Analysis of the salary distribution in Luxembourg

A finite mixture model approach

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joint work with
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Outline

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1. Nagin's Finite Mixture Model
2. The Luxemburgish salary trajectories

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4. Generalization of the basic model

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This is still an inter-individual model, but unlike other classical models such as standard growth curve models, it allows the existence of subpolulations with completely different behaviors.

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Aim of the analysis: Find r groups of trajectories of a given kind (for instance polynomials of degree 4, $P(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4$.

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- mixture : population composed of a mixture of unobserved groups

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Hence,

$$\begin{aligned} y_{i_t} &= S_{min} \quad \text{si} \quad y^*_{i_t} < S_{min}, \\ y_{i_t} &= y^*_{i_t} \quad \text{si} \quad S_{min} \leq y^*_{i_t} \leq S_{max}, \\ y_{i_t} &= S_{max} \quad \text{si} \quad y^*_{i_t} > S_{max}, \end{aligned}$$

where S_{min} and S_{max} dennote the minimum and maximum of the censored normal distribution.

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$$p^j(y_{i_t} = S_{max}) = 1 - \Phi\left(\frac{S_{max} - \beta^j t_{i_t}}{\sigma}\right). \quad (7)$$

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Software:

SAS-based Proc Traj procedure
by Bobby L. Jones (Carnegie Mellon University).

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Mplus package by L.K. Muthén and B.O Muthén.

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Fewer groups are required to specify a satisfactory model.

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3. can create the illusion of non-existing groups.

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Rule:

The bigger the BIC, the better the model!

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To be classified into a small group, an individual really needs to be strongly consistent with it.

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OCC_j should be greater than 5 for all groups.

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Diagonostic 4: Confidence Intervals for Group Membership Probabilities

The confidence intervals for group membership probabilities estimates should be narrow, i.e. standard deviation of π_j should be small.

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- marital status

The data : second dataset

Salaries of all workers in Luxembourg which began to work in Luxembourg between 1980 and 1990 at an age less than 30 years.

1.303.010 salary lines corresponding to 85.049 workers.

Some sociological variables:

- gender (male, female)
- nationality and residentship
- sector of activity
- year of birth
- age in the first year of professional activity
- marital status
- year of birth of children

Dataset transformations

Mathematica programming

Dataset transformations

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- 1 row per year $\rightarrow$ 1 row per worker

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- taking out the years without work up to a maximum of five years

LUXEMBOURG SCHOOL OF FINANCE

Dataset transformations

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- 1 row per year $\rightarrow$ 1 row per worker
- selection of the period we are interested in
- taking out the years without work up to a maximum of five years
- selection of the people who worked at least during 20 years

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Transformations in SPSS

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Transformations in SPSS

- the number of months problem!
- elimination of all the workers who had monthly salaries above 15.000

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Transformations in SPSS

- the number of months problem!
- elimination of all the workers who had monthly salaries above 15.000
- creation of the time variables necessary for the Proc Traj procedure

Proc Traj procedure

Selection of the time period for macroeconomic reasons

Proc Traj procedure

Selection of the time period for macroeconomic reasons (Crisis in the steel industry and emergence of the financial market place of Luxembourg)

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20 years of work for workers beginning their carrier between 1982 and 1987

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Proc Traj Macro:

Proc Traj procedure

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20 years of work for workers beginning their carrier between 1982 and 1987

Proc Traj Macro:

```
DATA TEST;
    INPUT ID O1-O20 T1-T20;
    CARDS;
data
RUN;
```

Proc Traj procedure

Selection of the time period for macroeconomic reasons (Crisis in the steel industry and emergence of the financial market place of Luxembourg)

20 years of work for workers beginning their carrier between 1982 and 1987

Proc Traj Macro:

```
DATA TEST;
    INPUT ID O1-O20 T1-T20;
    CARDS;
data
RUN;

PROC TRAJ DATA=TEST OUTPLOT=OP OUTSTAT=OS OUT=OF
OUTEST=OE ITDETAIL;
    ID ID; VAR O1-O20; INDEP T1-T20;
    MODEL CNORM; MAX 8000; NGROUPS 6; ORDER 4 4 4 4 4 4;
RUN;
```

