a superalgebra with generator of spin $1, 3/2, 3/2, 2, 2, 5/2, 5/2, 3, \dots$, i.e. an $\mathcal{N} = 2$ super W -algebra.

If we consider the case of large N , we have to be careful. First define the currents of the $\mathfrak{su}(N)_k$ subalgebra by $K^a(z)$, those of the $\mathfrak{su}(M)_k$ subalgebra by $J^a(z)$ and the remaining currents carry the representation $\mathfrak{f}_N \otimes \bar{\mathfrak{f}}_M \oplus \bar{\mathfrak{f}}_N \otimes \mathfrak{f}_M$, where \mathfrak{f}_L denotes the fundamental representation of $\mathfrak{su}(L)$ and the bared expression its conjugate. We can thus label them by $I^{Ai}, I^{iA}, i = 1, \dots, N$ and $A = 1, \dots, M$. Further, the fermions $:\psi_{iA}\psi_{Aj}^\dagger:$ define $\mathfrak{u}(N)$ currents of level M , whose $\mathfrak{su}(N)_M$ subalgebra currents (the traceless ones) we denote by k^a . Analogously, the fermions $j^{AB} =: \psi_{iA}\psi_{Bi}^\dagger:$ define $\mathfrak{u}(M)$ currents of level N ,

$$j^{AB}(z)j^{CD}(w) \sim \frac{N\delta_{AD}\delta_{BC}}{(z-w)^2} + \frac{\delta_{BC}j^{AD}(w) - \delta_{AD}j^{CB}(w)}{(z-w)} \tag{3.24}$$

whose $\mathfrak{su}(M)_N$ subalgebra currents we denote by j^α . In addition, the $\mathfrak{u}(1)$ defined by the fermions is denoted by h and the one of the affine algebra by H . The coset algebra is then defined to be the subalgebra that commutes with the currents

$$K^a(z) + k^a(z), \quad J^a(z) + j^\alpha(z) \tag{3.25}$$

and with the $\mathfrak{u}(1)$ -current $H + h$.

Consider now the large N, k limit with $\lambda = N/(N + M + k)$ fixed and M finite. For finite N , the choice of basis of currents is irrelevant, but we require to take the limit in such a way, that currents have finite norm. The invariants under $K^a(z) + k^a(z)$ as the level is $k + M$ become in the large k limit the $\mathfrak{SU}(N)$ -group invariants as before. But for the $J^a + j^\alpha$ we have to be more careful as the level of both J^a and j^α is taken to infinity; we have to rescale the currents

$$X^\alpha(z) = \frac{J^\alpha(z) + j^\alpha(z)}{\sqrt{k + N}} \tag{3.26}$$

and also $H + h$ has to be rescaled by $1/\sqrt{k + N}$. Now, the fields

$$\frac{1}{N} \sum_{i=1}^N : \partial^a \psi_{iA}(z) \partial^b I^{Bi}(z) : , \quad \frac{1}{N} \sum_{i=1}^N : \partial^a I^{iA}(z) \partial^b \psi_{Bi}^\dagger(z) : \tag{3.27}$$

commute with $X^\alpha(z)$ and the rescaled $\mathfrak{u}(1)$ -current, and they have finite norm. These give us two $\mathfrak{U}(M)$ -valued fermionic fields of dimension $3/2, 5/2, \dots$. The J^a together with H form $\mathfrak{u}(M)$ of level k ; let us denote those currents by matrix indices J^{AB} . Then

$$\sqrt{\frac{1-\lambda}{k}} J^{AB}(z) - \sqrt{\frac{\lambda}{N}} j^{AB}(z) \tag{3.28}$$

commute with the X^{AB} . In summary, in the limit $k, N \rightarrow \infty$, we found $U(M)$ -valued coset fields of dimension $1, 3/2, 3/2, 2, 5/2, 5/2, 3, 7/2, \dots$ of finite norm. We remark that these are certainly not all coset fields.

We are interested in taking the limit, where the norm of fields scales as N . We thus rescale our coset fields (3.27) by \sqrt{N} ,

$$G_{ab}^{AB} = \frac{1}{\sqrt{N}} \sum_{i=1}^N : \partial^a \psi_{iA}(z) \partial^b I^{Bi}(z) :, \quad \bar{G}_{ab}^{AB} = \frac{1}{\sqrt{N}} \sum_{i=1}^N : \partial^a I^{iA}(z) \partial^b \psi_{Bi}^\dagger(z) :. \quad (3.29)$$

But now, some OPE computations reveal that we get closed operator product algebra, if we include the fields

$$J^{AB}, \quad V_{ab}^{AB} = \sum_{i=1}^N : \partial^a \psi_{iA}(z) \partial^b \psi_{Bi}^\dagger(z) :, \quad W_{ab}^{AB} = \frac{1}{N} \sum_{i=1}^N : \partial^a I^{iA}(z) \partial^b I^{iB}(z) :. \quad (3.30)$$

That is we get $U(M)$ -valued fields of dimension $1, 1, 3/2, 3/2, 2, 2, \dots$ as desired. Let us provide one example of an OPE illustrating this:

$$G_{00}^{AB}(z) \bar{G}_{00}^{CD}(w) \sim \frac{k \delta_{AD} \delta_{BC}}{(z-w)^3} + \frac{\delta_{AD} J^{BC}(w) + \frac{k}{N} \delta_{BC} V_{00}^{AD}(w)}{(z-w)^2} + \frac{\delta_{AD} W_{00}^{BC}(w) + \frac{k}{N} \delta_{BC} V_{10}^{AD}(w)}{(z-w)}. \quad (3.31)$$

Finally, we mention that one can decouple one $u(1)$ -current.

4 Bosonic Grassmannian model

In this section, we study the bosonic Grassmannian model (1.1)

$$\frac{\mathfrak{su}(N+M)_k}{\mathfrak{su}(N)_k \oplus \mathfrak{su}(M)_k \oplus \mathfrak{u}(1)_\kappa} \quad (4.1)$$

with $\kappa = kNM(N+M)$. We consider the 't Hooft limit where $N, k \rightarrow \infty$, but keeping M and the 't Hooft parameter λ_M defined in (1.3) finite. In the previous section, we have observed that the coset reduces to the system of MN complex free bosons at $\lambda_M = 1$, and the $U(M)$ invariant condition is not assigned to construct conserved currents. In the next subsection, we first introduce the bosonic Grassmannian coset (4.1) and then in section 4.2 we see how the “ $U(M)$ deconfinement” is realized in the coset language. In section 4.3 and 4.4 we reproduce the gravity partition function from the coset model (4.1) in the $\lambda_M = 1$ limit.

4.1 Primary states

In the Grassmannian coset (4.1), the denominator $\mathfrak{su}(M) \oplus \mathfrak{su}(N) \oplus \mathfrak{u}(1)$ is embedded into $\mathfrak{su}(N+M)$. The way of embedding is determined by those of $SU(M) \times SU(N) \times U(1)$ into $SU(N+M)$. We use

$$v_1(v, u, w) = \begin{pmatrix} w^N u & 0 \\ 0 & \bar{w}^M v \end{pmatrix} \in SU(N+M), \quad (4.2)$$

where $u \in \text{SU}(M)$, $v \in \text{SU}(N)$ and $w \in \text{U}(1)$. In terms of $\text{su}(N+M)$, the $\text{u}(1)$ current K is expressed as $\text{diag}(N, \dots, N, -M, \dots, -M)$, which has the OPE $K(z)K(0) \sim kMN(N+M)z^{-2}$.

The states of the Grassmannian model (4.1) are labeled by the representations of the algebra appearing in the coset expression. Let Λ_L be the highest-weights of $\widehat{\text{su}}(L)$ of level k . We then use the convention that we also denote the corresponding irreducible highest-weight module by Λ_L . For $u(1)_k$ we use $m \in \mathbb{Z}_k$. Then the states of the coset are obtained by the module decomposition as

$$\Lambda_{N+M} = \bigoplus_{\Lambda_N, \Lambda_M, m} (\Lambda_{N+M}; \Lambda_N, \Lambda_M, m) \otimes \Lambda_N \otimes \Lambda_M \otimes m. \quad (4.3)$$

Using the orthogonal basis ϵ_i ($i = 1, 2, \dots, N+M$) with $\epsilon_i \cdot \epsilon_j = \delta_{i,j}$, the highest weights are expressed as

$$\begin{aligned} \Lambda_{N+M} &= \sum_{i=1}^{N+M} \lambda_i^{N+M} \epsilon_i - \frac{|\Lambda_{N+M}|}{N+M} \sum_{j=1}^{N+M} \epsilon_j, \\ \Lambda_M &= \sum_{i=1}^M \lambda_i^M \epsilon_i - \frac{|\Lambda_M|}{M} \sum_{j=1}^M \epsilon_j, \quad \Lambda_N = \sum_{i=1}^N \lambda_i^N \epsilon_{i+M} - \frac{|\Lambda_N|}{N} \sum_{j=1}^N \epsilon_{j+M}. \end{aligned} \quad (4.4)$$

Here λ_j^L is the number of boxes in the j -th row of the Young diagram corresponding to Λ_L , and $|\Lambda_L|$ is the sum of the boxes of the Young diagram. The $\text{u}(1)$ charge is embedded in $\text{su}(N+M)$ as

$$\omega_m = \frac{m}{NM(N+M)} \left(N \sum_{i=1}^M \epsilon_i - M \sum_{j=1}^N \epsilon_{j+M} \right) \quad (4.5)$$

which satisfy $\omega_m(K) = m$. The embedding is possible only when

$$\Lambda_{N+M} - \Lambda_M - \Lambda_N - \omega_m \in \Delta_{N+M} \quad (4.6)$$

with Δ_{N+M} as the root system of $\text{su}(N+M)$. This condition leads to selection rules as

$$\frac{|\Lambda_{N+M}|}{N+M} - \frac{|\Lambda_M|}{M} + \frac{m}{M(N+M)} \in \mathbb{Z}, \quad \frac{|\Lambda_{N+M}|}{N+M} - \frac{|\Lambda_N|}{N} - \frac{m}{N(N+M)} \in \mathbb{Z}, \quad (4.7)$$

where m is defined modulo κ . In principle, we have to take into account field identifications [48], but for large k limit we can neglect it.

The conformal dimension of the state $(\Lambda_{N+M}; \Lambda_N, \Lambda_M, m)$ is

$$h = n + h_{\Lambda_{M+N}}^{M+N,k} - h_{\Lambda_N}^{N,k+M} - h_{\Lambda_M}^{M,k+N} - h_m \quad (4.8)$$

with an integer n . Here

$$h_{\Lambda}^{L,K} = \frac{C_2(\Lambda_L)}{K+L}, \quad h_m = \frac{m^2}{2\kappa} \quad (4.9)$$

with $C_2(\Lambda_L)$ as the second Casimir of $\text{su}(L)$. Notice that $h_{\Lambda_M}^{M,k+N} = 0$ in the large k limit, thus the states of all possible Λ_M degenerate.

4.2 States dual to bulk matter fields

In order to construct the primary states of the full CFT, we need to take a product of chiral and anti-chiral primary states. In the free limit with $\lambda_M = 1$, we constructed operators as in (3.14), where the $U(N)$ invariant condition is assigned, but $U(M)$ indices are still free. We would like to see how this condition is realized in the coset description.

We consider the coset of the form $\mathfrak{g}/\mathfrak{h}$, then the coset primary fields may be given as Ψ_{RL} , where R and L are representations of algebras \mathfrak{g} and \mathfrak{h} . The fields can be constructed by those of affine algebras $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$ [49–52]. For $\hat{\mathfrak{g}}$, we define $\chi_{\hat{\mathfrak{g}},RL}^i$ which are in the affine algebra built from R and transforms in L . Here $i = 1, 2, \dots, \dim L$. The grade in the affine algebra is related to the integer n in (4.8). For $\hat{\mathfrak{h}}$ we define $\chi_i^{\hat{\mathfrak{h}},\bar{L}}$, which transform in \bar{L} . Then the coset fields are

$$\Psi_{RL} = \sum_{i=1}^{\dim L} \chi_{\hat{\mathfrak{g}},RL}^i \chi_i^{\hat{\mathfrak{h}},\bar{L}}. \quad (4.10)$$

For our case, we should set

$$\mathfrak{g} = \mathfrak{su}(N+M)_k, \quad \mathfrak{h} = \mathfrak{su}(N)_k \oplus \mathfrak{su}(M)_k \oplus \mathfrak{u}(1)_\kappa. \quad (4.11)$$

Let us consider the states dual to the bulk matter fields. Generic primary states of the coset are given by the products of the two sectors as

$$(\Lambda_{N+M}; \Lambda_N, \Lambda_M, m) \otimes (\Lambda'_{N+M}; \Lambda'_N, \Lambda'_M, m'). \quad (4.12)$$

The two simplest examples are

$$(f; 0, f, N) \otimes (\bar{f}, 0, \bar{f}, N), \quad (0; \bar{f}, f, N+M) \otimes (0; f, \bar{f}, N+M), \quad (4.13)$$

where f (\bar{f}) denotes (anti-)fundamental representation. There are also complex conjugated ones with exchanging f and \bar{f} . We choose the simplest sets for Λ_{N+M}, Λ_N and $\Lambda'_{N+M}, \Lambda'_N$, but Λ_M, Λ'_M and m, m' are chosen such as to satisfy the selection rules (4.7) in the 't Hooft limit. See (4.27) and (4.28) below.

The field corresponding to the first state in (4.13) is

$$\Psi_{(f;0,f,N)} = \sum_{i=1}^M \chi_{\mathfrak{su}(N+M), (f;0,f,N)}^i \chi_i^{\mathfrak{su}(M), \bar{f}} \chi^{\hat{\mathfrak{u}}(1), N}. \quad (4.14)$$

With large $k \rightarrow \infty$, but finite M , we expect that the $\mathfrak{su}(M)$ sector decouples. Thus the chiral sector may be given by

$$\chi_{\hat{\mathfrak{g}}, (f;0,f,N)}^i \chi^{\hat{\mathfrak{u}}(1), N} \quad (4.15)$$

with label $i = 1, 2, \dots, M$ for the fundamental representation of $\mathfrak{su}(M)$. Multiplying the anti-chiral sector we have (i, \bar{j}) labels with $\bar{j} = 1, 2, \dots, M$ for the anti-fundamental representation of $\mathfrak{su}(M)$. Thus the state can be identified with a scalar field with $U(M)$ Chan-Paton factor. The field corresponding to the second state in (4.13) is

$$\Psi_{(0;\bar{f},f,N+M)} = \sum_{l=1}^N \sum_{i=1}^M \chi_{\mathfrak{su}(N+M), (0;\bar{f},f,N+M)}^{l,i} \chi_l^{\mathfrak{su}(N), f} \chi_i^{\mathfrak{su}(M), \bar{f}} \chi^{\hat{\mathfrak{u}}(1), N+M}. \quad (4.16)$$

Decoupling the $\mathfrak{su}(M)$ sector, the field becomes

$$\sum_{l=1}^N \chi_{\mathfrak{su}(N+M), (0; \bar{f}, f, N+M)}^{l, i} \chi_l^{\mathfrak{su}(N), f} \chi^{\hat{u}(1), N+M}. \quad (4.17)$$

Multiplying the anti-chiral sector, the field is dual to another bulk scalar field with $U(M)$ Chan-Paton factor.

We may see the relation to the free limit in section 3 as follows. Above we assumed that the $\mathfrak{su}(M)$ sector decouples in the large k limit. However, for the $\mathfrak{su}(N)$ sector, we first take the 't Hooft limit with $N, k \rightarrow \infty$ and $\lambda_M = k/(N+k+M)$ finite, then take $\lambda_M \rightarrow 1$ limit. In this limit, the $\mathfrak{su}(N)$ sector becomes free, but the $SU(N)$ invariant constraint should be left, see [44]. The state in (4.17) may reduce to the expression in section 3 by integrating over $\chi_l^{\mathfrak{su}(N), f}$.

4.3 The character of the chiral part

The CFT is given by the product of chiral and anti-chiral sectors, and the combination is fixed such that the torus amplitude is modular invariant. One of the conditions is $h - \bar{h} \in \mathbb{Z}$ for bosonic states, where h and \bar{h} are the conformal dimension of the states for the chiral and anti-chiral sectors. For finite k , this is a quite strong condition and a typical one is given by a diagonal modular invariant, where charge conjugated representations are paired. However, for infinite k , we have seen that states with all possible Λ_M have the same conformal weight. Therefore, the above condition is satisfied even without using the diagonal modular invariant with respect to the Λ_M label. In this subsection we examine the chiral part of the partition function. In the next subsection we consider how we should pair the chiral and the anti-chiral parts, where we first take the $\lambda_M = 1$ limit of the 't Hooft parameter in (1.3), and then discuss the generalization with $\lambda_M \neq 1$.

The characters of $\mathfrak{su}(L)_k$ are denoted as

$$\text{ch}_{\Lambda}^{L, k}(q, e^H) = \text{tr}_{\Lambda} q^{L_0} e^H. \quad (4.18)$$

Here L_0 is the zero mode of Virasoro generator and H is an element of Cartan subalgebra of $\mathfrak{su}(L)$. The character of $\mathfrak{u}(1)_\kappa$ is

$$\Theta_m^\kappa(q, w) = \text{tr}_m q^{L_0} w^{J_0} = \sum_{l \in \mathbb{Z}} w^{m+\kappa l} \frac{q^{\frac{1}{2\kappa}(m+\kappa l)}}{\prod_{n=1}^{\infty} (1 - q^n)}. \quad (4.19)$$

Then the character of the coset (4.1)

$$b_{\tilde{\Xi}}^{N, M, k}(q) = \text{tr}_{\tilde{\Xi}} q^{L_0}, \quad \tilde{\Xi} = (\Lambda_{N+M}; \Lambda_N, \Lambda_M, m) \quad (4.20)$$

is given by the decomposition (4.3) as

$$\text{ch}_{\Lambda_{N+M}}^{N+M, k}(q, \iota_1(u, v, w)) = \sum_{\Lambda_N, \Lambda_M, m} b_{\tilde{\Xi}}^{N, M, k}(q) \text{ch}_{\Lambda_N}^{N, k}(q, v) \text{ch}_{\Lambda_M}^{M, k}(q, u) \Theta_m^\kappa(q, w). \quad (4.21)$$

The embedding is defined in (4.2).

