Robustness of groups and trajectories in Nagin's finite mixture model

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joint work with
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& Bruno LOVAT (University Nancy II)

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Outline

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This is still an inter-individual model, but unlike other classical models such as standard growth curve models, it allows the existence of subpolulations with completely different behaviors.

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Aim of the analysis: Find r groups of trajectories of a given kind (for instance polynomials of degree 4, $P(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4$.)

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- finite : sums across a finite number of groups
- mixture : population composed of a mixture of unobserved groups

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Software:

SAS-based Proc Traj procedure
by Bobby L. Jones (Carnegie Mellon University).

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Finally,

$$L = \frac{1}{\sigma} \prod_{i=1}^{N} \sum_{j=1}^{r} \frac{e^{\theta_j}}{\sum_{j=1}^{r} e^{\theta_j}} \prod_{t=1}^{T} \phi\left(\frac{y_{i_t} - \beta^j t_{i_t}}{\sigma}\right). \tag{4}$$

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Rule:

The bigger the BIC, the better the model!

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To be classified into a small group, an individual really needs to be strongly consistent with it.

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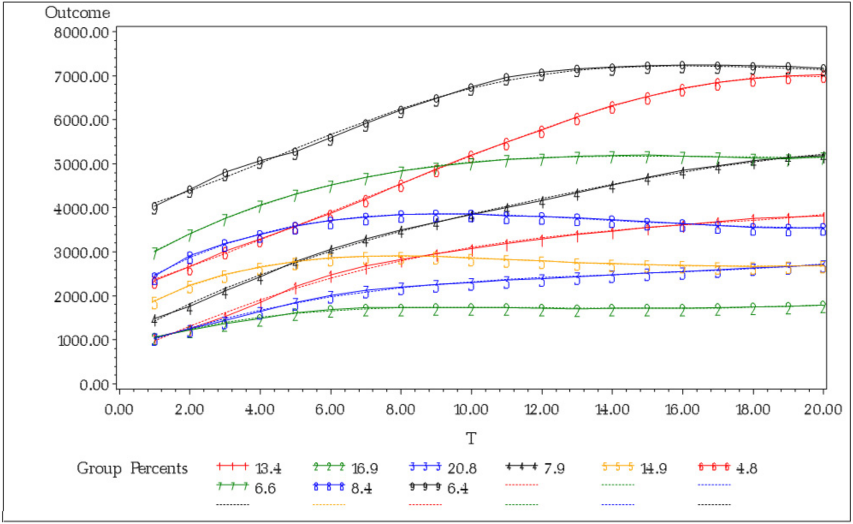

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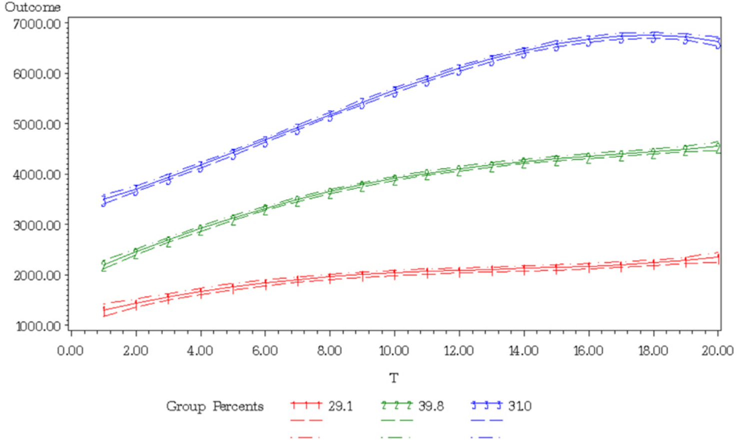
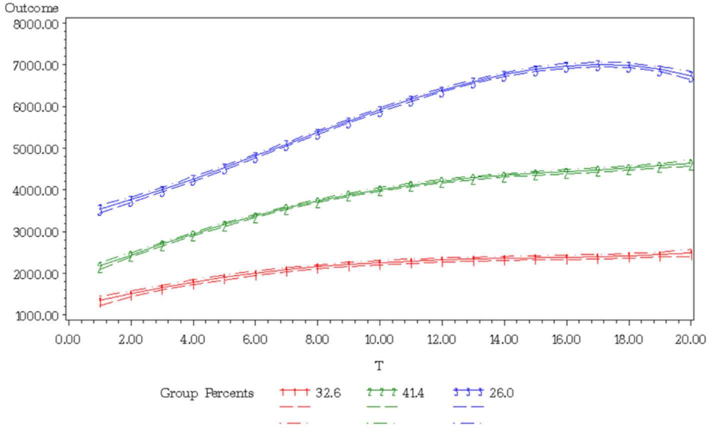
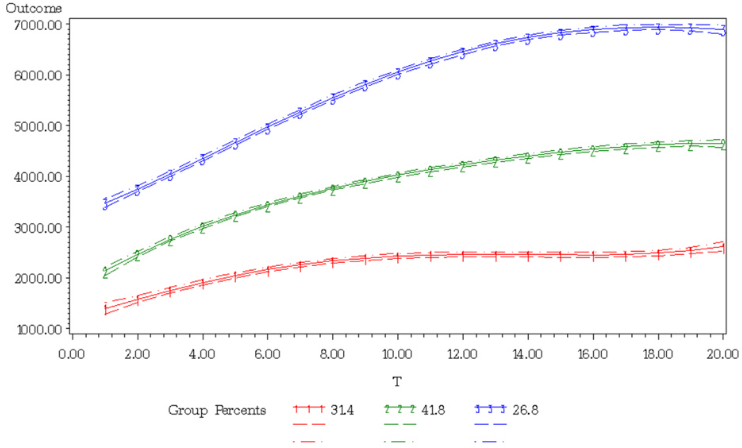
Result for 3 groups : workers beginning their career in 1982