LUXEMBOURG SCHOOL OF FINANCE

Result for 9 groups (dataset 1)

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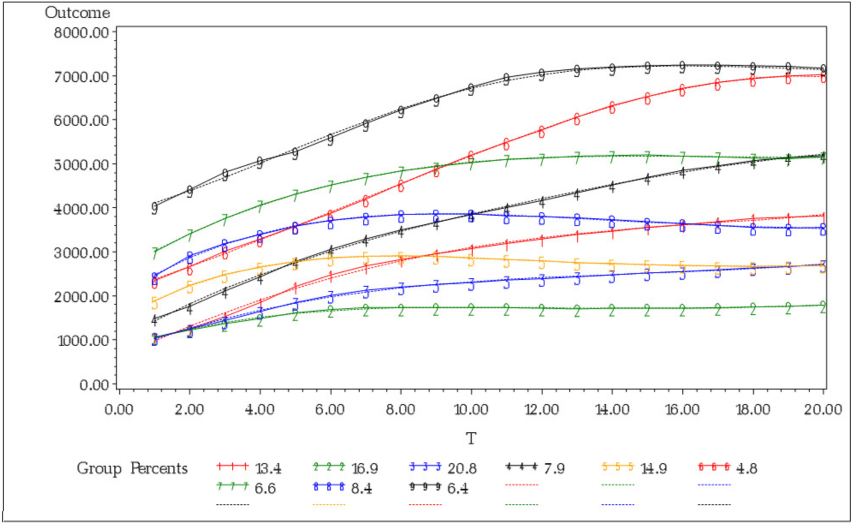


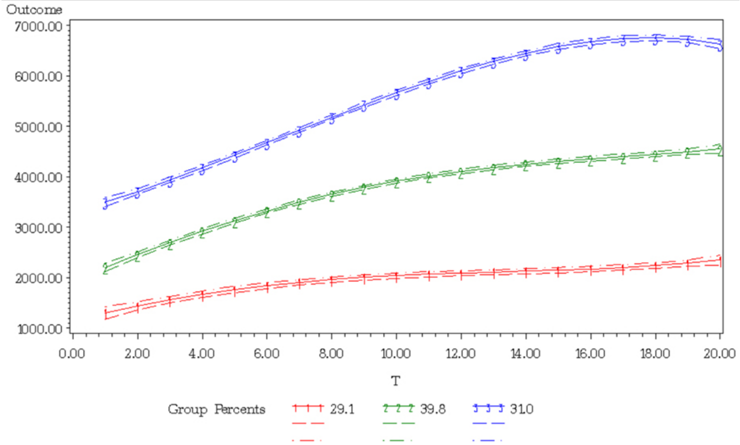
Results for 9 groups (dataset 1)

Maximum Likelihood Estimates
Model: Censored Normal (CNORM)

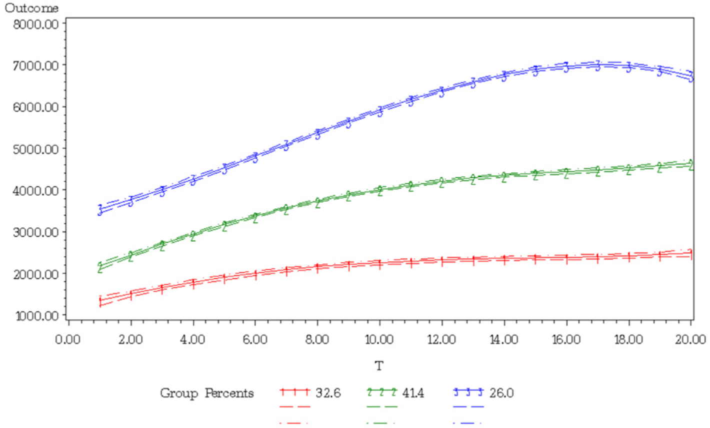
Group	Parameter	Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
1	Intercept	589.03067	18.46813	31.894	0.0000
	Linear	387.72145	11.31617	34.263	0.0000
	Quadratic	-14.36621	2.12997	-6.745	0.0000
	Cubic	-0.01563	0.15109	-0.103	0.9176
	Quartic	0.00856	0.00358	2.395	0.0166
2	Intercept	784.79156	15.75939	49.798	0.0000
	Linear	277.63602	9.78078	28.386	0.0000
	Quadratic	-28.36731	1.83236	-15.481	0.0000
	Cubic	1.17739	0.12972	9.076	0.0000
	Quartic	-0.01635	0.00307	-5.330	0.0000
3	Intercept	709.28728	15.90545	44.594	0.0000
	Linear	318.88029	8.97949	35.512	0.0000
	Quadratic	-21.54540	1.69611	-12.703	0.0000
	Cubic	0.62010	0.12002	5.167	0.0000
	Quartic	-0.00440	0.00284	-1.554	0.1203