Let us study the large k behavior of the coset character (4.20). At large k , the $\mathfrak{su}(L)_k$ character behaves as (see, e.g., [34, 35, 53])

$$\mathrm{ch}_{\Lambda}^{L,k}(q, e^H) \sim \frac{q^{h_{\Lambda}^{L,k}} \mathrm{ch}_{\Lambda}^L(e^H)}{\prod_{n=1}^{\infty} [(1 - q^n)^{L-1} \prod_{\alpha \in \Delta_L} (1 - q^n e^{\alpha(H)})]}, \quad (4.22)$$

where Δ_L represent the roots of $\mathfrak{su}(L)$ and $\mathrm{ch}^L(e^H)$ is the character of the finite Lie algebra $\mathfrak{su}(L)$. Similarly, the $\mathfrak{u}(1)_\kappa$ character behaves as

$$\Theta_m^\kappa(q, w) \sim \frac{q^{h_m^\kappa} w^m}{\prod_{n=1}^{\infty} (1 - q^n)}. \quad (4.23)$$

Combining all the factors in the coset (4.1), we have

$$\mathrm{ch}_{\Lambda_{N+M}}^{N+M}(\iota_1(u, v, w)) \vartheta(q, \iota_2(u, v, w)) = \sum_{\Lambda_N, \Lambda_M, m} a_{\Xi}^{N,M}(q) \mathrm{ch}_{\Lambda_N}^N(v) \mathrm{ch}_{\Lambda_M}^M(u) w^m, \quad (4.24)$$

where $a_{\Xi}^{N,M}(q)$ is related to the leading term at large k as

$$b_{\Xi}^{N,M,k}(q) \sim q^{h_{\Xi}^{N,M,k}} a_{\Xi}^{N,M}(q), \quad h_{\Xi}^{N,M,k} = h_{\Lambda_{N+M}}^{N+M,k} - h_{\Lambda_N}^{N,k} - h_{\Lambda_M}^{M,k} - h_m^\kappa. \quad (4.25)$$

The factor

$$\vartheta(q, \iota_2(u, v, w)) = \prod_{n=0}^{\infty} \prod_{i=1}^N \prod_{A=1}^M \frac{1}{(1 - \bar{w}^{N+M} \bar{u}_A v_i q^{n+1})(1 - w^{N+M} u_A \bar{v}_i q^{n+1})} \quad (4.26)$$

can be interpreted as the partition function of NM complex free bosons $\phi_{iA}, \phi_{jB}^\dagger$ with $i = 1, 2, \dots, N$ and $A = 1, 2, \dots, M$. Here ϕ_{iA} belongs to $(\mathfrak{f}, \bar{\mathfrak{f}})$ for $\mathfrak{su}(N) \oplus \mathfrak{su}(M)$ and has $-(M+N)$ charge for $\mathfrak{u}(1)$. Moreover, ϕ_{jB}^\dagger belongs to $(\bar{\mathfrak{f}}, \mathfrak{f})$ for $\mathfrak{su}(N) \oplus \mathfrak{su}(M)$ and has $(M+N)$ charge for $\mathfrak{u}(1)$.

At large N it is convenient to express the Λ_L of $\mathfrak{su}(L)$ in terms of a pair of two Young diagrams as $\mathbf{\Lambda}_L = (\Lambda_L^l, \Lambda_L^r)$. Denoting $|\mathbf{\Lambda}_L|_- = |\Lambda_L^r| - |\Lambda_L^l|$, the selection rule (4.7) uniquely fixes [35]

$$m = N|\mathbf{\Lambda}_{N+M}|_- - (N+M)|\mathbf{\Lambda}_N|_- \quad (4.27)$$

in the large N limit. This also leads to

$$|\Lambda_M| = |\mathbf{\Lambda}_{N+M}|_- - |\mathbf{\Lambda}_N|_- \quad (4.28)$$

modulo M . The w -dependent factor in (4.24) may be removed as

$$\tilde{\mathrm{ch}}_{\mathbf{\Lambda}_{M+N}}^{N+M}(\iota_1(u, v, w)) \vartheta(q, \iota_2(u, v, w)) = \sum_{\Lambda_N, \Lambda_M} a_{\mathbf{\Lambda}_{M+N}; \mathbf{\Lambda}_N, \mathbf{\Lambda}_M}^{N,M}(q) \tilde{\mathrm{ch}}_{\mathbf{\Lambda}_N}^N(v \bar{w}^{N+M}) \mathrm{ch}_{\Lambda_M}^M(u) \quad (4.29)$$

by rewriting the $\mathfrak{su}(L)$ character by the $\mathfrak{u}(L)$ one as

$$\mathrm{ch}_{\mathbf{\Lambda}_L}^L(v) w^{|\mathbf{\Lambda}_L|_-} = \tilde{\mathrm{ch}}_{\mathbf{\Lambda}_L}^L(vw). \quad (4.30)$$

In the right hand side, the pair of two Young diagrams labels the irreducible representation of $\mathfrak{u}(L)$.

4.4 The character at the free limit

Let us set $\Lambda_{M+N} = 0$, then we have

$$\vartheta(q, \nu_2(u, v, w)) = \sum_{\Lambda_N, \Lambda_M} a_{0; \Lambda_N, \Lambda_M}^{N, M}(q) \widetilde{\text{ch}}_{\Lambda_N}^N(v \bar{w}^{N+M}) \text{ch}_{\Lambda_M}^M(u). \quad (4.31)$$

Thus this sector reduces to MN complex free bosons in the bi-fundamental representation of $\text{su}(N) \oplus \text{su}(M)$, and the free bosons are organized in terms of representations of $\mathfrak{u}(N) \oplus \mathfrak{su}(M)$ in the coset model (4.1). This is consistent with the fact that the coset reduces to the free boson system in the $\lambda_M = 1$ limit.⁷ For the other value of λ_M , we should also consider the sector with $\Lambda_{M+N} \neq 0$ as we will discuss below. In this subsection, we mainly study the partition function of the free boson theory as a coset partition function in the $\lambda_M = 1$ limit. Here the $\text{U}(N)$ invariant condition is assigned, but the $\text{SU}(M)$ symmetry is treated as a global symmetry like a flavor symmetry. Later we discuss the case with $\lambda_M \neq 1$.

We denote the bosonic modes by $j_{aA} = \partial\phi_{aA}$ with $a = 1, 2, \dots, N$ as the index of the $\mathfrak{u}(N)$ fundamental representation and $A = 1, 2, \dots, M$ as the index of $\mathfrak{su}(M)$ anti-fundamental representation. There are also complex conjugates as $j_{aA}^\dagger = \partial\phi_{aA}^\dagger$. The full Fock space of the free system is spanned by

$$\prod_l j_{-t_l-1}^{\dagger a_l C_l} \prod_m j_{-u_m-1}^{b_m D_m} |0\rangle, \quad (4.32)$$

where t_l, u_m are non-negative integers and $|0\rangle$ is the vacuum state. The $\mathfrak{u}(N)$ invariants are then given by the linear combinations of the ‘‘basic’’ invariants as [47]

$$\prod_{t, u=0}^{\infty} \left(\sum_{a=1}^N j_{-t-1}^{\dagger a A t u} j_{-u-1}^{a B t u} \right)^{L_{tu}} |0\rangle,$$

where L_{tu} are non-negative integers. Since there are M^2 times more bi-linears than the ones for $M = 1$, the contribution from the $\mathfrak{u}(N)$ invariants is computed as (see (3.68) of [35])

$$Z_0^{\text{CFT}}(q) = \left(\prod_{s=2}^{\infty} Z_B^{(s)} \right)^{M^2}, \quad (4.33)$$

which is the one-loop partition function for the massless gauge sector of the gravity theory, see (2.7). Recall that some of the spin 1 gauge fields decouple at $\lambda_M = 1$.

For $\mathfrak{u}(N)$ non-invariants, we should pair chiral and anti-chiral parts.⁸ From the experience of the free limit in section 3, we propose to combine

$$j_{-t-1}^{\dagger a A} \bar{j}_{-u-1}^{a B}, \quad \bar{j}_{-t-1}^{\dagger a A} j_{-u-1}^{a B}, \quad (4.34)$$

⁷A similar relation between a free boson theory and the $\lambda = N/(N+k) \rightarrow 0$ limit of the coset (1.4) is obtained in [44]. See also [54, 55].

⁸For more proper treatment, see appendix C.

where the $u(N)$ index a is the same (but not summed over here) for the chiral and anti-chiral sectors. In other words, for a fixed $su(M)$ label A , states of the form

$$\prod_{l=1}^{n_j} j_{-u_l-1}^{a_l A} |0\rangle \quad (4.35)$$

produces a $u(N)$ tensor of the shape Λ^l . Then we take a pair with the anti-chiral state from $\bar{j}^{\dagger b_m B}$ corresponding to the $u(N)$ tensor of the shape $\bar{\Lambda}^l$. If we sum over all possible u_l while keeping $n_j = |\Lambda^l|$ fixed, then the partition function is computed as (see (2.50) of [35] for a free fermion system)

$$|\text{ch}_{\Lambda^l}(U(1))|^2, \quad (4.36)$$

where the character is defined in (2.12). Since we have M^2 sets with $A, B = 1, 2, \dots, M$, we have

$$\begin{aligned} Z^{\text{CFT}}(q) &= \left(\sum_{\Lambda_N} |\text{ch}_{\Lambda_N^l}(U(1)) \text{ch}_{\Lambda_N^r}(U(1))|^2 \right)^{M^2} Z_0^{\text{CFT}}(q) \\ &= (Z_0 Z_{\text{scalar}}^1)^{M^2}, \end{aligned} \quad (4.37)$$

where we have taken into account the $u(N)$ invariants discussed above. This CFT partition function is the same as the bulk one, but including only one complex scalar with $h_+ = 1$. This is consistent with the argument in section 3.

From the experience of the $M = 1$ case in [34, 35], we should include the sectors with $\Lambda_{N+M} \neq 0$ for $\lambda_M \neq 1$, and the dual gravity theory should have two complex scalars with two different boundary conditions. The detailed analysis can be found in appendix C (but only for the supersymmetric case). For $\lambda_M \neq 1$, the decoupling of spin 1 gauge fields does not occur on the gravity side, and this effect can be easily included in the CFT side. These decoupled spin 1 gauge fields correspond to extra $su(M)$ currents in the CFT as discussed in section 3.4. The negative-mode contribution to the character is simply given by

$$\text{ch}'_M(q, e^H) = \frac{1}{\prod_{n=1}^{\infty} \prod_{\alpha \in \Delta_M} (1 - q^n e^{\alpha(H)})}. \quad (4.38)$$

Taking $H \rightarrow 0$ limit, we have

$$|\text{ch}'^M(q, 1)|^2 = \left(\prod_{n=1}^{\infty} \frac{1}{|1 - q^n|^2} \right)^{M^2-1} = (Z_B^{(1)})^{M^2-1}, \quad (4.39)$$

which reproduces the extra factor in (2.15).

5 Supersymmetric Grassmannian model

We now move on to study the supersymmetric Grassmannian model (1.5)

$$\frac{\text{su}(N+M)_k \oplus \text{so}(2NM)_1}{\text{su}(N)_{k+M} \oplus \text{su}(M)_{k+N} \oplus \text{u}(1)_\kappa} \quad (5.1)$$

with $\kappa = MN(N+M)(N+M+k)$. This coset is proposed to be dual to the $\mathcal{N} = 2$ higher spin supergravity in [6] with $U(M)$ Chan-Paton factor. We study the 't Hooft limit of the coset where $N, k \rightarrow \infty$ while keeping M and the 't Hooft parameter (1.7) finite. In the next section we review the supersymmetric Grassmannian model (5.1), see e.g. [56]. In section 5.2 we see that the coset is related to a system with MN complex free bosons and MN complex free fermions in the language of the partition function. In section 5.3 we reproduce the gravity partition function from the $\lambda = 0$ limit of the coset (5.1).

5.1 Primary states

In the coset (5.1) the denominator $\mathfrak{su}(M) \oplus \mathfrak{su}(N) \oplus \mathfrak{u}(1)$ is embedded into $\mathfrak{su}(N+M) \oplus \mathfrak{so}(2MN)$. The way of embedding is determined by those of $SU(M) \times SU(N) \times U(1)$ into $SU(N+M)$ and $SO(2NM)$. For $SU(N+M)$ we use (4.2). For $\mathfrak{so}(2NM)_1$ we may use the system of $2NM$ free Majorana fermions, which can be given by NM free complex fermions. We denote the fermions as ψ_{iA} and the complex conjugates as ψ_{iA}^\dagger . Here i is the index for the $\mathfrak{su}(N)$ fundamental representation and A is the index for the $\mathfrak{su}(M)$ anti-fundamental representation. The $\mathfrak{u}(1)$ charges for the fermions are set as $-(N+M)$ for ψ_{iA} and $(N+M)$ for ψ_{iA}^\dagger . We denote the corresponding embedding as $\iota_2(u, v, w)$.

The states of the Grassmannian model (5.1) are labeled by the representations of the algebras appearing in the coset expression. We denote Λ_L as the highest weight of the representation for $\mathfrak{su}(L)$ and $m \in \mathbb{Z}_\kappa$ for $\mathfrak{u}(1)_\kappa$ as in the bosonic case. For affine $\mathfrak{so}(2NM)_1$ we use $\omega = -1, 0, 1, 2$, where $\omega = 0, 2$ denotes the identity and the vector representations and $\omega = 1, -1$ denotes the spinor and the co-spinor representations. Then the states of the coset are obtained by the decomposition as

$$\Lambda_{N+M} \otimes \omega = \bigoplus_{\Lambda_N, \Lambda_M, m} (\Lambda_{N+M}, \omega; \Lambda_N, \Lambda_M, m) \otimes \Lambda_N \otimes \Lambda_M \otimes m. \quad (5.2)$$

There are selection rules as (see, e.g., [56])

$$\begin{aligned} m &= N|\Lambda_{N+M}| - (N+M)|\Lambda_N| + N(N+M)(a + \frac{1}{2}M\epsilon), \\ m &= -M|\Lambda_{N+M}| + (N+M)|\Lambda_M| + M(N+M)(b + \frac{1}{2}N\epsilon) \end{aligned} \quad (5.3)$$

modulo κ . Here a and b are integers defined modulo $M(N+M+k)$ and $N(N+M+k)$, respectively, and $\epsilon = 0$ for $\omega = 0, 2$ and $\epsilon = 1$ for $\omega = \pm 1$. We have to take field identifications in account [48], but for large k limit we can neglect it. The conformal dimension of the state $(\Lambda_{N+M}, \omega; \Lambda_N, \Lambda_M, m)$ is

$$h = n + h_{\Lambda_{M+N}}^{M+N, k} + h_\omega^{2MN} - h_{\Lambda_N}^{N, k+M} - h_{\Lambda_M}^{M, k+N} - h_m \quad (5.4)$$

with an integer n and (4.9). Moreover, $h_\omega^{2L} = \omega/4$ for $\omega = 0, 2$ and $h_\omega^{2L} = L/8$ for $\omega = \pm 1$. In the following we consider the sector denoted by NS, which is the direct sum of the identity and the vector representations as in [9, 35].

5.2 The character of chiral part

As in the bosonic case, we examine first the chiral part of the partition function and later consider how we should pair the chiral and the anti-chiral parts. For the $2NM$ free fermions we use

$$\begin{aligned} \theta(q, \nu_2(u, v, w)) &= \text{tr}_{\text{NS}} q^{L_0} \nu_2(u, v, w) \\ &= \prod_{n=0}^{\infty} \prod_{i=1}^N \prod_{A=1}^M (1 + \bar{w}^{N+M} \bar{u}_A v_i q^{n+\frac{1}{2}}) (1 + w^{N+M} u_A \bar{v}_i q^{n+\frac{1}{2}}), \end{aligned} \quad (5.5)$$

where u_A, v_i are eigenvalues of u, v . Then the character of the coset (1.5)

$$sb_{\tilde{\Xi}}^{N,M,k}(q) = \text{tr}_{\tilde{\Xi}} q^{L_0}, \quad \tilde{\Xi} = (\Lambda_{N+M}; \Lambda_N, \Lambda_M, m) \quad (5.6)$$

is given by the decomposition (5.2) as

$$\begin{aligned} \text{ch}_{\Lambda_{N+M}}^{N+M,k}(q, \nu_1(u, v, w)) \theta(q, \nu_2(u, v, w)) \\ = \sum_{\Lambda_N, \Lambda_M, m} sb_{\tilde{\Xi}}^{N,M,k}(q) \text{ch}_{\Lambda_N}^{N,k+M}(q, v) \text{ch}_{\Lambda_M}^{M,k+N}(q, u) \Theta_m^\kappa(q, w) \end{aligned} \quad (5.7)$$

with (4.18) and (4.19). At large k we have

$$\text{ch}_{\Lambda_{N+M}}^{N+M}(\nu_1(u, v, w)) s\vartheta(q, \nu_2(u, v, w)) = \sum_{\Lambda_N, \Lambda_M, m} sa_{\tilde{\Xi}}^{N,M}(q) \text{ch}_{\Lambda_N}^N(v) \text{ch}_{\Lambda_M}^M(u) w^m, \quad (5.8)$$

where we have used (4.22) and (4.23). Here $sa_{\tilde{\Xi}}^{N,M}(q)$ is related to the leading term at large k as

$$sb_{\tilde{\Xi}}^{N,M,k}(q) \sim q^{h_{\tilde{\Xi}}^{N,M,k}} sa_{\tilde{\Xi}}^{N,M}(q), \quad h_{\tilde{\Xi}}^{N,M,k} = h_{\Lambda_{N+M}}^{N+M,k} - h_{\Lambda_N}^{N,k+M} - h_{\Lambda_M}^{M,k+N} - h_m^\kappa. \quad (5.9)$$

Moreover, we have used

$$s\vartheta(q, \nu_2(u, v, w)) = \prod_{n=0}^{\infty} \prod_{i=1}^N \prod_{A=1}^M \frac{(1 + \bar{w}^{N+M} \bar{u}_A v_i q^{n+\frac{1}{2}}) (1 + w^{N+M} u_A \bar{v}_i q^{n+\frac{1}{2}})}{(1 - \bar{w}^{N+M} \bar{u}_A v_i q^{n+1}) (1 - w^{N+M} u_A \bar{v}_i q^{n+1})}. \quad (5.10)$$

This is the partition function of MN complex free bosons and MN complex free fermions in the bi-fundamental representations of $\text{su}(N) \oplus \text{su}(M)$. At large N , we may rewrite this as

$$\tilde{\text{ch}}_{\Lambda_{N+M}}^{N+M}(\nu_1(u, v, w)) s\vartheta(q, \nu_2(u, v, w)) = \sum_{\Lambda_N, \Lambda_M} sa_{\Lambda_{M+N}; \Lambda_N, \Lambda_M}^{N,M}(q) \tilde{\text{ch}}_{\Lambda_N}^N(v \bar{w}^{N+M}) \text{ch}_{\Lambda_M}^M(u). \quad (5.11)$$

Here we set $m = N|\Lambda_{N+M}|_- - (N+M)|\Lambda_N|_-$ as in (4.27) and used (4.30).