Result for 3 groups : workers beginning their career in 1983



Result for 3 groups : workers beginning their career in 1984



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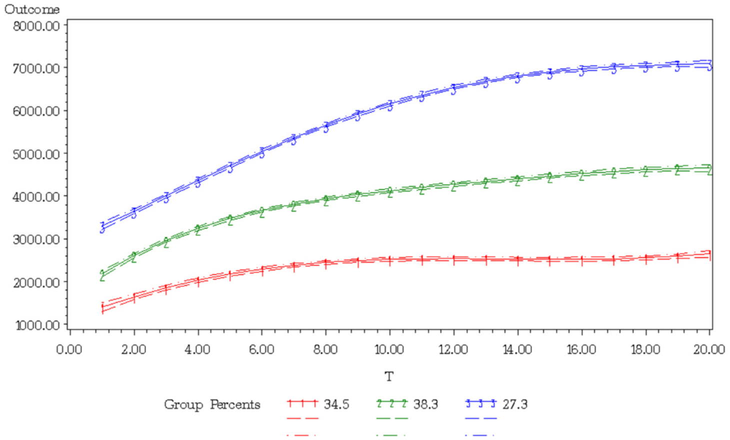

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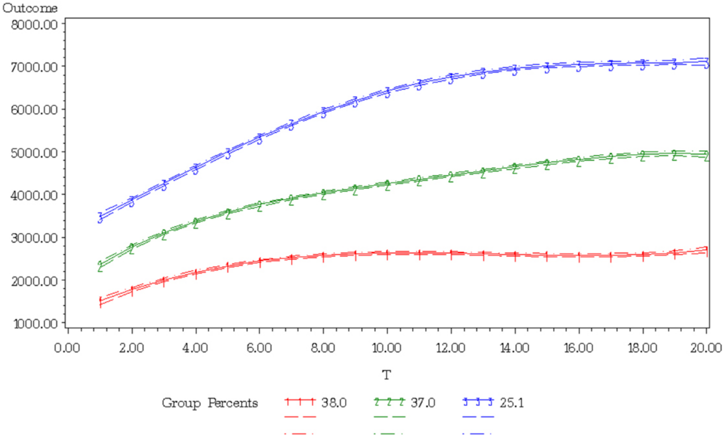
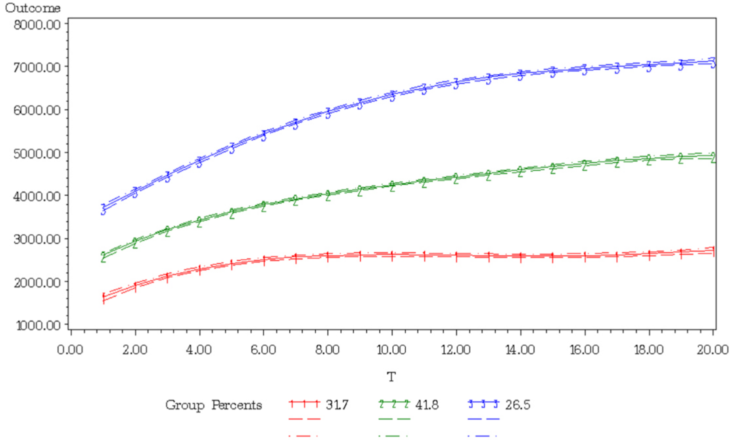


Result for 3 groups : workers beginning their career in 1987



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The statistical shape analysis approach

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Remark:

This apporach is just useful to compare a whole set of models.