Outline

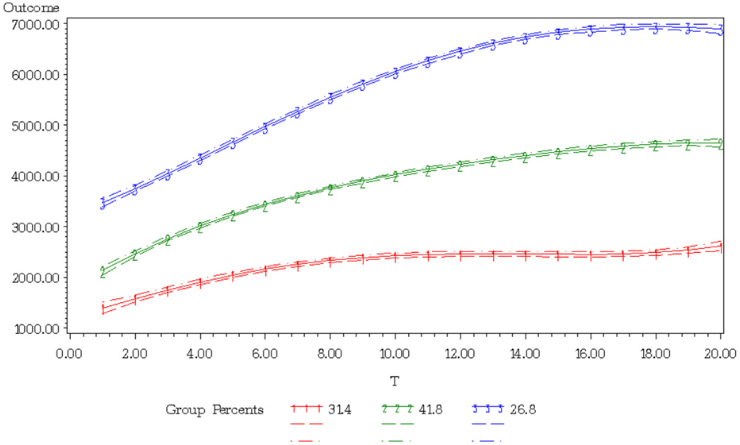
Result for 3 groups (dataset 2): workers beginning their career in 1982



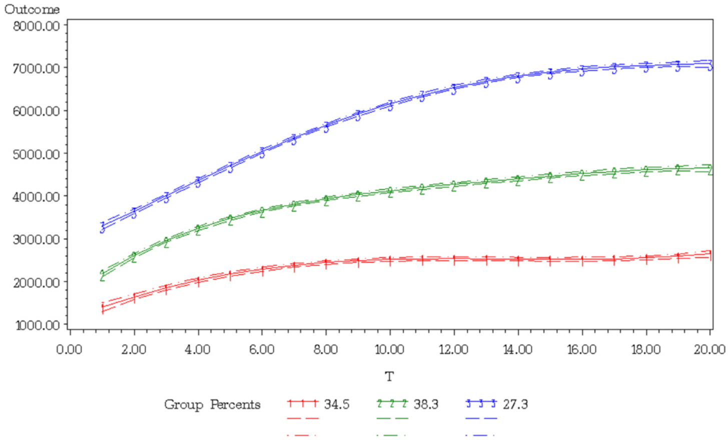
Result for 3 groups (dataset 2): workers beginning their career in 1983



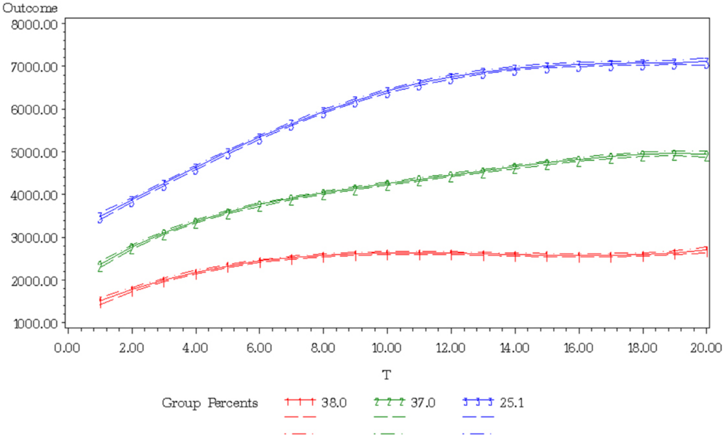
Result for 3 groups (dataset 2): workers beginning their career in 1984