5.3 The character at the free limit

Setting $\Lambda_{N+M} = 0$, we have

$$s\vartheta(q, \iota_2(u, v, w)) = \sum_{\Lambda_N, \Lambda_M} sa_{0; \Lambda_N, \Lambda_M}^{N, M}(q) \tilde{\text{ch}}_{\Lambda_N}^N(v\bar{w}^{N+M}) \text{ch}_{\Lambda_M}^M(u). \quad (5.12)$$

Therefore $sa_{0; \Lambda_N, \Lambda_M}^{N, M}(q)$ can be obtained by decomposing the free system in terms of the $u(N)$ representation Λ_N and the $su(M)$ representation Λ_M . We need to assign the $U(N)$ invariant condition, but we treat $SU(M)$ as a global symmetry. The free system corresponds to the case with $\lambda = 0$ of the 't Hooft parameter (1.7). For $\lambda \neq 0$, we should consider the sectors with $\Lambda_{N+M} \neq 0$ as in the bosonic case, see appendix C.

We denote the fermionic and bosonic modes by ψ_{aA} and j_{aA} with $a = 1, 2, \dots, N$ as the index of the $u(N)$ fundamental representation and $A = 1, 2, \dots, M$ as the index of the $u(M)$ anti-fundamental representation. There are also complex conjugates ψ_{aA}^\dagger and j_{aA}^\dagger . The full Fock space of the free system is spanned by

$$\prod_j \psi_{-r_j - \frac{1}{2}}^{\dagger a_j A_j} \prod_k \psi_{-s_k - \frac{1}{2}}^{b_k B_k} \prod_l j_{-t_l - 1}^{\dagger c_l C_l} \prod_m j_{-u_m - 1}^{d_m D_m} |0\rangle, \quad (5.13)$$

where r_j, s_k, t_l, u_m are non-negative integers. The $u(N)$ invariants are then given by the linear combinations of the ‘‘basic’’ invariants as

$$\begin{aligned} & \prod_{r, s=0}^{\infty} \left(\sum_{a=1}^N \psi_{-r - \frac{1}{2}}^{\dagger a A_{rs}^1} \psi_{-s - \frac{1}{2}}^{a B_{rs}^1} \right)^{K_{rs}} \prod_{t, u=0}^{\infty} \left(\sum_{a=1}^N j_{-t-1}^{\dagger a A_{tu}^2} j_{-u-1}^{a B_{tu}^2} \right)^{L_{tu}} \\ & \times \prod_{t, s=0}^{\infty} \left(\sum_{a=1}^N j_{-t-1}^{\dagger a A_{ts}^3} \psi_{-s - \frac{1}{2}}^{a B_{ts}^3} \right)^{P_{ts}} \prod_{r, u=0}^{\infty} \left(\sum_{a=1}^N \psi_{-r - \frac{1}{2}}^{\dagger a A_{ru}^4} j_{-u-1}^{a B_{ru}^4} \right)^{Q_{ru}} |0\rangle, \end{aligned} \quad (5.14)$$

where K_{rs}, L_{tu} are non-negative integers while $P_{ts}, Q_{ru} = 0, 1$. Since there are M^2 times more bi-linears than the ones for $M = 1$, the contribution from $u(N)$ invariants are (see (3.68) of [35])

$$\mathcal{Z}_0^{\text{CFT}}(q) = \left(\prod_{s=2}^{\infty} Z_B^{(s)} (Z_F^{(s-1/2)})^2 Z_B^{(s-1)} \right)^{M^2}, \quad (5.15)$$

which is the one-loop partition function for the massless gauge sector of the gravity theory, see (2.17).

As in the bosonic case, we take pairs such that they are charge conjugates with $u(N)$, but we do not have to take care for the $su(M)$ sector. From the experience of the free limit in section 3, we propose to combine

$$\begin{aligned} & \psi_{-r - \frac{1}{2}}^{\dagger a A} \bar{\psi}_{-s - \frac{1}{2}}^{a B}, & \bar{\psi}_{-r - \frac{1}{2}}^{\dagger a A} \psi_{-s - \frac{1}{2}}^{a B}, & j_{-t-1}^{\dagger a A} \bar{j}_{-u-1}^{a B}, & \bar{j}_{-t-1}^{\dagger a A} j_{-u-1}^{a B}, \\ & j_{-t-1}^{\dagger a A} \bar{\psi}_{-s - \frac{1}{2}}^{a B}, & \bar{j}_{-t-1}^{\dagger a A} \psi_{-s - \frac{1}{2}}^{a B}, & \psi_{-r - \frac{1}{2}}^{\dagger a A} \bar{j}_{-u-1}^{a B}, & \bar{\psi}_{-r - \frac{1}{2}}^{\dagger a A} j_{-u-1}^{a B}, \end{aligned} \quad (5.16)$$

where the $u(N)$ index a is the same for chiral and anti-chiral sector. Following the analysis for the bosonic case and section 3.4 of [35], we have

$$\begin{aligned} \mathcal{Z}^{\text{CFT}}(q) &= \left(\sum_{\Lambda_N} |\text{sch}_{(\Lambda_N^l)^t}(\mathcal{U}(\frac{1}{2})) \text{sch}_{(\Lambda_N^r)^t}(\mathcal{U}(\frac{1}{2}))|^2 \right)^{M^2} \mathcal{Z}_0^{\text{CFT}}(q) \\ &= \left(\mathcal{Z}_0 \mathcal{Z}_{\text{matter}}^{1/2} \right)^{M^2}. \end{aligned} \quad (5.17)$$

Here the supercharacter $\text{sch}_\Lambda(\mathcal{U}_1)$ is defined in (2.23). The partition function reproduces the gravity one, but with matter fields dual to an $\mathcal{N} = 2$ multiplet, and this is consistent with the argument for the free theory in section 3.

For $\lambda \neq 0$, we need to consider the sectors with $\Lambda_{N+M} \neq 0$. Moreover, we have to include bosonic $\text{su}(M)$ currents dual to the spin 1 gauge fields generated by $V_0^{(1)+} \otimes t_a$ with $\text{sl}(M)$ generator t_a . This amounts to multiplying with the negative mode contribution of $\text{su}(M)$ currents (4.39), which is consistent with the gravity partition function in (2.25). In appendix C we show that the gravity partition function with matter fields dual to two sets of $\mathcal{N} = 2$ multiplet in (2.25) can be reproduced by the supersymmetric Grassmannian model with $\lambda \neq 0$ once extra $\text{su}(M)$ factor is added as in (1.9).

6 Supersymmetric Grassmannian model as a product theory

In this section we again consider the Grassmannian Kazama-Suzuki model, but now in a BRST approach. In the first section we construct the Grassmannian model as a product theory of WZNW models with a BRST constraint. In section 6.2 we introduce a first order formulation for the product theory in order to compare with the free theory which is done in section 6.3. In section 6.4 all the currents of spin-one, 3/2 and two are constructed explicitly in the large N limit, and the OPEs of the extended supercharges are calculated. Finally, in section 6.5 the OPEs of the spin-one and spin-3/2 currents are derived from the bulk side, including non-linear terms, and they are successfully compared to the results from the CFT side.

6.1 The product theory

Following [57] we can write a supersymmetric coset model $\mathfrak{g}/\mathfrak{h}$ as a product theory. Here we assume that \mathfrak{g} has invariant bi-linear form Ω which restricted to \mathfrak{h} is non-degenerate. We choose a basis $\{t_\alpha\}$ of \mathfrak{g} such that the basis of \mathfrak{h} is given by the subset $\{t_a\}$ and the remaining generators outside the coset are denoted with barred Greek letters $\{\bar{t}_{\bar{\alpha}}\}$. The product theory is, after removing Kugo-Ojima quartets, consisting of the WZNW model $(\mathfrak{g}, \Omega - \frac{1}{2}\kappa^{(\mathfrak{g})})$ where $\kappa_{\alpha\beta}^{(\mathfrak{g})} = f_{\alpha\gamma}{}^\delta f_{\beta\delta}{}^\gamma$ is the Killing metric, the WZNW model $(\mathfrak{h}, -\Omega|_{\mathfrak{h}} - \frac{1}{2}\kappa^{(\mathfrak{h})})$, $\dim \mathfrak{h}$ ghosts b_a, c^a and, finally, $\dim \mathfrak{g} - \dim \mathfrak{h}$ fermions $\psi_{\bar{\alpha}}$ with metric $\Omega_{\bar{\alpha}\bar{\beta}}$. The spectrum is the BRST cohomology generated by the current

$$j_{\text{BRST}} = (J_a + \tilde{J}_a)c^a + \frac{1}{2}\Omega^{\bar{\alpha}\bar{\gamma}} f_{\alpha\bar{\alpha}}{}^{\bar{\beta}} \psi_{\bar{\beta}} \psi_{\bar{\gamma}} c^a - \frac{1}{2} f_{ab}{}^c b_c c^a c^b, \quad (6.1)$$

where J_a are currents of the \mathfrak{g} WZNW model in the \mathfrak{h} -direction and \tilde{J}_a of the \mathfrak{h} WZNW model.

For the Grassmannian Kazama-Suzuki model we start with $\Omega = (k + M + N) \text{tr}$, and the content is thus as follows

- A $\widehat{\text{su}}(N+M)_k$ WZNW model with matrix element g and central charge $c[M+N, k] = k((N+M)^2 - 1)/(N+M+k)$.
- A $\widehat{\text{su}}(N)_{-2N-M-k}$ WZNW model with matrix element h_N and central charge $c[N, -2N-M-k] = (2N+M+k)(N^2 - 1)/(N+M+k)$.
- A $\widehat{\text{su}}(M)_{-N-2M-k}$ WZNW model with matrix element h_M and central charge $c[M, -N-2M-k] = (N+2M+k)(M^2 - 1)/(N+M+k)$.
- A scalar ϕ_h with level $-NM(N+M)(M+N+k)$, that is with action $S = -\frac{NM(N+M)(N+M+k)}{4\pi} \int d^2z \partial\phi_h \bar{\partial}\phi_h$ and central charge $c_{\text{scalar}} = 1$.
- $2NM$ fermions $\psi^A_i, \bar{\psi}^i_A$ with $i = 1, \dots, N, A = 1, \dots, M$, OPEs (after renormalization) $\psi^A_i(z) \bar{\psi}^j_B(w) \sim \delta_i^j \delta_A^B$ and total central charge $c_{\text{ferm.}} = NM$.
- Three families of ghost systems $c_{(N)}^a, b_{(N)a}, c_{(M)}^d, b_{(M)d}, c_{(\phi)}, b_{(\phi)}$ with $a = 1, \dots, N^2 - 1$ and $d = 1, \dots, M^2 - 1$. The total central charge is $c_{\text{ghosts}} = -2(N^2 + M^2 - 1)$.

We can check that the total central charge is

$$\begin{aligned} c &= c[M+N, k] + c[N, -2N-M-k] + c[M, -N-2M-k] + c_{\text{scalar}} + c_{\text{ferm.}} + c_{\text{ghosts}} \\ &= \frac{3NMk}{N+M+k}, \end{aligned} \quad (6.2)$$

which agrees with (1.6).

6.2 First order formulation

To compare with the free theory (see appendix B and below) it is useful to introduce a first order formalism for the $\widehat{\text{su}}(N+M)_k$ WZNW model which has the action

$$S_k^{\text{WZNW}}(g) = -\frac{k}{4\pi} \int_{\Sigma} d^2z \text{tr}(g^{-1} \partial g, g^{-1} \bar{\partial} g) - \frac{k}{24\pi} \int_B \text{tr}(g^{-1} dg [g^{-1} dg, g^{-1} dg]) \quad (6.3)$$

with $\partial B = \Sigma$. We make an $\mathfrak{sl}(N+M)$ like 3-decomposition of the group element $g = g_1 g_0 g_{-1}$ where

$$g_1 = e^{\gamma^i t_{i, N+A}}, \quad g_0 = e^{i\phi t_{\phi}} \begin{pmatrix} g_N & 0 \\ 0 & g_M \end{pmatrix}, \quad g_{-1} = e^{\tilde{\gamma}^A t_{N+A, i}}. \quad (6.4)$$

Here $(t_{IJ})_{KL} = \delta_{IK} \delta_{JL}$, $I, J, K, L = 1, \dots, N+M$, $i = 1, \dots, N$, $A = 1, \dots, M$, $t_{\phi} = M \sum_{i=1}^N t_{ii} - N \sum_{l=1}^M t_{N+l, N+l}$, and g_N, g_M are respectively group elements of $\mathfrak{su}(N), \mathfrak{su}(M)$.

We can now use the Polyakov-Wiegmann identity

$$S_k^{\text{WZNW}}(gh) = S_k^{\text{WZNW}}(g) + S_k^{\text{WZNW}}(h) - \frac{k}{2\pi} \int d^2z \text{tr}(g^{-1} \bar{\partial} g \partial h h^{-1}) \quad (6.5)$$

to express the action as

$$S_k^{\text{WZNW}}(g_1 g_0 g_{-1}) = S_k^{\text{WZNW}}(h_N) + S_k^{\text{WZNW}}(h_M) + \frac{kNM(N+M)}{4\pi} \int d^2z \partial\phi \bar{\partial}\phi - \frac{k}{2\pi} \int d^2z \text{tr}(g_1^{-1} \bar{\partial} g_1 g_0 \partial g_{-1} (g_{-1})^{-1} g_0^{-1}), \quad (6.6)$$

where

$$-\frac{k}{2\pi} \int d^2z \text{tr}(g_1^{-1} \bar{\partial} g_1 g_0 \partial g_{-1} g_{-1}^{-1} g_0^{-1}) = -\frac{k}{2\pi} \int d^2z e^{-(N+M)i\phi} \bar{\partial} \gamma^i{}_A (g_M)^A{}_B \partial \bar{\gamma}^B{}_j (g_N^{-1})^j{}_i. \quad (6.7)$$

We can now integrate in dimension one fields $\beta^A{}_i$ and $\bar{\beta}^i{}_A$, and we propose that the action takes the following first order form

$$S^{\text{WZNW}}(\phi, \gamma, \beta, \bar{\gamma}, \bar{\beta}) = S_0 + S_{\text{int}}, \quad (6.8)$$

where

$$S_0 = S_{k+M}^{\text{WZNW}}(h_N) + S_{k+N}^{\text{WZNW}}(h_M) + \frac{1}{2\pi} \int d^2z \left(\frac{(k+N+M)NM(N+M)}{2} \partial\phi \bar{\partial}\phi + \frac{iNM(N+M)}{8} \sqrt{g} \mathcal{R} \phi + \beta^A{}_i \bar{\partial} \gamma^i{}_A + \bar{\beta}^i{}_A \partial \bar{\gamma}^A{}_j \right) \quad (6.9)$$

and

$$S_{\text{int}} = \frac{1}{2\pi k} \int d^2z e^{(N+M)i\phi} \beta^A{}_i (g_N)^i{}_j \bar{\beta}^j{}_B (g_M^{-1})^B{}_A. \quad (6.10)$$

Here the renormalization can be read of from the equations of motion

$$\begin{aligned} \beta^A{}_i &= -k e^{-(N+M)i\phi} (g_M)^A{}_B \partial \bar{\gamma}^B{}_j (g_N^{-1})^j{}_i, \\ \bar{\beta}^i{}_A &= -k e^{-(N+M)i\phi} (g_N^{-1})^i{}_j \bar{\partial} \gamma^j{}_B (g_M)^B{}_A, \end{aligned} \quad (6.11)$$

which shows that each of the NM $\beta^A{}_i$ comes with a path integral measure contribution of

$$\delta S = \frac{(N+M)^2}{4\pi} \int d^2z \partial\phi \bar{\partial}\phi + \frac{(N+M)i}{16\pi} \int d^2z \sqrt{g} \mathcal{R} \phi, \quad (6.12)$$

and for the WZNW terms we e.g. have that for each $i = 1, \dots, N$, $\beta^A{}_i$ will increase the level of the $\text{su}(M)$ WZNW model with one. We choose not to renormalize ϕ since in this formulation it has radius 1 for N, M mutually prime. One can check that the central charge and conformal dimensions are correct for this action.