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That means that the mean shape is defined as the shape with the smallest variance of all shapes in a group of objects.

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The test hypotheses are:

$$\begin{array}{ll} \textit{Hypothesis:} & H_0 : P = Q \\ \textit{Alternative:} & H_1 : P \neq Q \end{array}$$

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4. Comparing the u_0-value to all possible u-values. Computing the rank (small u-value mean a small rank).
5. Calculate the p-value for H_0. $p_{r=i} = \frac{1}{\binom{N}{n}}$ for $i = 1, \ldots, \binom{N}{n}$, where r is the rank for which we assume a uniform distribution.

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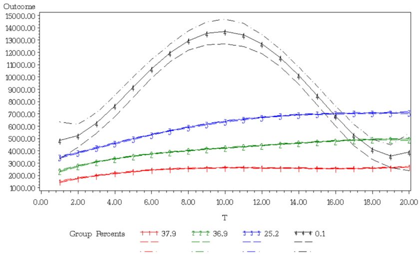

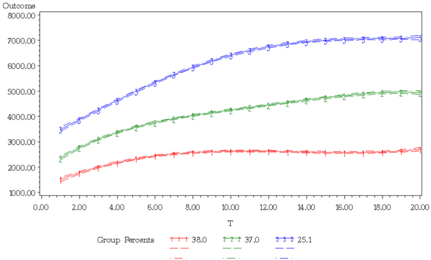

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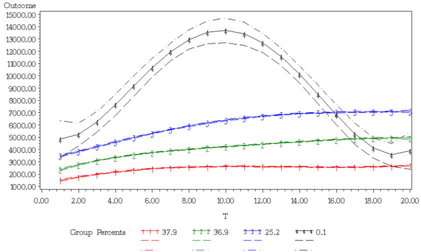

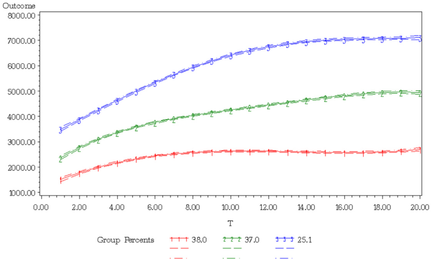

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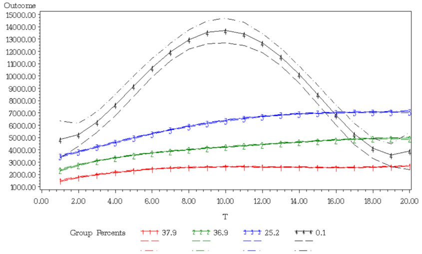

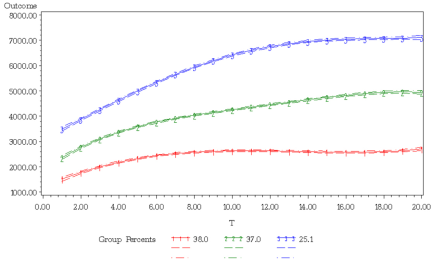

Shape Analysis says yes, but are they really?

The statistical shape analysis approach

Alternative methodology

To avoid this kind of situation, one can take the estimated parameters of the model as landmarks and perform a statistical "shape" analysis on these.

The classical statistics approach

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Compare the estimated parameters:

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- Performing the Wald test to see if the parameters differ between two models.

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Compare the estimated parameters:

- Performing the Wald test to see if the parameters differ between two models.
- Compare the confidence intervals of the parameters and see if they have an intersection.

Functional Data Analysis Approach

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Compare the set of trajectories as functions:

Functional Data Analysis Approach

Compare the set of trajectories as functions:

Consider a metrical space on the continuous functions defined on the time interval of the trajectories and use tests on functional data to analyze the time stability of the results.

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