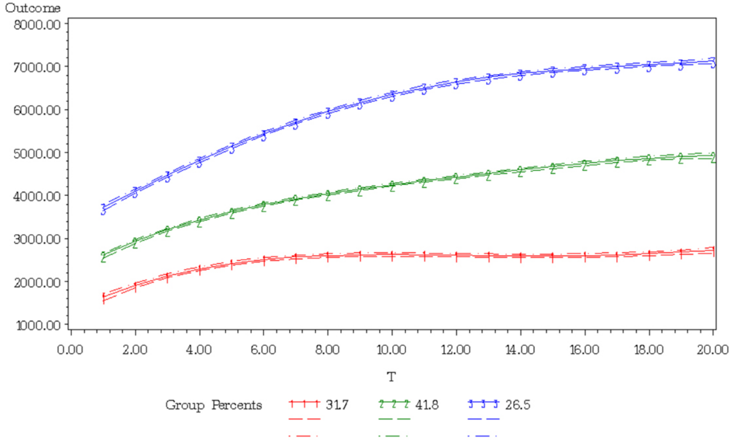
Result for 3 groups (dataset 2): workers beginning their career in 1985



Result for 3 groups (dataset 2): workers beginning their career in 1986



Result for 3 groups (dataset 2): workers beginning their career in 1987



The statistical shape analysis approach

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Comparing the geometrical figure of the trajectories

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$\longrightarrow$ statistical shape analyis:

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Compute the mean shape of the different results.

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Comparing the geometrical figure of the trajectories

$\longrightarrow$ statistical shape analyis:

Compute the mean shape of the different results.

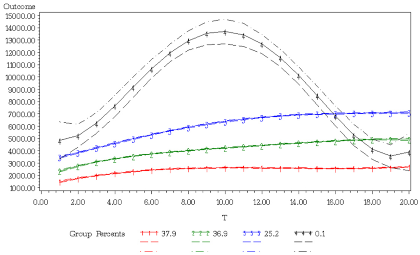
Use Ziezold's test for every set of trajectories to see if it is significantly different from the mean set of trajectories.

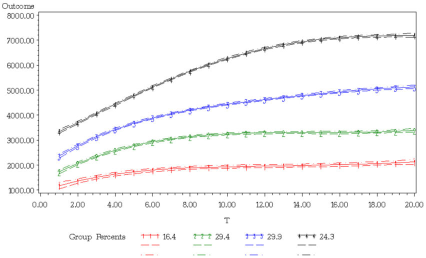
The statistical shape analysis approach

Are these set of trajectories different?

The statistical shape analysis approach

Are these set of trajectories different?





The statistical shape analysis approach

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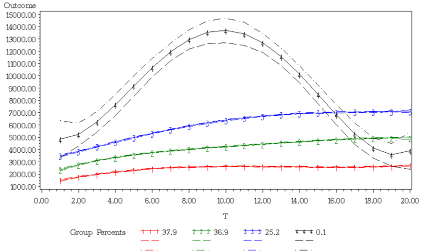

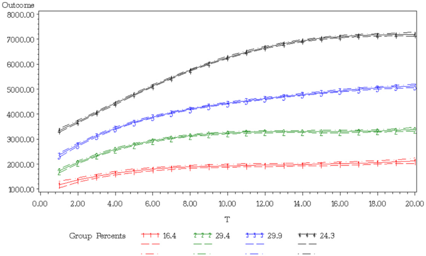

Shape Analysis says yes!

The statistical shape analysis approach

Are these set of trajectories different?