6.3 The free higher spin CFT

In [36] a free CFT consisting of a $\beta\gamma$ and a bc system was constructed representing the linear supersymmetric $sw_\infty[\lambda]$ algebra. We propose that the extension to general M is done by introducing $\beta^A{}_i, \gamma^i{}_A$ -systems with conformal weights $(1 - \lambda/2, \lambda/2)$ and $b^A{}_i, c^i{}_A$ -systems

with weights $(1/2 - \lambda/2, 1/2 + \lambda/2)$ where $i = 1, \dots, N$ and $A = 1, \dots, M$, and the systems have standard OPEs

$$\gamma^i_A(z)\beta^B_j(w) \sim \frac{\delta^B_A \delta^i_j}{z-w}, \quad b^A_i(z)c^j_B(w) \sim \frac{\delta^A_B \delta^j_i}{z-w}. \quad (6.13)$$

We can then define the currents (s integer)

$$\begin{aligned} J_{\text{lin}}^{(s)+}[t_a] &= -(-1)^{s-1} \sum_{i=0}^{s-1} a^i(s, \lambda) \partial^{s-1-i} \{ \beta^B_l(t_a)^A (\partial^i \gamma^l_A) \} \\ &\quad - (-1)^{s-1} \sum_{i=0}^{s-1} a^i(s, \lambda + 1) \partial^{s-1-i} \{ b^B_l(t_a)^A (\partial^i c^l_A) \}, \\ \tilde{J}_{\text{lin}}^{(s)-}[t_a] &= (-1)^{s-1} \frac{s-1+\lambda}{2s-1} \sum_{i=0}^{s-1} a^i(s, \lambda) \partial^{s-1-i} \{ \beta^B_l(t_a)^A (\partial^i \gamma^l_A) \} \\ &\quad - (-1)^{s-1} \frac{s-\lambda}{2s-1} \sum_{i=0}^{s-1} a^i(s, \lambda + 1) \partial^{s-1-i} \{ b^B_l(t_a)^A (\partial^i c^l_A) \}, \\ J_{\text{lin}}^{(s-1/2)\pm}[t_a] &= \pm (-1)^{s-1/2} \sum_{i=0}^{s-1} \alpha^i(s, \lambda) \partial^{s-1-i} \{ (\partial^i \gamma^l_A)(t_a)^A b^B_l \} \\ &\quad - (-1)^{s-1/2} \sum_{i=0}^{s-2} \beta^i(s, \lambda) \partial^{s-2-i} \{ (\partial^i c^l_A)(t_a)^A \beta^B_l \} \end{aligned} \quad (6.14)$$

where $t_a \in \mathfrak{u}(M)$ and where

$$\begin{aligned} a^i(s, \lambda) &= \binom{s-1}{i} \frac{(-\lambda - s + 2)_{s-1-i}}{(s+i)_{s-1-i}}, \\ \alpha^i(s, \lambda) &= \binom{s-1}{i} \frac{(-\lambda - s + 2)_{s-1-i}}{(s+i-1)_{s-1-i}}, \\ \beta^i(s, \lambda) &= \binom{s-2}{i} \frac{(-\lambda - s + 2)_{s-2-i}}{(s+i)_{s-2-i}}. \end{aligned} \quad (6.15)$$

We propose that these currents generate the extended linear $sw_\infty[\lambda]$ algebra with $J_{\text{lin}}^{(s)\pm}[t_a]$ corresponding to the generator⁹ $V^{(s)\pm} \otimes t_a$. Indeed for $M = 1$ it reduces to the system in [36] up to the traced i -index. Note here that we have a factor of 2 on λ compared to [36], and for later use we have taken $\beta \mapsto i\gamma$, and $\gamma \mapsto i\beta$ and $b \leftrightarrow c$, and finally used the anti-automorphism $\sigma(V_m^{(s)\pm}) = (\pm)^{2s} (-1)^{s-1} V_m^{(s)\pm}$. There is however one subtlety: In the $M = 1$ case the $J_{\text{lin}}^{(1/2)+} = -J_{\text{lin}}^{(1/2)-}$ and $J_{\text{lin}}^{(1)+}$ form a decoupled short multiplet of the remaining algebra, however in this case we know we cannot decouple the extra spin-one generator (see app. A), but we have to decouple the spin-half generator.

⁹The tilde on the generator $\tilde{V}^{(s)-}$ is to denote that it is not directly $V^{(s)-}$ in our basis (see app. A for notation), but the generator orthogonal to $V^{(s)+}$ under the supertrace metric.

Let us now go back to our product theory for the Grassmannian using the first order formalism for the $\mathfrak{su}(N+M)$ factor. The point is now that if we ignore the background charge for ϕ , the model takes the form of the β^A_i, γ^i_A -systems, the corresponding fermions $\psi^A_i, \bar{\psi}^i_A$ and three topological $\mathfrak{g}/\mathfrak{g}$ models, namely the ones based on $\hat{u}(1)_{(k+N+M)(N+M)NM}$ and (supersymmetric) $\widehat{\mathfrak{su}}(N)_{N+M+k}$ and $\widehat{\mathfrak{su}}(M)_{N+M+k}$. The models are then related by the interaction term. This splitting is also respected by the BRST current (6.1). The reason is that it only contains the $\mathfrak{su}(N+M)$ currents in the $\mathfrak{su}(N)$, $\mathfrak{su}(M)$ $u(1)$ directions. These currents are in the first order formalism

$$\begin{aligned} J_a^{(N+M,k)} &= J_a^{(N,k+M)} + : \beta^A_i(t_a)^i_j \gamma^j_A : , & \text{for } t_a \in \mathfrak{su}(N) \subset \mathfrak{su}(N+M) , \\ J_d^{(N+M,k)} &= J_d^{(M,k+N)} - : \beta^A_i(t_d)^B_A \gamma^i_B : , & \text{for } t_d \in \mathfrak{su}(M) \subset \mathfrak{su}(N+M) , \\ J_{t_\phi}^{(N+M,k)} &= J_\phi^{(1, NM(N+M)(N+M+k))} + (M+N)\beta^A_i \gamma^i_A \end{aligned} \quad (6.16)$$

where $J_a^{(N+M,k)}$ is the current of $\widehat{\mathfrak{su}}(N+M)_k$ and $J_a^{(N,k+M)}$ is the current of $\widehat{\mathfrak{su}}(N)_{k+M}$ etc. Note that the $\beta\gamma$ -currents here generate $\widehat{\mathfrak{su}}(N)_{-M}$ and $\widehat{\mathfrak{su}}(M)_{-N}$ respectively. For completeness the remaining currents $J_{t_{i,N+A}}, J_{t_{N+A,i}}$ take the following form

$$\begin{aligned} J_{t_{i,N+A}}^{(N+M,k)} &= \beta^A_i , \\ J_{t_{N+A,i}}^{(N+M,k)} &= J_{t_{N+A,N+B}} \gamma^i_B - J_{t_{j,i}} \gamma^j_{A-} : \beta^B_j \gamma^j_A \gamma^i_B : - k \partial \gamma^i_A \\ &= J_d^{(M,k+N)} g_{(M)}^{de} \text{tr}(t_e t_{N+A,N+B}) \gamma^i_B - J_a^{(N,k+M)} g_{(N)}^{ab} \text{tr}(t_b t_{j,i}) \gamma^j_A \\ &\quad - \frac{1}{MN} J_\phi^{(1, NM(N+M)(N+M+k))} \gamma^i_{A-} : \beta^B_j \gamma^j_A \gamma^i_B : - k \partial \gamma^i_A , \end{aligned} \quad (6.17)$$

with right-nested normal ordering. Where $J_{t_{N+A,N+B}}$ and $J_{t_{j,i}}$ are combinations of $\mathfrak{su}(N)$, $\mathfrak{su}(M)$ and $\partial\phi$ currents.

We can now write the BRST currents (6.1) as

$$j_{\text{BRST}} = K_a c^a , \quad (6.18)$$

where K_a are currents of level zero given explicitly by

$$\begin{aligned} K_a &= J_a^{(N,k+M)} + \beta^A_i(t_a)^i_j \gamma^j_A + \tilde{J}_a^{(N,-k-2N-M)} + \psi^A_i(t_a)^i_j \bar{\psi}^j_A + \frac{1}{2} f_{ab}{}^c b_c c^b , \\ &\quad \text{for } t_a \in \mathfrak{su}(N) , \\ K_d &= J_d^{(M,k+N)} - \beta^A_i(t_d)^B_A \gamma^i_B + \tilde{J}_d^{(M,-k-N-2M)} - \psi^B_i(t_d)^A_B \bar{\psi}^i_A + \frac{1}{2} f_{de}{}^f b_f c^e , \\ &\quad \text{for } t_d \in \mathfrak{su}(M) , \\ K_\phi &= J_\phi^{(1, NM(N+M)(N+M+k))} + (M+N)\beta^A_i \gamma^i_A + \tilde{J}_\phi^{(1, -NM(N+M)(N+M+k))} \\ &\quad + (M+N)\psi^A_i \bar{\psi}^i_A , \\ &\quad \text{for } t_\phi \in u(1) . \end{aligned} \quad (6.19)$$

For finite N, M, k this means that BRST invariant currents built out of β^A_i, γ^j_B would need contraction in both the i and A index.

Let us first consider the case $\lambda = 0$, i.e. k infinite with $N/k \rightarrow 0$ and M kept finite. If it were not for the possible infinite number of fields we could localize the action to its classical value. The background charge is further effectively zero in this limit (if we take $N\lambda \rightarrow 0$) and the $u(1)$ systems are thus together with the $su(N)$ and $su(M)$ factors topological, at least when acting in certain states. The left over $\beta\gamma$ systems have exactly the right dimensions compared to the free field theory generating the linear W -algebra.

For general λ we have to be more careful to get the correct dimensions for the $\beta\gamma$ systems, since we cannot obtain this by simple field redefinitions. However, we can do it by adding the following BRST exact term to the action

$$Q_{\text{BRST}} \cdot \frac{\lambda}{(M+N)4\pi} \int d^2z \bar{\partial} \ln |\rho|^2 b_\phi = \frac{\lambda}{(M+N)4\pi} \int d^2z \bar{\partial} \ln |\rho|^2 K_\phi, \quad (6.20)$$

which introduces background charges for the $u(1)$ scalars, the $\beta\gamma$ ghosts and the fermions. Here $\sqrt{g}R = -4\partial\bar{\partial} \ln |\rho|^2$. Indeed, if we bosonize as

$$\beta^A_i = \partial Y^A_i e^{-X^A_i + Y^A_i}, \quad \gamma^i_A = e^{X^A_i - Y^A_i}, \quad : \beta^A_i \gamma^i_A := \partial X^A_i, \quad (6.21)$$

$$\psi^A_i = e^{-Z^A_i}, \quad \bar{\psi}^i_A = e^{Z^A_i}, \quad : \psi^A_i \bar{\psi}^i_A := -\partial Z^A_i, \quad (6.22)$$

and use

$$\begin{aligned} J_\phi^{(1, NM(N+M)(N+M+k))} &= -iNM(N+M)(N+M+k)\partial\phi, \\ \tilde{J}_\phi^{(1, -NM(N+M)(N+M+k))} &= iNM(N+M)(N+M+k)\partial\phi_h, \end{aligned} \quad (6.23)$$

we see that the total background charge for X^A_i becomes $-1/2 + \lambda/2$ and for Z^A_i it becomes $-\lambda/2$, i.e. the conformal dimensions for the $\beta\gamma$ and bc system take the wanted values. The field ϕ now has background charge $iNM^2/2$, and ϕ_h has background charge $iN^2M/2$. We can calculate the central charge and check that we get $c = 3NM(1-\lambda) - 3\lambda M^2 = 3NMk/(N+M+k)$ again. Thus the free theory is naturally embedded in the product theory.

In taking the large k limit, we can choose to take $\beta \mapsto k\beta$ (since it basically had this k in its definition). This means that the action for the $\beta\gamma$ systems also localize to the classical solutions. To keep the currents finite, we then divide them with k . We thus also have to divide j_{BRST} with k . In all, we have the same expression for the currents (6.16), but where the OPEs of γ with β now is proportional to $1/k$ and hence vanishing. We can thus forget the constraints from the $\widehat{su}(M)_{-N}$ currents, but we cannot forget the constraints from the $\widehat{su}(N)_{-M}$ currents. The reason is that we have an infinite amount of currents. If we, as an example, want to know if $\beta^{A_0}_{i_0} \gamma^{j_0}_{B_0}$ is in the BRST cohomology, we see that it will be mapped to a state with an infinite number of fields, due to the infinite number of generators $t_a \in su(N) \subset su(N+M)$, divided by k which is a state of undetermined norm. To ensure invariance we thus can only consider currents of the form $\sum del^n \beta^A_i \gamma^i_B$, i.e. precisely of the form suggested in (6.14) generating the higher spin algebra. However, due to the derivatives these will only be BRST-invariant up to derivatives of the ghosts, and will not keep the interaction term invariant. In the next section we will show how at low spins we can dress the free currents with $su(N)$, $su(M)$ and $u(1)$ currents such that they are approximately BRST invariant in the 't Hooft limit.

6.4 Currents

We now want to extend the free currents from last subsection such that they obey BRST invariance and keep the interaction term invariant, i.e. that they are true symmetries of the theory. We denote the currents dual to $V_m^{(s)\sigma}$ on the bulk side by $J^{(s)\sigma}$. In principle, we already know some BRST invariant currents, namely, the $u(1)$ current, the supercharges and the stress-energy tensor. These can be expressed in their full versions including the ghosts or BRST-equivalent in the coset description where they are constructed only out of currents from the $su(N + M)$ factor and the fermions.

Our strategy will be to basically follow the latter coset description. That is, we write up the most general linear combination of terms of a given spin, not including the fields coming from the gauging, and also not including the ghosts, and demand BRST invariance. That is BRST invariance up to finite contributions in the $su(M)$ direction. It is natural to expect that such a coset description is possible, but it is not a priori clear and we will have one exception below.

To ensure invariance of the interaction term in the action we can only build higher spin currents from the fermions b^A_i, c^i_A , and from the full $\hat{su}(M + N)_k$ currents including the $\beta\gamma$ terms, which we now denote $J_a^{(N)}, J_d^{(M)}, J_\phi, J^A_i \equiv J_{t_{i,N+A}}$ and $J^i_A \equiv J_{t_{N+A,i}}$. Here upper indices transform in the fundamental representation, and lower indices in the anti-fundamental representation. We can create the currents using products and derivatives of these constituents, but we have to contract all i indices to ensure the first order OPEs with the BRST currents are zero, and in the case of finite N, M, k both i and A indices should be contracted. To organize the fermionic currents and easily calculate their OPEs, it is a good idea to introduce the $u(1)$ level MN current $I_\phi = b^A_i c^i_A$, the $su(N)$ level M current $I_a^{(N)} = b^A_i (t_a)^i_j c^j_A$, and the $su(M)$ level N current $I_d^{(M)} = -b^A_i (t_d)^B_A c^i_B$ which are all primaries. The BRST currents are now simply of the form

$$j_{\text{BRST}} = \sum_{su(N)} (J_a^{(N)} + I_a^{(N)}) c^a + \sum_{su(M)} (J_d^{(M)} + I_d^{(M)}) c^d + (J_\phi + (M + N)I_\phi) c^\phi + \dots \quad (6.24)$$

where the extra terms are from the denominator-factors and ghosts.

Spin-one

Let us start with the spin-one currents. For $t_d \in su(M)$ we have the most general linear combination

$$J^{(1)} = \sum_d N_1^d J_d^{(M)} + \sum_d N_2^d I_d^{(M)} + N_1^\phi J_\phi + N_2^\phi I_\phi. \quad (6.25)$$

The second order of the OPEs with the currents K_a from (6.18) are generically divergent in the large N limit, and can spoil BRST invariance even when dividing the BRST current with k . We thus demand that these second order terms are zero, and this fixes the coefficient $N_2^d = -\frac{k}{N} N_1^d$. We thus get one independent solution for each $t_d \in su(M)$

$$\begin{aligned} J^{(1)-}[t_d] &= -\lambda J_d^{(M)} + \frac{\lambda k}{N} I_d^{(M)} \\ &= \lambda \beta^B_i (t_d)^A_B \gamma^i_A - \lambda J_d^{(M,k+N)} - (1 - \lambda) b^B_i (t_d)^A_B c^i_A, \end{aligned} \quad (6.26)$$

where we have chosen an overall normalization λ . We see that the result is exactly as in (6.14) with the simple addition of the $\text{su}(M)$ current. In the $\text{u}(1)$ case we of course have to add J_ϕ instead

$$\begin{aligned} J^{(1)-}[1] &= \frac{\lambda}{M+N} J_\phi - \frac{\lambda k}{M+N} I_\phi \\ &= \lambda \beta^A{}_l \gamma^l{}_A + \frac{\lambda}{M+N} J_\phi^{(1, NM(N+M)(N+M+k))} - (1-\lambda) b^A{}_l c^l{}_A. \end{aligned} \quad (6.27)$$

This current is BRST invariant also in the finite case, where the coefficient in front of the bc current is $-\lambda k/(M+N)$. This is up to normalization exactly the $\text{u}(1)$ charge of the $\mathcal{N}=2$ algebra from [59].

We now introduce the notation $J_{1_M}^{(M)} = -J_\phi/(M+N)$ and $I_{1_M}^{(M)} = -I_\phi$ since these currents in the large N limit have the same OPEs as we expect for the enlargement of $J_d^{(M)}$ to $\text{u}(M)$. We can then compactly write

$$J^{(1)-}[t_d] = -\lambda J_d^{(M)} + (1-\lambda) I_d^{(M)} \quad (6.28)$$

for all $t_d \in \text{u}(M)$. The action of the $\text{su}(M)$ currents K_d on these currents is just first order OPEs giving the $\text{su}(M)$ rotation. The BRST current thus has the following OPE

$$\frac{1}{k} j_{\text{BRST}}(z) J^{(1)-}[t_d](w) \sim \frac{\sum_{\text{u}(M)} f_{ed}^g J^{(1)-}[t_g] c^e(w)}{k(z-w)}. \quad (6.29)$$

The OPEs of $J_1^{(1)}[t_d]$ with themselves obviously do not close, and we need a second set of spin one currents. If we now transcend the coset formalism and allow factors of $\tilde{J}_d^{(M, -k-N-2M)}$ in our linear combinations, we also get the following possibility in the large N limit

$$\begin{aligned} J^{(1)+}[t_d] &= J_d^{(M)} + I_d^{(M)} + \tilde{J}_d^{(M, -k-N-2M)} \\ &= -\beta^B{}_i (t_d)^A{}_B \gamma^i{}_A + J_d^{(M, k+N)} - b^B{}_i (t_d)^A{}_B c^i{}_A + \tilde{J}_d^{(M, -k-N-2M)}. \end{aligned} \quad (6.30)$$

This again fits exactly with (6.14) with the simple addition of the $\text{su}(M)$ currents, and its OPE with the BRST current is like in (6.29). Note that we could also make an operator with $t_d = 1_d$, but since this is a central element, we do not need to include it on the bulk side, and from the point of view of the CFT it would be $K_\phi/(M+N)$ which goes like K_ϕ/k and is thus exact. Also note that the field $J^{(1)+}[t_d]$ would not be seen in a standard coset analysis since we used $\tilde{J}_d^{(M, -k-N-2M)}$. Obviously, this does not help for the closure of the $J^{(1)-}[t_d]$ currents, however, in the large N limit the field $\tilde{J}_d^{(M, -k-N-2M)}$ becomes classical, and we will from now on remove it by hand. In this case the algebra will close.

Spin-3/2 currents, large N case

We now consider the spin-3/2 generators. Let us first note that we have no spin-1/2 BRST invariant generator, and thus the short multiplet of spin-1/2 and spin-one generators

present in the $M = 1$ free case (6.14) gets truncated to the top component. Spin-3/2 generators in the large N limit can be built from $b^B{}_i J^i{}_A$ and $c^i{}_A J^B{}_i$. We suggest

$$J^{(3/2)\pm}[t_d] = \pm \frac{1}{k+N} b^B{}_i (t_d)^A{}_B J^i{}_A + c^i{}_B (t_d)^B{}_A J^A{}_i, \quad (6.31)$$

where $t_d \in \mathfrak{u}(M)$. The coefficients have been chosen to fit with (6.14), when we only keep the $\beta\gamma$ - and bc -systems. In the comparison we have to remove a term of the form $\gamma\partial b$ in the free theory current which cannot appear in the CFT (like the spin-1/2 currents) and a term in the CFT of the form $\gamma\gamma\beta b$ which cannot appear in the free theory. In this case, and with the chosen coefficients we get a perfect match. Finally, we can also recast this is ordinary supercharge notation as $\mathfrak{u}(M)$ extensions of the supercurrents

$$G^\pm[t_d] = \frac{1}{\sqrt{2}} (J^{(3/2)+}[t_d] \pm J^{(3/2)-}[t_d]), \quad (6.32)$$

where $G^\pm[1_M]$ are the actual supercurrents since these are BRST invariant - also in the finite case.

Spin-two currents finite case

Let us now consider the spin-two currents. We will first consider the case of finite N, M, k . We need invariance in the i and A indices. The possible terms in the spin-two operator are thus

$$\begin{aligned} & J_\phi J_\phi, \quad g_{(N)}^{ab} J_a^{(N)} J_b^{(N)}, \quad g_{(M)}^{de} J_d^{(M)} J_e^{(M)}, \quad J^A{}_i J^i{}_A, \quad \partial J_\phi, \\ & J_\phi I_\phi, \quad g_{(N)}^{ab} J_a^{(N)} I_b^{(N)}, \quad g_{(M)}^{de} J_d^{(M)} I_e^{(M)}, \\ & I_\phi I_\phi, \quad g_{(N)}^{ab} I_a^{(N)} I_b^{(N)}, \quad T_{bc}, \quad \partial I_\phi, \end{aligned} \quad (6.33)$$

where $g_{ab} = \text{tr } t_a t_b$. For the fermions we have used that there are only two independent terms for terms with four fermions. The reason is the normal ordering for the four fermion terms can be chosen as $((\beta^A{}_i \gamma^i{}_B)(\beta^C{}_j \gamma^j{}_D))$ up to terms containing two fermions with one derivative. There are now only two ways to contract the $\mathfrak{su}(M)$ indices. Explicitly we find that

$$g_{(M)}^{de} I_d^{(M)} I_e^{(M)} = 2(M+N)T_{bc} - \frac{M+N}{MN} I_\phi I_\phi - g_{(N)}^{ab} I_a^{(N)} I_b^{(N)}, \quad (6.34)$$

which is seen using the relation

$$g_{(M)}^{de} I_d^{(M)} I_e^{(M)} = ((b^A{}_i c^i{}_B)(b^B{}_j c^j{}_C)) - \frac{1}{M} I_\phi I_\phi. \quad (6.35)$$

There is however one caveat: When $M = 1$ we see that $g_{(N)}^{ab} I_a^{(N)} I_b^{(N)}$ can be rewritten in terms of $I_\phi I_\phi$.

We can now use the well-known OPE for a simple Lie algebra

$$J_a(z) g^{bc} J_b J_c(w) \sim 2(k + \hat{g}) \frac{J_a(w)}{(z-w)^2} \quad (6.36)$$

where \hat{g} is the dual Coxeter number and k the level. We can now easily get the possible spin-two symmetries. Firstly we have

$$\begin{aligned}
J_1^{(2)} &= T_{\text{su}(N+M)} - \frac{g_{(N)}^{ab} J_a^{(N)} J_b^{(N)}}{2(k+N)} - \frac{g_{(M)}^{de} J_d^{(M)} J_e^{(M)}}{2(k+M)} - \frac{J_\phi J_\phi}{2NM(N+M)k} \\
&= \frac{1}{2(k+N+M)} (2J^A_i J^i_A - \partial J_\phi - \frac{1}{NMk} J_\phi J_\phi - \frac{M}{k+N} g_{(N)}^{ab} J_a^{(N)} J_b^{(N)} \\
&\quad - \frac{N}{k+M} g_{(M)}^{de} J_d^{(M)} J_e^{(M)}) . \tag{6.37}
\end{aligned}$$

For the invariance the following OPEs are useful

$$\begin{aligned}
J_a^{(N)}(z) J^A_i J^i_A(w) &\sim \frac{M J_a^{(N)}}{(z-w)^2} , \\
J_d^{(M)}(z) J^A_i J^i_A(w) &\sim \frac{N J_d^{(M)}}{(z-w)^2} , \\
J_\phi(z) J^A_i J^i_A(w) &\sim \frac{(N+M)NMk}{(z-w)^3} + \frac{(N+M)J_\phi}{(z-w)^2} . \tag{6.38}
\end{aligned}$$

We can also get a spin-two current including the fermions:

$$\begin{aligned}
J_2^{(2)} &= T_{bc} - \left(\frac{1}{NM(N+M)k} J_\phi I_\phi - \frac{1}{2NM(N+M)k^2} J_\phi J_\phi \right) \\
&\quad - \left(\frac{1}{k} g_{(N)}^{ab} J_a^{(N)} I_b^{(N)} - \frac{M}{2(k+N)k} g_{(N)}^{ab} J_a^{(N)} J_b^{(N)} \right) \\
&\quad - \left(\frac{1}{k} g_{(M)}^{de} J_d^{(M)} I_e^{(M)} - \frac{N}{2(k+M)k} g_{(M)}^{de} J_d^{(M)} J_e^{(M)} \right) . \tag{6.39}
\end{aligned}$$

Surprisingly, a third spin-two current is also possible when $M > 1$ where the term $g_{(N)}^{ab} I_a^{(N)} I_b^{(N)}$ also can be used

$$J_3^{(2)} = \frac{1}{2(M+N)} g_{(N)}^{ab} I_a^{(N)} I_b^{(N)} - \left(\frac{1}{k} g_{(N)}^{ab} J_a^{(N)} I_b^{(N)} - \frac{M}{2(k+N)k} g_{(N)}^{ab} J_a^{(N)} J_b^{(N)} \right) . \tag{6.40}$$

These are all the possible spin-two operators which we see as follows. Given a general linear combination of the operators in (6.33), we can use $J_1^{(2)}, J_2^{(2)}, J_3^{(2)}$ to ensure that the coefficients of $J^A_i J^i_A$, T_{bc} and $g_{(N)}^{ab} I_a^{(N)} I_b^{(N)}$ are zero. Further we can use $J^{(1)-}$ and $\partial J^{(1)-}$ to set the coefficient of $I_\phi I_\phi$ and ∂I_ϕ to zero. The coefficient of ∂J_ϕ must then be zero since it is the only term with a third order OPE with K_ϕ . It is easy to see that the remaining terms can not give invariants of the BRST operator.

We now want to consider the large N limit. From the bulk side we expect to find two $u(M)$ extended spin-two currents. It might seem worrisome that we found three spin-two currents in the finite N case, however we see that $J_3^{(2)}$ can be rewritten in terms of the invariant contraction of the extra spin one currents $J^{(1)-}[t_a]$, and thus should not be counted as a new current.

Spin-two currents, large N case

In the large N limit, we again do not have to worry about the first order OPEs with K_d for $t_d \in \text{su}(M)$. We thus have more possible spin-two operators

$$\begin{aligned}
& J_\phi J_\phi, J_\phi J_d^{(M)}, g_{(N)}^{ab} J_a^{(N)} J_b^{(N)}, J_d^{(M)} J_e^{(M)} + J_e^{(M)} J_d^{(M)}, J^A_i J^i_B, \partial J_\phi, \partial J_d^{(M)}, \\
& J_\phi I_\phi, J_\phi I_d^{(M)}, J_d^{(M)} I_\phi, g_{(N)}^{ab} J_a^{(N)} b^A_i (t_b)^i_j c^j_B, J_d^{(M)} I_e^{(M)}, \\
& I_\phi I_\phi, I_\phi I_d^{(M)}, I_d^{(M)} I_e^{(M)} + I_e^{(M)} I_d^{(M)}, \partial I_\phi, \partial I_d^{(M)}, (\partial b^A_i) c^i_B - b^A_i \partial c^i_B. \quad (6.41)
\end{aligned}$$

Products of non-commuting currents have been kept in symmetric combinations for linear independence since we also include their derivatives. We can now find the following $\text{u}(M)$ extended spin-two operators for $t_d \in \text{su}(M)$

$$\begin{aligned}
J_1^{(2)}[t_d] &= J^B_i (t_d)^A_B J^i_A + \frac{N(k+M)}{2k+M} \partial J_d^{(M)} + \frac{1}{Mk} J_\phi J_d^{(M)} - \frac{N}{2k+M} J_f^{(M)} \text{tr}(t^f t_d t^g) J_g^{(M)} \\
&= J^B_i (t_d)^A_B J^i_A + \frac{N}{2} \partial J_d^{(M)} + \frac{1}{Mk} J_\phi J_d^{(M)} - \frac{N}{2(2k+M)} J_f^{(M)} \text{tr}(t_d \{t^f, t^g\}) J_g^{(M)}, \quad (6.42)
\end{aligned}$$

and for $t_d = 1$ we, of course, get the same result as above

$$\begin{aligned}
J_1^{(2)}[1_M] &= J^A_i J^i_A - \frac{1}{2} \partial J_\phi - \frac{1}{2NMk} J_\phi J_\phi - \frac{M}{2(k+N)} g_{(N)}^{ab} J_a^{(N)} J_b^{(N)} \\
&\quad - \frac{N}{2(k+M)} g_{(M)}^{de} J_d^{(M)} J_e^{(M)}. \quad (6.43)
\end{aligned}$$

We can also write this in a unified form for all $t_d \in \text{u}(M)$ with the notation from (6.28) in the large N limit (we will also denote explicitly when the sum over generators is for $\text{u}(M)$ and not only $\text{su}(M)$)

$$\begin{aligned}
J_1^{(2)}[t_d] &= J^B_i (t_d)^A_B J^i_A + \frac{N}{2} \partial J_d^{(M)} - \frac{\text{tr } t_d}{2(k+N)} g_{(N)}^{ab} J_a^{(N)} J_b^{(N)} \\
&\quad - \frac{N}{4k} \sum_{\text{u}(M) \text{ generators}} J_f^{(M)} \text{tr}(t_d \{t^f, t^g\}) J_g^{(M)}. \quad (6.44)
\end{aligned}$$

Here the following OPEs for $t_d \in \text{u}(M)$ are useful

$$\begin{aligned}
J_\phi(z) J^B_i (t_d)^A_B J^i_A(w) &\sim \frac{(M+N)Nk \text{tr}(t_d)}{(z-w)^3} + (M+N) \frac{\text{tr}(t_d) \frac{1}{M} J_\phi - N \text{tr}(t_d t^f) J_f}{(z-w)^2}, \\
J_a^{(N)}(z) J^B_i (t_d)^A_B J^i_A(w) &\sim \frac{\text{tr}(t_d) J_a^{(N)}}{(z-w)^2}, \\
J_e^{(M)}(z) J^B_i (t_d)^A_B J^i_A(w) &\sim \frac{-kN \text{tr}(t_e t_d)}{(z-w)^3} + \frac{-\text{tr}(t_e t_d) \frac{1}{M} J_\phi + N \text{tr}(t_d t_e t^f) J_f^{(M)}}{(z-w)^2}, \\
&\quad + \frac{J^B_i([t_e, t_d])^A_B J^i_A}{z-w}, \quad (6.45)
\end{aligned}$$

and

$$\begin{aligned}
J_e^{(M)}(z)J_f^{(M)}\text{tr}(t^f t_d t^g)J_g^{(M)}(w) &\sim \frac{kM\text{tr}(t_e t_d)}{(z-w)^3} \\
&+ \frac{k\text{tr}(\{t_e, t_d\}t^f)J_f^{(M)} + M\text{tr}(t_e t_d t^f)J_f^{(M)} + \text{tr}(t_d)J_e^{(M)}}{(z-w)^2} \\
&+ \frac{J_f^{(M)}\text{tr}(t^f[t_e, t_d]t^g)J_g^{(M)}}{z-w}
\end{aligned} \tag{6.46}$$

where we have used the $\text{sl}(N)$ identity $\sum_d t_d A t^d = \text{tr} A 1_M - \frac{1}{M}A$. This gives vanishing OPEs with K_a, K_ϕ whereas for K_e with $t_e \in \text{su}(M)$ we have first order OPEs, and we get

$$\frac{1}{k}j_{\text{BRST}}(z)V_1^{(2)}[t_d](w) \sim \frac{\sum_e V_1^{(2)}[[t_e, t_d]]c^e(w)}{k(z-w)}, \tag{6.47}$$

as in (6.29).

Considering again a general linear combination of the operators in (6.41) we can use $J_1^{(2)}[t_d]$ to assume that we do not have any $J^A_i J^i_B$ terms. Further, we can use products and derivatives of $J^{(1)-}[t_a]$ to get that the only pure fermion terms are $(\partial b^A_i)c^i_B - b^A_i \partial c^i_B$. We can thus create the second $\mathfrak{u}(M)$ extended operator as

$$\begin{aligned}
J_2^{(2)}[t_d] &= (\partial b^B_i)(t_d)^A{}_B c^i_A - b^B_i(t_d)^A{}_B \partial c^i_A + \frac{2}{NMk}J_\phi I_d^{(M)} + \frac{2}{Mk}J_d^{(M)}I_\phi - \frac{2}{Mk^2}J_\phi J_d^{(M)} \\
&- \frac{2}{k}g_{(N)}^{ab}J_a^{(N)}b^B_i(t_b)^i{}_j(t_d)^A{}_B c^j_A \\
&- \frac{1}{k}(J_f^{(M)}\text{tr}(t_d\{t^f, t^g\})I_g^{(M)} - \frac{N}{2k+M}J_f^{(M)}\text{tr}(t_d\{t^f, t^g\})J_g^{(M)})
\end{aligned} \tag{6.48}$$

for $t_d \in \text{su}(M)$ and as before

$$\begin{aligned}
J_2^{(2)}[1_M] &= 2T_{bc} - 2\left(\frac{1}{NMk}J_\phi I_\phi - \frac{1}{2NMk^2}J_\phi J_\phi\right) \\
&- 2\left(\frac{1}{k}g_{(N)}^{ab}J_a^{(N)}I_b^{(N)} - \frac{M}{2(k+N)k}g_{(N)}^{ab}J_a^{(N)}J_b^{(N)}\right) \\
&- 2\left(\frac{1}{k}g_{(M)}^{de}J_d^{(M)}I_e^{(M)} - \frac{N}{2(k+M)k}g_{(M)}^{de}J_d^{(M)}J_e^{(M)}\right).
\end{aligned} \tag{6.49}$$

Finally, we again write this in unified form for $t_d \in \mathfrak{u}(M)$ with the notation from (6.28) in the large N limit

$$\begin{aligned}
J_2^{(2)}[t_d] &= (\partial b^B_i)(t_d)^A{}_B c^i_A - b^B_i(t_d)^A{}_B \partial c^i_A \\
&- \frac{2}{k}g_{(N)}^{ab}J_a^{(N)}b^B_i(t_b)^i{}_j(t_d)^A{}_B c^j_A + \frac{\text{tr} t_d}{(k+N)k}g_{(N)}^{ab}J_a^{(N)}J_b^{(N)} \\
&- \frac{1}{k}\sum_{\mathfrak{u}(M)\text{ generators}}(J_f^{(M)}\text{tr}(t_d\{t^f, t^g\})I_g^{(M)} - \frac{N}{2k}J_f^{(M)}\text{tr}(t_d\{t^f, t^g\})J_g^{(M)}).
\end{aligned} \tag{6.50}$$

Here we have used that for $t_d = 1_M$, $T_{bc}[t_d] \equiv \frac{1}{2}(\partial b^B{}_i)(t_d)^A{}_B c^i{}_A - \frac{1}{2}b^B{}_i(t_d)^A{}_B \partial c^i{}_A$ is simply T_{bc} and for $t_d \in \text{su}(M)$ we have the OPEs

$$\begin{aligned}
I_\phi(z)T_{bc}[t_d](w) &\sim \frac{-I_d^{(M)}}{(z-w)^2}, \\
I_a^{(N)}(z)T_{bc}[t_d](w) &\sim \frac{b^B{}_i(t_a)^i{}_j(t_d)^A{}_B c^j{}_A}{(z-w)^2}, \\
I_e^{(M)}(z)T_{bc}[t_d](w) &\sim \frac{\frac{1}{2}\text{tr}\{t_e, t_d\}t^f I_f^{(M)} - \frac{1}{M}\text{tr}(t_e t_d)I_\phi}{(z-w)^2} + \frac{T_{bc}[[t_e, t_d]]}{z-w}. \tag{6.51}
\end{aligned}$$

Also note that we have a relation similar to (6.47).