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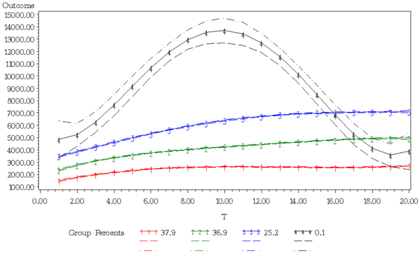

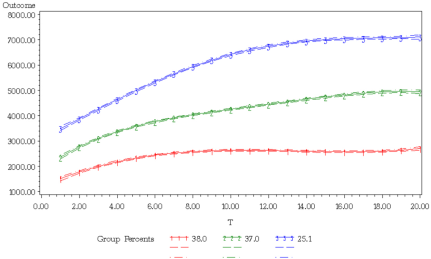

The statistical shape analysis approach

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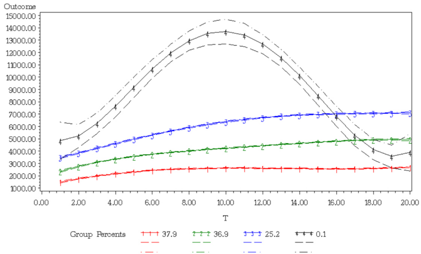
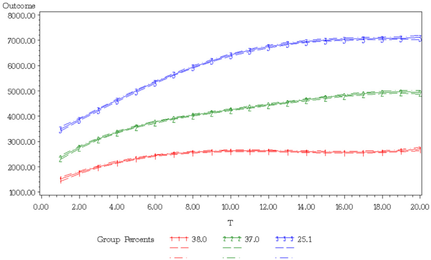
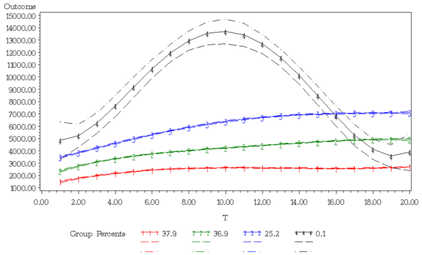


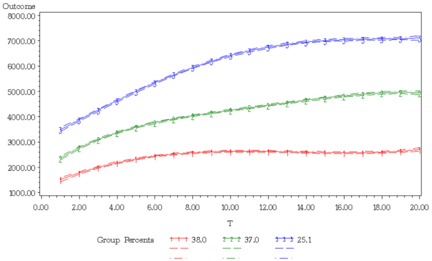


Shape Analysis says yes,

The statistical shape analysis approach

Are these set of trajectories different?





Shape Analysis says yes, but are they really?

The classical statistics approach

The classical statistics approach

Compare the estimated parameters:

The classical statistics approach

Compare the estimated parameters:

- Performing the Wald test to see if the parameters differ between two models.

The classical statistics approach

Compare the estimated parameters:

- Performing the Wald test to see if the parameters differ between two models.
- Compare the confidence intervals of the parameters and see if they have an intersection.

Functional Data Analysis Approach

Functional Data Analysis Approach

Compare the set of trajectories as functions:

Functional Data Analysis Approach

Compare the set of trajectories as functions:

Consider a metrical space on the continuous functions defined on the time interval of the trajectories and use tests on functional data to analyze the time stability of the results.

Outline

LUXEMBOURG SCHOOL OF FINANCE

Predictors of trajectory group membership

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x_i : vector of variables potentially associated with group membership (measured before t_1).

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Multinomial logit model:

$$\pi_j(x_i) = \frac{e^{x_i \theta_j}}{\sum_{k=1}^{r} e^{x_i \theta_k}}, \tag{14}$$

where θ_j denotes the effect of x_i on the probability of group membership.

LUXEMBOURG SCHOOL OF FINANCE

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where θ_j denotes the effect of x_i on the probability of group membership.

$$L = \frac{1}{\sigma}\prod_{i=1}^{N}\sum_{j=1}^{r}\frac{e^{x_i\theta_j}}{\sum_{k=1}^{r} e^{x_i\theta_k}}\prod_{t=1}^{T}\phi\left(\frac{y_{i_t} - \beta^j t_{i_t}}{\sigma}\right). \tag{15}$$

Group membership probabilities

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The Wald test which indicates whether any number of ocefficients is significally different, allows the statistical testing of the predictors.

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Confidence intervals for the probabilities of group membership can be computed by a parametric bootstrap technique.

Group membership probabilities: macro

Proc Traj Macro:

Group membership probabilities: macro

Proc Traj Macro:

```
DATA TEST;
      INPUT ID O1-O20 T1-T20 NATIO SEXE;
      CARDS;
data
RUN;
```

Group membership probabilities: macro

Proc Traj Macro:

```
DATA TEST;
      INPUT ID O1-O20 T1-T20 NATIO SEXE;
      CARDS;
data
RUN;

PROC TRAJ DATA=TEST OUTPLOT=OP OUTSTAT=OS OUT=OF
OUTEST=OE ITDETAIL;
      ID ID; VAR O1-O20; INDEP T1-T20;
      MODEL CNORM; MAX 15000; NGROUPS 3; ORDER 4 4 4;
      RISK NATIO SEXE; RUN;
```

LUXEMBOURG SCHOOL OF FINANCE

Group membership probabilities: results

LUXEMBOURG SCHOOL OF FINANCE

Group membership probabilities: results

Maximum Likelihood Estimates
Model: Censored Normal (CNORM)

Group	Parameter	Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
1	Intercept	1275.04779	62.99065	20.242	0.0000
	Linear	389.21202	38.92642	9.999	0.0000
	Quadratic	-39.15162	7.30285	-5.361	0.0000
	Cubic	1.59498	0.51761	3.081	0.0021
	Quartic	-0.02140	0.01226	-1.746	0.0808
2	Intercept	2222.80424	54.84704	40.527	0.0000
	Linear	424.05294	34.28352	12.369	0.0000
	Quadratic	-37.06840	6.43629	-5.759	0.0000
	Cubic	1.86457	0.45611	4.088	0.0000
	Quartic	-0.03661	0.01079	-3.392	0.0007
3	Intercept	3320.52407	69.05348	48.086	0.0000
	Linear	404.79252	43.10582	9.391	0.0000
	Quadratic	-4.60135	8.09472	-0.568	0.5697
	Cubic	-0.80156	0.57341	-1.398	0.1622
	Quartic	0.02479	0.01356	1.828	0.0675
	Sigma	931.29644	3.56268	261.403	0.0000
	Group membership				
1	Constant	(0.00000)	.	.	.
2	Constant	0.42351	0.11171	3.791	0.0002
	NATIO	0.11996	0.02712	4.424	0.0000
	SEXE	-0.76451	0.12195	-6.269	0.0000
3	Constant	0.13833	0.11999	1.153	0.2490
	NATIO	0.21875	0.03024	7.233	0.0000
	SEXE	-2.08600	0.15090	-13.823	0.0000

BIC=-285105.7 (N=34320) BIC=-285072.7 (N=1716) AIC=-285012.8 L=-284990.8

LUXEMBOURG SCHOOL OF FINANCE

Adding covariates to the trajectories (1)

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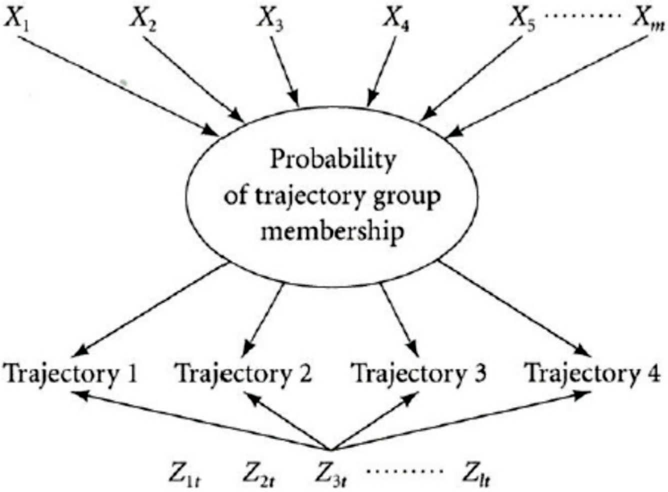

Adding covariates to the trajectories (2)

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Y^*: latent variable measured by Y.