It is easy to see that it is not possible to construct other BRST invariant operators.

For the comparison to the bulk in the next subsection, let us here calculate the OPE of the extended supercurrents from (6.32)

$$\begin{aligned}
G^+[t_a](z)G^-[t_b](0) &\sim \frac{\frac{2}{3}c\text{tr}(t_a t_b)}{Mz^3} + \frac{2\lambda(1-\lambda)f_{ab}{}^c J^{(1)+}[t_c] + ((1-2\lambda)f_{ab}{}^c + s_{ab}{}^c)J^{(1)-}[t_c]}{z^2} \\
&+ \frac{(s_{ab}{}^c - f_{ab}{}^c)\frac{1}{k+N}J_1^{(2)}[t_c] + (s_{ab}{}^c + f_{ab}{}^c)\frac{1-\lambda}{2}J_2^{(2)}[t_c]}{z} \\
&+ \frac{\lambda(1-\lambda)f_{ab}{}^c \partial J^{(1)+}[t_c] + \frac{1}{2}((1-2\lambda)f_{ab}{}^c + s_{ab}{}^c)\partial J^{(1)-}[t_c]}{z} \\
&- \frac{\lambda f_{ab}{}^c s^f{}_g((1-\lambda)J^{(1)+} - J^{(1)-})[t_f]((1-\lambda)J^{(1)+} - J^{(1)-})[t_g]}{2kz} \\
&- \frac{(f^g{}_b{}^c(s_{ac}{}^f + f_{ac}{}^f) + \frac{1}{2}(s_{ab}{}^c + f_{ab}{}^c)f^g{}_c)(\lambda J^{(1)+} + J^{(1)-})[t_f]((1-\lambda)J^{(1)+} - J^{(1)-})[t_g]}{(k+N)z}, \tag{6.52}
\end{aligned}$$

where the sums are over $\mathfrak{u}(M)$, $f_{ab}{}^c$ are the structure constants and $s_{ab}{}^c t_c = \{t_a, t_b\}$. The expression simplifies in the case of the actual supercurrents,

$$G^+[1_M](z)G^-[1_M](0) \sim \frac{\frac{2}{3}c}{z^3} + \frac{J^{(1)-}[1_M]}{z^2} + \frac{\frac{2}{k+N}J_1^{(2)}[1_M] + (1-\lambda)J_2^{(2)}[1_M] + \partial J^{(1)-}[1_M]}{z}. \tag{6.53}$$

From this we see that the Virasoro tensor is

$$T = \frac{1}{k+N}J_1^{(2)}[1_M] + \frac{1-\lambda}{2}J_2^{(2)}[1_M]. \tag{6.54}$$

We note that the field $J^{(1)-}[1_M]$ is primary, whereas the fields $J^{(1)-}[t_a]$ in the $\text{su}(M)$ directions will be non-primary

$$J^{(1)-}[t_a](z)T(0) \sim \frac{J^{(1)-}[t_a]}{z^2} - \frac{1-\lambda}{k} \frac{f_a{}^{bc} J_b^{(M)} I_c^{(M)}}{z}. \tag{6.55}$$

Of course, the fields $J^{(1)+}[t_a]$ are all non-primary having vanishing OPE with the stress-energy tensor.

The relation of $J_{1,2}^{(2)}$ to the basis $J^{(2)\pm}$ is not as straightforward in this case and will contain products of the spin-one operators, as we will see in next subsection. Also the relation to (6.14) is difficult due to normal ordering issues.

6.5 OPEs of currents from the bulk/boundary correspondence

In this section we will derive some of the boundary OPEs from the bulk side, and check the results with the currents derived in last subsection.

Let us first see which OPEs we get to linear order from the bulk side. These are most easily obtained in the following way using the notation and results of [30, 58]: We split the gauge field into the AdS₃ part A_{AdS} and a small deformation Ω

$$A = A_{\text{AdS}} + \Omega . \quad (6.56)$$

Here

$$A_{\text{AdS}} = e^\rho V_1^2 dz + V_0^2 d\rho . \quad (6.57)$$

The coupling between the bulk and the boundary is then

$$\exp \left(- \frac{1}{2\pi} \int d^2 z [(\Omega_{\bar{z}})^{(s)\sigma}]|_{\text{bdry}} J^{(s)\sigma} \right) . \quad (6.58)$$

The change under (gauge) transformations of the field is

$$\delta A = d\Lambda + [A, \Lambda]_* . \quad (6.59)$$

As proposed in [58] the gauge transformation

$$\Lambda^{(s)\pm} = \epsilon_s^\pm \sum_{n=1}^{2s-1} \frac{1}{(n-1)!} (-\partial)^{n-1} \Lambda^{(s)}(z) e^{(s-n)\rho} V_{s-n}^{(s)\pm} \quad (6.60)$$

of some operator \mathcal{O} corresponds on the bulk side to

$$\frac{1}{2\pi i} \oint_0 dz \Lambda^{(s)}(z) J^{(s)\pm}(z) \mathcal{O}(0) . \quad (6.61)$$

Indeed for $\Lambda^{(s)}(z) = z^{s-1-m}$, $m = -s+1, \dots, s-1$ these are the global transformations. To calculate the OPEs of currents with themselves, we can then proceed as follows. First we create a current insertion $J^{(s)\sigma}$ at $z=0$ by using (6.60) with $\Lambda^{(s)} = 1/z$ on the identity operator, respectively, on the AdS solution. This gives the following fields on the bulk side

$$\begin{aligned} \Omega_z^{(s)\pm} &= \epsilon \frac{1}{(2s-2)!} \partial^{2s-1} \Lambda^{(s)}(z) e^{-(s-1)\rho} V_{-(s-1)}^{(s)\pm} , \\ \Omega_{\bar{z}}^{(s)\pm} &= \epsilon \sum_{n=1}^{2s-1} \frac{1}{(n-1)!} (-\partial)^{n-1} \bar{\partial} \Lambda^{(s)}(z) e^{(s-n)\rho} V_{s-n}^{(s)\pm} \sim \epsilon 2\pi \delta^{(2)}(z-w) e^{(s-1)\rho} V_{s-1}^{(s)\pm} + \dots , \\ \Omega_\rho^{(s)\pm} &= 0 , \end{aligned} \quad (6.62)$$

where the important part is the leading term in $\Omega_{\bar{z}}^{(s)\pm}$ which is a delta function, indeed giving the wanted insertion. This is a solution to the linearized equation of motion

$$d\Omega + A_{\text{AdS}} \wedge_* \Omega + \Omega \wedge_* A_{\text{AdS}} = 0 . \quad (6.63)$$

To find the OPE with $J^{(s')\sigma'}$ we now perform the transformation (6.60) with $\Lambda^{(s)} = 1, z, \dots, z^{s+s'-1}$. When we are not dealing with a global symmetry we have to remember that A_{AdS} is not kept invariant, which gives the central extension on the CFT side. This gives an extra term (coming from varying both the bulk and the necessary extra boundary term, see [30])

$$\delta S = -\frac{k_{\text{CS}}}{2\pi} \int d^2 z e^{2\rho} \text{str}(\Omega_z \delta \Omega_z) . \quad (6.64)$$

Here the supertrace is the supertrace over higher spin algebra and the trace over the $\text{su}(M)$ part with a prefactor $1/M$. The supertrace is normalized to have $\text{str}(V_1^{(2)+} V_{-1}^{(2)+}) = -1$, i.e. the total supertrace has $\text{str}(V_1^{(2)+} \otimes 1_M, V_{-1}^{(2)+} \otimes 1_M) = -1$. An explicit formula was found in [58]

$$\begin{aligned} & \text{str}(P^\pm V_m^s, P^\pm V_{-m}^s) \\ &= -\frac{2(-1)^{s-m-1} \Gamma(s+m) \Gamma(s-m)}{\lambda_+(1-\lambda_+)(2s-2)!} \frac{\Gamma(s) \sqrt{\pi}}{4^s \Gamma(s+1/2)} (1-\lambda_\pm)_{s-1} (1+\lambda_\pm)_{s-1} \lambda_\pm \end{aligned} \quad (6.65)$$

for $s \in \mathbb{Z}$ and

$$\begin{aligned} & \text{str}(P^\pm V_m^s, P^\mp V_{-m}^s) \\ &= -\frac{2(-1)^{s-m-1} \Gamma(s+m) \Gamma(s-m)}{\lambda_+(1-\lambda_+)(2s-2)!} \frac{\Gamma(s-\frac{1}{2}) \sqrt{\pi}}{4^s \Gamma(s)} (1-\lambda_+)_{s-\frac{1}{2}} (1+\lambda_+)_{s-\frac{3}{2}} \lambda_+ \end{aligned} \quad (6.66)$$

for $s \in \mathbb{Z} + 1/2$. Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the ascending Pochhammer symbol.

We can now use the above formulas to consider the OPEs of the $\text{su}(M)$ current $J^{(1)+}[t_a]$ dual to $1 \otimes t_a$. Remembering the special definition $V_0^{(1)-} = \frac{1}{2}(1-2\lambda+k)$, we define $J^{(1)-}[t_a]$ to be dual to $\frac{1}{2}(1-2\lambda+k) \otimes t_a$. Here k is, of course, the operator in the higher spin algebra, not the level. We then find that $J^{(1)+}[t_a]$ has a first order OPE with all operators except itself

$$J^{(1)+}[t_a](z) J^{(s)\pm}[t_b](0) \sim \frac{f_{ab}^c J^{(s)\pm}[t_c]}{z} , \quad (6.67)$$

and

$$J^{(1)+}[t_a](z) J^{(1)+}[t_b](0) \sim \frac{2k_{\text{CS}} \text{tr } t_a t_b}{\lambda(1-\lambda) M z^2} + \frac{f_{ab}^c J^{(1)+}[t_c]}{z} , \quad (6.68)$$

which follows from $\text{str}(1) = -2/(\lambda(1-\lambda))$ and $\text{str}(1-2\lambda+k) = 0$. This fits perfectly with our currents. These were indeed the currents, which had only first order OPEs with the $\text{su}(M)$ current and transforming like (6.67), see eqs. (6.29) and (6.47). Also the fact that the found $\text{su}(M)$ current has level $k+N$ fits perfectly since

$$c \sim 3\lambda(1-\lambda)M(N+k) . \quad (6.69)$$

We have here used that

$$c = 6k_{\text{CS}} , \quad (6.70)$$

as we will see below.

We can see in general that a central term is only happening between two operators of the same spin and has the form

$$J^{(s)\sigma}[t_a](z)J^{(s)\sigma'}[t_b](0) \sim -\frac{(2s-1)c \operatorname{str}(V_{s-1}^{(s)\sigma'} V_{-s+1}^{(s)\sigma}) \operatorname{tr} t_a t_b}{6Mz^{2s}} + \dots \quad (6.71)$$

For the second spin one current $J^{(1)-}[t_a]$, defined as above, we then easily get

$$J^{(1)-}[t_a](z)J^{(1)-}[t_b](0) \sim \frac{c \operatorname{tr} t_a t_b}{3Mz^2} + \frac{\lambda(1-\lambda)f_{ab}^c J^{(1)+}[t_c] + (1-2\lambda)f_{ab}^c J^{(1)-}[t_c]}{z} \quad (6.72)$$

Here we have used $\frac{1}{4} \operatorname{str}((1-2\lambda+k)(1-2\lambda+k)) = -2$ and the commutator (A.9). This again fits perfectly with our currents (6.28). The OPE of $J^{(1)-}[t_a]$ with the higher spin operators take the form

$$\begin{aligned} J^{(1)-}[t_a](z)J^{(s)\pm}[t_b](0) &\sim \frac{f_{ab}^c((\frac{1}{2}-\lambda)J^{(s)\pm}[t_c] + \frac{1}{2}J^{(s)\mp}[t_c])}{z} && \text{for } s \in \mathbb{Z}, \\ &\sim \frac{(\frac{1}{2}-\lambda)f_{ab}^c J^{(s)\pm}[t_c] + \frac{1}{2}s_{ab}^c J^{(s)\mp}[t_c]}{z} && \text{for } s \in \mathbb{Z} + \frac{1}{2}, \end{aligned} \quad (6.73)$$

where $s_{ab}^c t_c = \{t_a, t_b\}$. This is indeed fulfilled for the fermionic currents (6.31).

We can now check the OPEs of the extended supercurrents $G^\pm[t_a] = (J^{(3/2)+}[t_a] \pm J^{(3/2)-}[t_a])/\sqrt{2}$. Following the linear approach we get

$$\begin{aligned} G^+[t_a](z)G^-[t_b](0) &\sim \frac{4k_{CS}g_{ab}}{Mz^3} + \frac{2\lambda(1-\lambda)f_{ab}^c J^{(1)+}[t_c] + ((1-2\lambda)f_{ab}^c + s_{ab}^c)J^{(1)-}[t_c]}{z^2} \\ &\quad + \frac{s_{ab}^c J^{(2)+}[t_c] + f_{ab}^c J^{(2)-}[t_c]}{z} \\ &\quad + \frac{\partial \left(\lambda(1-\lambda)f_{ab}^c J^{(1)+}[t_c] + ((\frac{1}{2}-\lambda)f_{ab}^c + \frac{1}{2}s_{ab}^c)J^{(1)-}[t_c] \right)}{z} + \mathcal{O} \end{aligned} \quad (6.74)$$

Here the novelty compared to $M=1$ is that also $J^{(2)-}[t_c]$ appears, see appendix A. Starting at this spin non-linear terms can occur, and these are denoted by \mathcal{O} , whereas the above OPEs containing the spin-one currents cannot have non-linear terms. The non-linear terms come with a normalization of $1/k_{CS}$ and such non-perturbative terms are not captured by the above method, only the linear terms. Up to these non-linear terms, we now see that we have perfect agreement with (6.52) and (6.53). From the central term, we see that the central charge need to be identified via the Brown-Henneaux relation (6.70). Further, we can actually fix the non-linear terms for the stress-energy tensor, assuming that the proposed bulk-boundary correspondence holds:

$$\begin{aligned} J^{(2)+}[1_M] &= T_{\text{CFT}} - \frac{3}{2c} J^{(1)-}[1]J^{(1)-}[1] \\ &\quad - \sum_{\text{su}(M)} \left(\frac{g^{de} I_d^{(M)} I_e^{(M)}}{2(N+M)} + \frac{g^{de} J_d^{(M)} J_e^{(M)}}{2(k+M)} - \frac{g^{de} J^{(1)+}[t_d] J^{(1)+}[t_e]}{2(N+k+M)} \right) \\ &\approx T_{\text{CFT}} - \frac{3M}{2c} \sum_{\text{u}(M)} g^{de} J^{(1)-}[t_d] J^{(1)-}[t_e] \end{aligned} \quad (6.75)$$

Here the last line is to leading N -order. To fix the terms we have used that we know from the bulk that T_{CFT} has zero OPE with all $J^{(1)+}[t_a]$, and the same is true for $J^{(2)+}[1_M]$ via (6.67) and hence also for the non-linear terms relating $J^{(2)+}[1_M]$ and T_{CFT} . Further, we know from the CFT that the currents $J^{(1)-}[t_a]$ are not quite primary and satisfy the OPE (6.55), however, $J^{(1)-}[t_a]$ and $J^{(2)+}[1_M]$ should have zero OPEs by (6.73). The suggested non-linear terms solve this.

To find the non-linear terms, we can use the bulk equations of motion via the method developed in [60] and used very nicely in the $M = 1$ case in [38]. We first introduce the gauge field a with the ρ -dependence adjointly removed

$$A = b^{-1}ab + b^{-1}db, \quad b = e^{\rho V_0^{(2)+}}. \quad (6.76)$$

We now consider the background with the current $-\int d^2z \mu(z, \bar{z}) J^{(t)\delta}[t_b]$ turned on, where $\mu = \epsilon 2\pi \delta^{(2)}(z - w)$ when comparing to (6.62). In this background the gauge field in the lowest weight gauge takes the form (including the AdS part)

$$\begin{aligned} a_z &= V_1^{(2)+} \otimes 1 - \frac{1}{k_{CS}} \left(\langle J^{(s)\sigma}[t_a] \rangle_0 + \langle J^{(s)\sigma}[t_a] \rangle_\mu \right) g^{V_{-s'+1}^{(s')\sigma'} \otimes t_{a'} V_{s-1}^{(s)\sigma} \otimes t_a} V_{-s'+1}^{(s')\sigma'} \otimes t_{a'}, \\ a_{\bar{z}} &= \sum_{m=-t+1}^{t-1} \mu_m^c V_m^{(t)\delta} \otimes t_c \end{aligned} \quad (6.77)$$

where $\langle J^{(s)\sigma}[t_a] \rangle_0$ are zeroth order in μ and hence holomorphic, and $\langle J^{(s)\sigma}[t_a] \rangle_\mu$ is first order. $g^{V_{-s'+1}^{(s')\sigma'} \otimes t_{a'} V_{s-1}^{(s)\sigma} \otimes t_a}$ is the inverse metric of the supertrace defined previously, and $\mu \equiv \mu_{t-1}^b$. The remaining μ_{t-1}^c for $c \neq b$ are zero, however the trailing terms μ_m^c with $m < t - 1$ can and will be non-zero. The equations of motion

$$\partial a_{\bar{z}} - \bar{\partial} a_z + [a_z, a_{\bar{z}}] = 0 \quad (6.78)$$

can be solved at linear order in μ for $\bar{\partial} \langle J^{(s)\sigma}[t_a] \rangle_\mu$ which is equivalent to determine

$$-\frac{1}{2\pi} \partial_{\bar{z}} \int d^2w \mu(w, \bar{w}) \langle J^{(s)\sigma}[t_a](z) J^{(t)\delta}[t_b](w) \rangle \quad (6.79)$$

on the CFT side. We can use this to confirm the above OPEs and to get the OPEs of the $u(M)$ -extended supercharges $G^\pm[t_a]$. Firstly

$$G^\pm[t_a](z) G^\pm[t_b](0) \sim 0, \quad (6.80)$$

which is certainly fulfilled by (6.32). And the non-trivial OPEs are

$$\begin{aligned} G^-[t_a](z) G^+[t_b](0) &\sim \frac{4k_{CS} g_{ab}}{Mz^3} + \frac{2A_{ab}}{z^2} \\ &+ \frac{s_{ab}^c J^{(2)+}[t_c] - f_{ab}^c J^{(2)-}[t_c] + \partial A_{ab} + \frac{M}{2k_{CS}} A^c_b A_{ac} + \mathcal{O}(t_a, t_b)}{z} \end{aligned} \quad (6.81)$$

where

$$A_{ab} = \lambda(1 - \lambda) f_{ab}^c J^{(1)+}[t_c] + \left(\left(\frac{1}{2} - \lambda \right) f_{ab}^c - \frac{1}{2} s_{ab}^c \right) J^{(1)-}[t_c]. \quad (6.82)$$

The term $\mathcal{O}(t_a, t_b)$ is added due to the fact that this bulk can not know about the quantum normal ordering issues of the non-linear term. Due to the OPEs of the spin-one operators which each other, we see that $\mathcal{O}(t_a, t_b)$ can be a the derivative of some combination of the spin-one operators, and has a normalization of the order $1/k_{CS}$.

The result fits perfectly with the linear analysis. Also, looking at $t_a = t_b = 1_M$ we see that we again get the $\mathcal{N} = 2$ superalgebra with

$$T_{\text{bulk}} = J^{(2)+}[1_M] + \frac{3M}{4k_{CS}} \sum_{\mathfrak{u}(M)} g^{de} J^{(1)-}[t_d] J^{(1)-}[t_e] + \frac{1}{2} \mathcal{O}(t_a, t_b) . \quad (6.83)$$

From (6.75) we conclude that the stress-energy tensor from the bulk is the same as the CFT Virasoro tensor, at least up to the derivative of a spin-one current which amounts to a twist.

Finally, we can also compare for currents in general $\mathfrak{u}(M)$ directions. First, the relation between $J^{(2)\pm}[t_a]$ and the currents $J_{1,2}^{(2)}[t_a]$ found in last section in eqs. (6.44) and (6.50) needs to be determined. It turns out that the relation is fixed by (6.67) and (6.73) with the normalizations fixed from comparison to (6.14) and demanding that the relation fits with the $\mathfrak{u}(1)$ case for $t_a = 1_M$. We then get

$$\begin{aligned} J^{(2)+}[t_a] &\approx \frac{1}{k+N} J_1^{(2)}[t_a] + \frac{1-\lambda}{2} J_2^{(2)}[t_a] - \frac{3M}{4c} \sum_{\mathfrak{u}(M)} s^{bc} J^{(1)-}[t_b] J^{(1)-}[t_c] , \\ J^{(2)-}[t_a] &\approx -\frac{1}{k+N} J_1^{(2)}[t_a] + \frac{1-\lambda}{2} J_2^{(2)}[t_a] - \frac{3M}{4c} \sum_{\mathfrak{u}(M)} s^{bc} J^{(1)-}[t_b] J^{(1)-}[t_c] , \end{aligned} \quad (6.84)$$

to leading order in the large N limit. We can now compare the result from the boundary side, (6.52), to the result from the bulk, (6.81). This indeed gives a match including the non-linear terms up to derivatives of spin-one currents scaling like $1/k$ which were undetermined from the bulk. This gives a strong confirmation of the proposed duality.

7 Conclusion and discussions

We have proposed that a bosonic higher spin gravity on AdS_3 with $U(M)$ Chan-Paton factor in [6] is dual to the two dimensional Grassmannian model (1.1) (or (1.8) after $\mathfrak{su}(M)$ factor added) in the 't Hooft limit with large N, k , but with M and the 't Hooft parameter λ_M (1.3) finite. It was argued in [15] that a higher spin gravity on AdS_4 with $U(M)$ Chan-Paton factor is dual to three dimensional $U(N)_k \times U(M)_{-k}$ Chern-Simons theory coupled with bi-fundamental matter $A_{i\alpha}, B_{\beta j}$ for large N, k , but finite M . Here we choose i, j for the $U(N)$ index and α, β for the $U(M)$ index. Then higher spin gauge fields with $U(M)$ Chan-Paton factor are dual to operators of the form $\sum_i A_{\alpha i} B_{i\beta}$. Thus the $U(N)$ invariant condition is assigned, but the $U(M)$ invariance is somehow broken by a “deconfinement.” We have prepared a similar system with NM complex free bosons $\phi_{iA}, \phi_{iA}^\dagger$ in two dimensions to construct operators (3.17) dual to higher spin gauge fields. As in the higher dimensional case, we assign only the $U(N)$ invariant condition and the $U(M)$ symmetry is treated as a global symmetry. We have shown that the free system has the

same higher spin symmetry as the asymptotic symmetry of bulk theory. The free system arises as the $\lambda_M = 1$ limit of the Grassmannian model (1.1), and the gravity partition function can be reproduced from the limit of the CFT.

We have also considered $\mathcal{N} = 2$ holography between the higher spin supergravity on AdS_3 with $U(M)$ Chan-Paton factor in [6] and the $\mathcal{N} = (2, 2)$ Grassmannian Kazama-Suzuki model (1.5) [24, 25] (or (1.9) after $\text{su}(M)$ factor added). We again need to take the 't Hooft limit with large N, k , but with M and the 't Hooft parameter λ (1.7) finite. We have shown that a free system with NM complex free bosons and NM complex free fermions has the desired higher spin symmetry, and the $\lambda = 0$ limit of the Grassmannian model reduces to the free system. We have constructed low spin currents for the Grassmannian model at generic λ , and discuss how the $U(M)$ deconfinement occurs at the 't Hooft limit. These currents are compared with those obtained from the bulk theory. The perfect match of the OPEs of the extended supercharges between the bulk and boundary theories is a very strong hint that the symmetries of the two sides match at all λ -values. The reason is that we expect the currents of spin-two, together with standard supersymmetry relations, to generate all the higher spin symmetries, and from equation (6.81) we see that the extended supercharges in turn generate all the spin-two currents except $J^{(2)-}[1]$. The one-loop partition function of the supergravity theory can be reproduced by 't Hooft limit of the coset (1.9).

A motivation to study the supersymmetric case is to see a relation between higher spin gauge theory and superstring theory. Supersymmetry can be introduced to the higher spin gravity on AdS_4 and it was claimed in [15] that a supersymmetric higher spin theory is dual to the $\mathcal{N} = 6$ $U(N)_k \times U(M)_{-k}$ Chern-Simons-matter theory (ABJ theory) [16, 17] with large N, k , but finite M . Since the ABJ theory with large N, k, M is dual to a superstring theory, we can see a relation between higher spin gauge theory and superstring theory. The states dual to superstrings are of the form $\text{tr}(AB \cdots AB)$, thus we need a kind of $U(M)$ deconfinement phase transition in terms of $\lambda^{\text{Bulk}} = M/N$ to go to states dual to higher spin fields. We have studied our holography in the expectation that a similar relation between higher spin gauge theory and superstring theory can be seen even in the lower dimensional case.¹⁰ For this purpose, we first need to find a superstring dual of the Grassmannian model. Moreover, we should investigate more on the $U(M)$ confinement/deconfinement phase transition of the Grassmannian model.

There are other open problems on the higher spin holography. For instance, it is natural to ask for a CFT dual to $\mathcal{N} = 1$ higher spin supergravity with $O(M)$ Chan-Paton factor. A candidate might be an appropriate orbifold of

$$\frac{\text{so}(2N + M)_k \oplus \text{so}(2NM)_1}{\text{so}(2N)_{k+M} \oplus \text{so}(M)_{k+2N}}. \quad (7.1)$$

For $M = 1$ the coset reduces to the one in [10]. We are also interested in what would happen away from the 't Hooft limit where we have to deal with quantum effects on the gravity side. If a consistent quantum gravity can be described only by superstring theory, then the Grassmannian coset (1.5) without extra $\text{su}(M)$ currents should be used as the

¹⁰See [12] for an example in this direction given quite recently.

dual CFT. However, in three dimensions, there is a possibility that the higher spin theory can be quantized consistently, and in that case the duality between the higher spin theory and the coset (1.9) should hold even at finite N .

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A $\mathfrak{gl}(M)$ extended higher spin algebra

For the supersymmetric higher spin algebra $\text{shs}[\lambda] \oplus \mathbb{C}$ we will use the notation of appendix B in [58]. The generators are $V_m^{(s)+} \equiv V_m^s$ and $V_m^{(s)-} = kV_m^s$, where $k^2 = 1$, where k is not to be confused with the level appearing in the dual CFT. The star products among the generators V_m^s can be expressed as

$$V_m^s * V_n^t = \frac{1}{2} \sum_{u=1,2,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda_k) V_{m+n}^{s+t-u} \quad (\text{A.1})$$

with $\lambda_k = (1 - \nu k)/2$ where $\nu = 1 - 2\lambda$. The star product for operators with k follows trivially, one simply has to remember that k anti-commutes with the half-integer spin operators. The particular values of the structure coefficients that we need in this paper, can be found in [58].

We know the following relations [58]

$$g_u^{st}(m, n; \lambda, k) = (-1)^{1+u} g_u^{ts}(n, m; \lambda, (-1)^{2(t+s)} k) , \quad (\text{A.2})$$

which means

$$[V_m^s, V_n^t]_* = \sum_{u=2,4,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda_k) V_{m+n}^{s+t-u} \quad \text{for } s, t \in \mathbb{Z} , \quad (\text{A.3})$$

and

$$\{V_m^s, V_n^t\}_* = \sum_{u=1,3,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda_k) V_{m+n}^{s+t-u} \quad \text{for } s, t \in \mathbb{Z} + 1/2 . \quad (\text{A.4})$$

Further we expect that $g_u^{st}(m, n; \lambda, k)$ is independent of k for u odd. This would then imply

$$[V_m^s, V_n^t]_* = \sum_{u=2,4,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda_k) V_{m+n}^{s+t-u} \quad \text{for } s \in \mathbb{Z}, t \in \mathbb{Z} + 1/2 . \quad (\text{A.5})$$

We now want to consider the $\mathfrak{gl}(M)$ extension of this, $(\text{shs}[\lambda] \oplus \mathbb{C}) \otimes \mathfrak{gl}(M)$. In this case we will, however, not find such nice relations. Denoting the generators of $\mathfrak{gl}(M)$ by t_a , we define the product as

$$(V_m^s \otimes t_a) * (V_n^t \otimes t_b) = V_m^s * V_n^t \otimes t_a t_b . \quad (\text{A.6})$$

We see that it is essential that $\mathfrak{gl}(M)$ is closed under matrix multiplication, and we could not have used e.g. $\mathfrak{sl}(M)$. By using the standard (anti-)commutator $[[,]]*$ we now get a Lie superalgebra. Denoting

$$[t_a, t_b] = f_{ab}{}^c t_c , \quad \{t_a, t_b\} = s_{ab}{}^c t_c , \quad (\text{A.7})$$

we have as example for the bosonic subalgebra

$$\begin{aligned} [V_m^s \otimes t_a, V_n^t \otimes t_b]_* &= \frac{1}{2} \sum_{u=2,4,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda_k) s_{ab}{}^c (V_{m+n}^{s+t-u}, t_c) \\ &+ \frac{1}{2} \sum_{u=1,3,\dots}^{s+t-|s-t|-1} g_u^{st}(m, n; \lambda_k) f_{ab}{}^c V_{m+n}^{s+t-u} \otimes t_c \quad \text{for } s, t \in \mathbb{Z} . \end{aligned} \quad (\text{A.8})$$

As a particular example, note that $(1, t_a)$ is not central, but only $(1, 1)$, and that the extension of the former only spin-one operator $\frac{1}{2}(\nu + k)$, where $\nu = 1 - 2\lambda$, now generates these elements

$$[(\nu + k) \otimes t_a, (\nu + k) \otimes t_b]_* = f_{ab}{}^c (\nu^2 + 1 + 2\nu k) \otimes t_c . \quad (\text{A.9})$$

B Extended $w_\infty[\lambda]$ algebras from ghost systems

In this paper, we have proposed that the Grassmannian coset (1.1) is dual to the bosonic higher spin theory with $U(M)$ Chan-Paton factor at generic λ . Here we consider a bit different system with NM sets of free bosons $(\gamma_{iA}, \beta_{iA})$ where $i = 1, 2, \dots, N$ and $A = 1, 2, \dots, M$. The conformal weights are $(\frac{1-\lambda}{2}, \frac{1+\lambda}{2})$ and OPEs are

$$\gamma_{iA}(z) \beta_{jB}(0) \sim \delta_{ij} \delta_{AB} \frac{1}{z} . \quad (\text{B.1})$$

From them we can define currents

$$[J^{(s)}(z)]_{AB} = \sum_{i=1}^N \sum_{l=0}^{s-1} a^l(s, \lambda) \partial^{s-1-l} \{(\partial^l \beta_{iA}) \gamma_{iB}\} , \quad (\text{B.2})$$

where

$$a^l(s, \lambda) = \binom{s-1}{i} \frac{(-\lambda - s + 1)_{s-1-l}}{(s+l)_{s-1-l}} \quad (\text{B.3})$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$. For $M=1$ they are given in [36]. We can construct the operators as

$$[\mathcal{O}(z, \bar{z})]_{AB} = \sum_{i=1}^N \gamma_{iA}(z) \otimes \bar{\gamma}_{iB}(\bar{z}), \quad [\tilde{\mathcal{O}}(z, \bar{z})]_{AB} = \sum_{i=1}^N \beta_{iA}(z) \otimes \bar{\beta}_{iB}(\bar{z}), \quad (\text{B.4})$$

whose conformal weights are $h = \frac{1-\lambda}{2}$ and $h = \frac{1+\lambda}{2}$, respectively. See [38] for $M=1$. The currents (B.2) generate a linear algebra $w_{1+\infty}[\lambda] \otimes \mathcal{M}$, where the wedge subalgebra of $w_{1+\infty}[\lambda]$ is $\text{hs}[\lambda] \oplus \mathfrak{u}(1)$. However, outside the wedge subalgebra we cannot decouple the $\mathfrak{u}(1) \otimes 1_M$ sector, so we cannot relate the asymptotic symmetry algebra of the dual gravity theory. In other words, the free theory is closely related to, but different from the CFT dual to the higher spin theory with $U(M)$ Chan-Paton factor.

Similar generators can be constructed as in (6.14) for the $\mathcal{N}=2$ supersymmetric case by introducing (b, c) systems along with (β, γ) systems above. The free system should have some direct relation to the Grassmannian coset (1.5), see section 6. In terms of the free system, the operators dual to the massive matter are expressed as

$$\begin{aligned} [\mathcal{O}_B^+(z, \bar{z})]_{AB} &= \sum_{i=1}^N \gamma_{iA}(z) \otimes \bar{\gamma}_{iB}(\bar{z}), & [\mathcal{O}_F^+(z, \bar{z})]_{AB} &= \sum_{i=1}^N c_{iA}(z) \otimes \bar{\gamma}_{iB}(\bar{z}), \\ [\mathcal{O}_F^-(z, \bar{z})]_{AB} &= \sum_{i=1}^N \gamma_{iA}(z) \otimes \bar{c}_{iB}(\bar{z}), & [\mathcal{O}_B^-(z, \bar{z})]_{AB} &= \sum_{i=1}^N c_{iA}(z) \otimes \bar{c}_{iB}(\bar{z}), \end{aligned} \quad (\text{B.5})$$

and their duals

$$\begin{aligned} [\tilde{\mathcal{O}}_B^+(z, \bar{z})]_{AB} &= \sum_{i=1}^N \beta_{iA}(z) \otimes \bar{\beta}_{iB}(\bar{z}), & [\tilde{\mathcal{O}}_F^+(z, \bar{z})]_{AB} &= \sum_{i=1}^N b_{iA}(z) \otimes \bar{\beta}_{iB}(\bar{z}), \\ [\tilde{\mathcal{O}}_F^-(z, \bar{z})]_{AB} &= \sum_{i=1}^N \beta_{iA}(z) \otimes \bar{b}_{iB}(\bar{z}), & [\tilde{\mathcal{O}}_B^-(z, \bar{z})]_{AB} &= \sum_{i=1}^N b_{iA}(z) \otimes \bar{b}_{iB}(\bar{z}). \end{aligned} \quad (\text{B.6})$$

For $M=1$, see [36, 38].