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$$y^*_{i_t} = \beta^j_0 + \beta^j_1 t + \beta^j_2 t^2 + \beta^j_3 t^3 + \beta^j_4 t^4 + \alpha^j_1 z_{1t} + ... + \alpha^j_L z_{Lt} + \varepsilon_{i_t}, \quad (16)$$

where $\varepsilon_{i_t} \sim \mathcal{N}(0, \sigma)$, σ being a constant standard deviation and z_{lt} are covariates that may depend or not upon time t.

Adding covariates to the trajectories (2)

Y^*: latent variable measured by Y.

$$y^*_{i_t} = \beta^j_0 + \beta^j_1 t + \beta^j_2 t^2 + \beta^j_3 t^3 + \beta^j_4 t^4 + \alpha^j_1 z_{1t} + ... + \alpha^j_L z_{Lt} + \varepsilon_{i_t}, \quad (16)$$

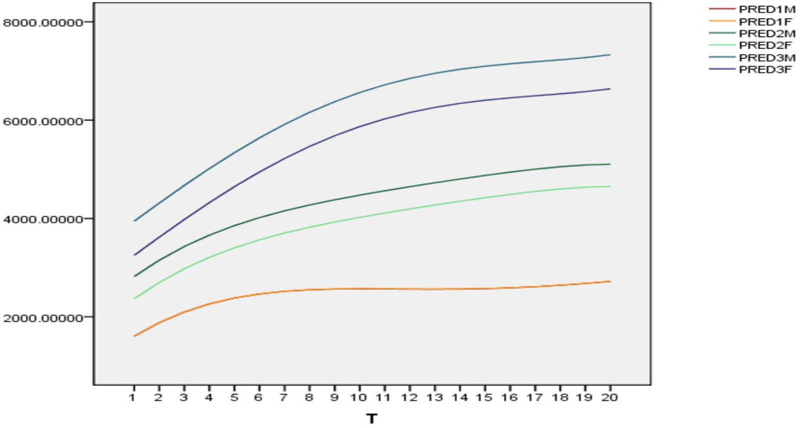
where $\varepsilon_{i_t} \sim \mathcal{N}(0, \sigma)$, σ being a constant standard deviation and z_{lt} are covariates that may depend or not upon time t.

Unfortunately the estimation of parameters α^j_l is not implemented in proc traj procedure; it is just possible to plot the impact of the covariates.

LUXEMBOURG SCHOOL OF FINANCE

Adding covariates to the trajectories (3)

Adding covariates to the trajectories (3)



Functional PLS regression with functional response

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Response $Y = \{Y_t\}_{t \in \mathcal{T}_Y}$ and predictor $X = \{X_t\}_{t \in \mathcal{T}_X}$ of functional type.

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Response $Y = \{Y_t\}_{t \in \mathcal{T}_Y}$ and predictor $X = \{X_t\}_{t \in \mathcal{T}_X}$ of functional type.

The Escoufier operators associated to X and Y are defined by

$$W^X Z = \int_{\mathcal{T}_X} \mathbb{E}(X_t Z) X_t dt, \quad \forall \text{ r.v } Z \tag{17}$$

and

$$W^Y Z = \int_{\mathcal{T}_Y} \mathbb{E}(Y_t Z) Y_t dt, \quad \forall \text{ r.v } Z.$$

Functional PLS regression with functional response

Response $Y = \{Y_t\}_{t \in \mathcal{T}_Y}$ and predictor $X = \{X_t\}_{t \in \mathcal{T}_X}$ of functional type.

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and

$$W^Y Z = \int_{\mathcal{T}_Y} \mathbb{E}(Y_t Z) Y_t dt, \quad \forall \text{ r.v } Z. \tag{18}$$

Theorem (C.Preda, J.S., 2011)

At each step of the PLS regression, the PLS components $t_h, h > 1$ are eigenvectors of the product of the two Esoufier operators i.e.

$$W^X W^Y t_h = \lambda_h. \tag{19}$$

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