C Comparison of partition function

In this appendix we continue to compare the gravity partition function with the CFT one. In particular, we include the case with $\mathbf{\Lambda}_{N+M} \neq 0$, which should be considered outside the free limit. We focus on the supersymmetric case as it is more interesting, but the bosonic case can be analyzed in almost the same way. In the main context, we closely examined how decoupling of $\text{su}(M)$ in the denominator occurs in the 't Hooft limit. Here we assume the decoupling from the beginning, and deal with the coset (1.9)

$$\frac{\text{su}(N+M)_k \oplus \text{so}(2NM)_1}{\text{su}(N)_{k+M} \oplus \mathfrak{u}(1)_\kappa} \quad (\text{C.1})$$

with $\kappa = NM(N+M)(N+M+k)$ in the 't Hooft limit, where $k, N \rightarrow \infty$ but M and $\lambda = N/(N+M+k)$ finite. Since we removed the $\text{su}(M)$, the states of the coset are then given by the decomposition (see (5.2) without the decoupling)

$$\Lambda_{N+M} \otimes \text{NS} = \bigoplus_{\Lambda_N, m} (\Lambda_{N+M}; \Lambda_N, m) \otimes \Lambda_N \otimes m. \quad (\text{C.2})$$

In the 't Hooft limit, the label m is fixed by the other labels as in (4.27), thus the states are labeled by $(\Lambda_{N+M}; \Lambda_N)$. We consider the diagonal modular invariant as

$$Z^{\text{CFT}}(q) = \lim_{N, k \rightarrow \infty} \sum_{\Lambda_{N+M}, \Lambda_N} |sb_{(\Lambda_{N+M}; \Lambda_N)}^{N, M, k}(q)|^2, \quad (\text{C.3})$$

where $sb_{(\Lambda_{N+M}; \Lambda_N)}^{N, M, k}(q)$ is the branching function of $(\Lambda_{N+M}; \Lambda_N)$. In appendix C.1 we rewrite the gravity partition function in the above form, and in appendix C.2 we show that the CFT partition function in the 't Hooft limit reproduces the gravity one.

C.1 Higher spin partition function

With $U(M)$ Chan-Paton factor the gravity partition function is given as (2.25)

$$Z_M^{\text{Bulk}} = (Z_0)^{M^2} (Z_B^{(1)})^{M^2-1} |Z_{\text{matter}}^{\lambda/2} Z_{\text{matter}}^{(1-\lambda)/2}|^{M^2}, \quad (\text{C.4})$$

where each contribution is expressed as in (2.17), (2.6) and (2.21). In order to compare with the CFT partition function, it is necessary to rewrite the matter part (2.21)

$$\left(Z_{\text{matter}}^h\right)^{1/2} = \prod_{n, m=0}^{\infty} \frac{(1 + q^{h+1/2+n} \bar{q}^{h+m})(1 + q^{h+n} \bar{q}^{h+1/2+m})}{(1 - q^{h+n} \bar{q}^{h+m})(1 - q^{h+1/2+n} \bar{q}^{h+1/2+m})} \quad (\text{C.5})$$

in a suitable form as suggested in [34, 35].

We utilize the formula (A.8) in [35]

$$\prod_{i, j=1}^{\infty} \frac{(1 - x_i \eta_j)(1 - y_i \xi_j)}{(1 - x_i y_j)(1 - \xi_i \eta_j)} = \sum_{\Lambda} s_{\Lambda}(x|\xi) s_{\Lambda}(y|\eta). \quad (\text{C.6})$$

Here $s_{\Lambda}(x|\xi)$ is defined as

$$s_{\Lambda}(x|\xi) = \sum_{T \in \text{STab}_{\Lambda}} \prod_{j \in T} X_{jj} (-1)^j \quad (\text{C.7})$$

with $x = (x_1, x_2, \dots)$, $\xi = (\xi_1, \xi_2, \dots)$ and

$$X_{2i, 2i} = x_{i+1}, \quad X_{2i+1, 2i+1} = \xi_{i+1}. \quad (\text{C.8})$$

Introducing M sets as $x^{(A)}, \xi^{(A)}, y^{(A)}, \eta^{(A)}$ ($A = 1, \dots, M$), we have

$$\begin{aligned} & \prod_{A, B=1}^M \prod_{i, j=1}^{\infty} \frac{(1 - x_i^{(A)} \eta_j^{(B)})(1 - y_i^{(A)} \xi_j^{(B)})}{(1 - x_i^{(A)} y_j^{(B)})(1 - \xi_i^{(A)} \eta_j^{(B)})} \\ &= \sum_{\Lambda} s_{\Lambda}(x^{(1)} \cup \dots \cup x^{(M)} | \xi^{(1)} \cup \dots \cup \xi^{(M)}) s_{\Lambda}(y^{(1)} \cup \dots \cup y^{(M)} | \eta^{(1)} \cup \dots \cup \eta^{(M)}). \end{aligned} \quad (\text{C.9})$$

Furthermore, we use (A.10) in [35]

$$s_\Lambda(x \cup y | \xi \cup \eta) = \sum_{\Xi, \Pi} c_{\Xi\Pi}^\Lambda s_\Xi(x | \xi) x_\Pi(y | \eta) \quad (\text{C.10})$$

with $c_{\Xi\Pi}^\Lambda$ as the Clebsh-Gordan coefficients. Applying the formula successively, we obtain

$$s_\Lambda(x^{(1)} \cup \dots \cup x^{(M)} | \xi^{(1)} \cup \dots \cup \xi^{(M)}) = \sum_{\Lambda_1, \dots, \Lambda_M} c_{\Lambda_1 \dots \Lambda_M}^\Lambda \prod_{A=1}^M s_{\Lambda_A}(x^{(A)} | \xi^{(A)}), \quad (\text{C.11})$$

where we have defined

$$c_{\Lambda_1 \dots \Lambda_M}^\Lambda = \sum_{\Xi_1, \dots, \Xi_{M-2}} c_{\Lambda_1 \Xi_1}^\Lambda \left(\prod_{A=1}^{M-3} c_{\Lambda_{A+1} \Xi_{A+1}}^{\Xi_A} \right) c_{\Lambda_{M-1} \Lambda_M}^{\Xi_{M-2}}. \quad (\text{C.12})$$

Setting

$$x_{i+1}^{(A)} = q^{h+i}, \quad \xi_{i+1}^{(A)} = -q^{h+1/2+i}, \quad y_{i+1}^{(A)} = \bar{q}^{h+i}, \quad \eta_{i+1}^{(A)} = -\bar{q}^{h+1/2+i} \quad (\text{C.13})$$

for all A , we arrive at

$$(\mathcal{Z}_{\text{matter}}^h)^{M^2/2} = \sum_{\Lambda} |\mathcal{B}_\Lambda^h|^2, \quad \mathcal{B}_\Lambda^h = \sum_{\Lambda_1, \dots, \Lambda_M} c_{\Lambda_1 \dots \Lambda_M}^\Lambda \prod_{A=1}^M \text{sch}_{\Lambda_A}(\mathcal{U}(h))$$

with the supercharacter defined in (2.23). Thus the gravity partition function (C.4) can be written as

$$Z_M^{\text{Bulk}} = (\mathcal{Z}_0)^{M^2} (\mathcal{Z}_B^{(1)})^{M^2-1} \sum_{\Lambda_{N+M}^l, \Lambda_{N+M}^r, \Lambda_N^l, \Lambda_N^r} \left| \mathcal{B}_{\Lambda_{N+M}^l}^{\lambda/2} \mathcal{B}_{\Lambda_{N+M}^r}^{\lambda/2} \mathcal{B}_{\Lambda_N^l}^{(1-\lambda)/2} \mathcal{B}_{\Lambda_N^r}^{(1-\lambda)/2} \right|^2. \quad (\text{C.14})$$

C.2 CFT partition function

Now that we consider the coset (C.1) with the decomposition (C.2), the branching function $sb_{\tilde{\Xi}}^{N,M,k}(q)$ with $\tilde{\Xi} = (\Lambda_{N+M}; \Lambda_N, m)$ is given by

$$\text{ch}_{\Lambda_{N+M}}^{N+M,k}(q, \iota_1(v, w)) \theta(q, \iota_2(v, w)) = \sum_{\Lambda_N, m} sb_{\tilde{\Xi}}^{N,M,k}(q) \text{ch}_{\Lambda_N}^{N,k+M}(q, v) \Theta_m^\kappa(q, w), \quad (\text{C.15})$$

where the characters are defined in (4.18), (4.19) and (5.5). Here the embeddings $\iota_i(v, w)$ ($i = 1, 2$) are defined with $u = 1$. At large k , we have

$$\text{ch}_{\Lambda_{N+M}}^{N+M}(\iota_1(v, w)) s\vartheta(q, \iota_2(v, w)) z_B^{(1)}(q) = \sum_{\Lambda_N, m} sa_{\tilde{\Xi}}^{N,M}(q) \text{ch}_{\Lambda_N}^N(v) w^m, \quad (\text{C.16})$$

with (5.10) and

$$sb_{\tilde{\Xi}}^{N,M,k}(q) \sim q^{h_{\tilde{\Xi}}^{N,M,k}} sa_{\tilde{\Xi}}^{N,M}(q), \quad h_{\tilde{\Xi}}^{N,M,k} = h_{\Lambda_{N+M}}^{N+M,k} - h_{\Lambda_N}^{N,k+M} - h_m^\kappa. \quad (\text{C.17})$$

We have introduced

$$z_B^{(1)}(q) = \text{ch}'_M(q, 1) = \left(\prod_{n=1}^{\infty} \frac{1}{1 - q^n} \right)^{M^2 - 1} \quad (\text{C.18})$$

as in (4.39). Setting $m = N|\mathbf{\Lambda}_{N+M}|_- - (N + M)|\mathbf{\Lambda}_N|_-$ as in (4.27), we can remove the factor w^m as

$$\tilde{\text{ch}}_{\mathbf{\Lambda}_{N+M}}^{N+M}(\iota_1(v, w)) s\vartheta(q, \iota_2(v, w)) = \sum_{\mathbf{\Lambda}_N} \tilde{s}a_{\mathbf{\Lambda}_{M+N}; \mathbf{\Lambda}_N}^{N, M}(q) \tilde{\text{ch}}_{\mathbf{\Lambda}_N}^N(v\bar{w}^{N+M}) \quad (\text{C.19})$$

by using $U(N)$ characters in (4.30). Here we have introduced

$$sa_{\mathbf{\Lambda}_{M+N}; \mathbf{\Lambda}_N}^{N, M}(q) = z_B^{(1)}(q) \tilde{s}a_{\mathbf{\Lambda}_{M+N}; \mathbf{\Lambda}_N}^{N, M}(q). \quad (\text{C.20})$$

With $\mathbf{\Lambda}_{M+N} = 0$, the above expression reduces to

$$s\vartheta(q, \iota_2(v, w)) = \sum_{\mathbf{\Lambda}_N, \mathbf{\Lambda}_M} \tilde{s}a_{0; \mathbf{\Lambda}_N}^{N, M}(q) \tilde{\text{ch}}_{\mathbf{\Lambda}_N}^N(v\bar{w}^{N+M}). \quad (\text{C.21})$$

Thus the function $\tilde{s}a_{0; \mathbf{\Lambda}_N}^{N, M}(q)$ can be obtained by decomposing free bosons and free fermions in terms of $U(N)$ representations. As in section 5.3, we obtain

$$\tilde{s}a_{0; 0}^M = \lim_{N \rightarrow \infty} \tilde{s}a_{0; 0}^{N, M} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \left(\frac{(1 + q^{n-1/2})^2}{(1 - q^n)(1 - q^{n-1})} \right)^{M^2}. \quad (\text{C.22})$$

For $\mathbf{\Lambda}_N \neq 0$, we consider the Fock space generated by $(A = 1, 2, \dots, M)$

$$\prod_{k=1}^{n_f} \psi_{-s_k - \frac{1}{2}}^{b_k A} \prod_{m=1}^{n_b} j_{-u_m - 1}^{d_m A} |0\rangle, \quad (\text{C.23})$$

which produces a $U(N)$ tensor of the shape Λ_A^l . Summing over all possible s_k and u_m with keeping $n_f + n_b = |\Lambda_A^l|$, the contribution to the branching function is written as (see (3.69) of [35])

$$\text{sch}_{(\Lambda_A^l)^t}(\mathcal{U}(\frac{1}{2})). \quad (\text{C.24})$$

Products of characters with $A = 1, \dots, M$ have to be taken, and the product of the representation Λ_A^l should behave as Λ_N^l . Thus we find

$$\tilde{s}a_{0; \mathbf{\Lambda}_N}^M = \lim_{N \rightarrow \infty} \tilde{s}a_{0; \mathbf{\Lambda}_N}^{N, M} = \mathcal{B}_{(\Lambda_N^l)^t}^{1/2} \mathcal{B}_{(\Lambda_N^r)^t}^{1/2} \tilde{s}a_{0; 0}^M. \quad (\text{C.25})$$

In order to consider the case with $\mathbf{\Lambda}_{N+M} \neq 0$, it is convenient to introduce the restriction functions $r_{\mathbf{\Lambda}\Phi}^{N, M}$ as

$$\tilde{\text{ch}}_{\mathbf{\Lambda}}^{N+M}(\iota_1(v, w)) = \sum_{\Phi} r_{\mathbf{\Lambda}\Phi}^{N, M} \tilde{\text{ch}}_{\Phi}^N(v\bar{w}^{N+M}). \quad (\text{C.26})$$

Then from (3.52) of [35] we have

$$\tilde{s}a_{\Lambda_{N+M};\Lambda_N}^{N,M}(q) = \sum_{\Phi,\Psi} r_{\Lambda_{N+M}\Phi}^{N,M} c_{\Phi\Lambda_N}^{(N)\bar{\Psi}} \tilde{s}a_{0;\Psi}^{N,M}(q), \quad (\text{C.27})$$

where $c_{\Phi\Lambda_N}^{(N)\bar{\Psi}}$ are $U(N)$ Clebsh-Gordan coefficients. We will use later that

$$\lim_{N \rightarrow \infty} r_{\Lambda\Phi}^{N,M} \rightarrow r_{\Lambda^l\Phi^l r_{\Lambda^r\Phi^r}}, \quad \lim_{N \rightarrow \infty} c_{\Lambda\Phi}^{(N)\bar{\Psi}} \rightarrow c_{\Lambda^l\Phi^r}^{\Psi^r} c_{\Lambda^r\Phi^l}^{\Psi^l}. \quad (\text{C.28})$$

and

$$\sum_{\Lambda} c_{\Xi\Phi}^{\Lambda} \mathcal{B}_{\Lambda}^{1/2} = \mathcal{B}_{\Xi}^{1/2} \mathcal{B}_{\Phi}^{1/2}, \quad (\text{C.29})$$

which comes from

$$\begin{aligned} & \sum_{\Lambda} c_{\Xi\Phi}^{\Lambda} s_{\Lambda}(x^{(1)} \cup \dots \cup x^{(M)} | \xi^{(1)} \cup \dots \cup \xi^{(M)}) \\ &= s_{\Xi}(x^{(1)} \cup \dots \cup x^{(M)} | \xi^{(1)} \cup \dots \cup \xi^{(M)}) s_{\Phi}(x^{(1)} \cup \dots \cup x^{(M)} | \xi^{(1)} \cup \dots \cup \xi^{(M)}). \end{aligned} \quad (\text{C.30})$$

Let us introduce

$$x^{(M+1)} = (w_1, \dots, w_M, 0, 0, \dots), \quad \xi^{(M+1)} = (0, 0, \dots), \quad (\text{C.31})$$

then $s_{\Pi}(x^{(M+1)} | \xi^{(M+1)})$ becomes a $U(M)$ character. Setting $w_A = 1$ for all A , we have

$$\begin{aligned} & s_{\Lambda}(x^{(1)} \cup \dots \cup x^{(M+1)} | \xi^{(1)} \cup \dots \cup \xi^{(M+1)}) \\ &= s_{\Lambda}((x^{(1)} \cup \{1\}) \cup \dots \cup (x^{(M)} \cup \{1\}) | \xi^{(1)} \cup \dots \cup \xi^{(M)}) \\ &= \sum_{\Pi} r_{\Lambda\Pi} s_{\Pi}(x^{(1)} \cup \dots \cup x^{(M)} | \xi^{(1)} \cup \dots \cup \xi^{(M)}), \end{aligned} \quad (\text{C.32})$$

where (C.10) has been used and $r_{\Lambda\Pi} = \sum_{\Xi} c_{\Xi\Pi}^{\Lambda} \dim \Xi$ in this case.¹¹ With the second equality and (A.13) of [35], we arrive at

$$\mathcal{B}_{\Lambda}^0 = \sum_{\Xi} r_{\Lambda\Xi} \mathcal{B}_{\Xi}^{1/2}. \quad (\text{C.33})$$

Noticing that

$$\lim_{N,k \rightarrow \infty} h_{\Xi}^{N,M,k} = \frac{\lambda}{2} (|\Lambda_{N+M}| - |\Lambda_N|) \quad (\text{C.34})$$

and $|\Lambda| = \sum_{A=1}^M |\Lambda_A|$ for non zero $c_{\Lambda_1 \dots \Lambda_A}^{\Lambda}$, the branching function is computed as

$$\lim_{N,k \rightarrow \infty} b_{\Lambda_{N+M};\Lambda_N}^{N,M,k} = \mathcal{B}_{\Lambda_{N+M}^l}^{\lambda/2} \mathcal{B}_{\Lambda_{N+M}^r}^{\lambda/2} \mathcal{B}_{(\Lambda_N^l)^t}^{(1-\lambda)/2} \mathcal{B}_{(\Lambda_N^r)^t}^{(1-\lambda)/2} \tilde{s}a_{0;0}^M(q) z_B^{(1)}(q). \quad (\text{C.35})$$

Considering the diagonal modular invariant

$$Z^{\text{CFT}}(q) = \lim_{N,k \rightarrow \infty} \sum_{\Lambda_{N+M};\Lambda_N} |b_{\Lambda_{N+M};\Lambda_N}^{N,M,k}|^2, \quad (\text{C.36})$$

we can show that the CFT partition function defined above reproduces the gravity one given in (C.14).

¹¹With $w_A = 1$ for all A , we have $\dim \Xi = s_{\Xi}(x^{(M+1)} | 0)$.

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