The weighted lattice polynomials as aggregation functions

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Abstract

We define the concept of weighted lattice polynomials as lattice polynomials constructed from both variables and parameters. We provide equivalent forms of these functions in an arbitrary bounded distributive lattice. We also show that these functions include the class of discrete Sugeno integrals and that they are characterized by a remarkable median based decomposition formula.

Key words: weighted lattice polynomial; lattice polynomial; bounded distributive lattice; discrete Sugeno integral.

1 Introduction

In lattice theory, *lattice polynomials* have been defined as well-formed expressions involving variables linked by the lattice operations $\wedge$ and $\vee$ in an arbitrary combination of parentheses; see e.g. Birkhoff [1, §II.5] and Grätzer [3, §I.4]. In turn, such expressions naturally define *lattice polynomial functions*. For instance,

$$p(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee x_3$$

is a 3-ary lattice polynomial function.

The concept of lattice polynomial function can be straightforwardly generalized by regarding some variables as "parameters", like in the 2-ary polynomial

$$p(x_1, x_2) = (c \vee x_1) \wedge x_2,$$

where c is a constant.

In this paper we investigate those "parameterized" polynomial functions, which we shall call *weighted lattice polynomial* functions. Particularly, we show that, in any bounded distributive lattice, those functions can be expressed in disjunctive and conjunctive normal forms. We also show that they include the discrete Sugeno integral [7], which has been extensively studied and used in the setting of nonlinear aggregation and integration. Finally, we prove that those functions can be characterized by means of a remarkable median based functional system of equations.

Throughout, we let L denote an arbitrary bounded distributive lattice with lattice operations $\wedge$ and $\vee$. We denote respectively by 0 and 1 the bottom and top elements of L. For any integer $n \geqslant 1$, we set $[n] := \{1, \ldots, n\}$ and, for any $S \subseteq [n]$, we denote by $\mathbf{e}_S$ the characteristic vector of S in $\{0,1\}^n$, that is, the n-dimensional vector whose ith component is 1, if $i \in S$, and 0, otherwise. Finally, since L is bounded, we naturally assume that

$$\bigvee_{x \in \varnothing} x = 0 \quad \text{and} \quad \bigwedge_{x \in \varnothing} x = 1.$$

2 Weighted lattice polynomials

Before introducing the concept of weighted lattice polynomial function, let us recall the definition of lattice polynomials; see e.g. Grätzer [3, §I.4].

Definition 2.1. Given a finite collection of variables $x_1, \ldots, x_n \in L$, a *lattice polynomial* in the variables $x_1, \ldots, x_n$ is defined as follows:

1. the variables $x_1, \ldots, x_n$ are lattice polynomials in $x_1, \ldots, x_n$;
2. if p and q are lattice polynomials in $x_1, \ldots, x_n$, then $p \wedge q$ and $p \vee q$ are lattice polynomials in $x_1, \ldots, x_n$;
3. every lattice polynomial is formed by finitely many applications of the rules (1) and (2).

When two different lattice polynomials p and q in the variables $x_1, \ldots, x_n$ represent the same function from L^n to L, we say that p and q are equivalent and we write $p = q$. For instance, $x_1 \vee (x_1 \wedge x_2)$ and x_1 are equivalent.

We now recall that, in a distributive lattice, any lattice polynomial function can be written in disjunctive and conjunctive normal forms, that is, as a join of meets and dually; see [1, §II.5].

Proposition 2.1. *Let $p : L^n \to L$ be any lattice polynomial function. Then there are integers $k, l \geqslant 1$ and families $\{A_j\}_{j=1}^k$ and $\{B_j\}_{j=1}^l$ of nonempty subsets of $[n]$ such that*

$$p(x) = \bigvee_{j=1}^{k} \bigwedge_{i \in A_j} x_i = \bigwedge_{j=1}^{l} \bigvee_{i \in B_j} x_i.$$

Equivalently, there are nonconstant set functions $\alpha : 2^{[n]} \to \{0,1\}$ and $\beta : 2^{[n]} \to \{0,1\}$, with $\alpha(\varnothing) = 0$ and $\beta(\varnothing) = 1$, such that

$$p(x) = \bigvee_{\substack{S \subseteq [n] \\ \alpha(S)=1}} \bigwedge_{i \in S} x_i = \bigwedge_{\substack{S \subseteq [n] \\ \beta(S)=0}} \bigvee_{i \in S} x_i.$$

The concept of lattice polynomial function can be generalized by regarding some variables as parameters. Based on this observation, we naturally introduce the *weighted lattice polynomial* functions as follows.

Definition 2.2. A function $p : L^n \to L$ is an n-ary *weighted lattice polynomial* (w.l.p.) function if there exists an integer $m \geqslant 0$, parameters $c_1, \ldots, c_m \in L$, and a lattice polynomial function $q : L^{n+m} \to L$ such that

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n, c_1, \ldots, c_m) \qquad (x_1, \ldots, x_n \in L).$$

Using Proposition 2.1, we can easily see that any w.l.p. function can be written in disjunctive and conjunctive normal forms; see also [4, 6].

Proposition 2.2. *Let $p : L^n \to L$ be any w.l.p. function. Then there are integers $k, l \geqslant 1$, parameters $a_1, \ldots, a_k, b_1, \ldots, b_l \in L$, and families $\{A_j\}_{j=1}^k$ and $\{B_j\}_{j=1}^l$ of subsets of $[n]$ such that*

$$p(x) = \bigvee_{j=1}^{k} \Big[a_j \wedge \bigwedge_{i \in A_j} x_i\Big] = \bigwedge_{j=1}^{l} \Big[b_j \vee \bigvee_{i \in B_j} x_i\Big].$$

Equivalently, there exist set functions $\alpha : 2^{[n]} \to L$ and $\beta : 2^{[n]} \to L$ such that

$$p(x) = \bigvee_{S \subseteq [n]} \Big[\alpha(S) \wedge \bigwedge_{i \in S} x_i\Big] = \bigwedge_{S \subseteq [n]} \Big[\beta(S) \vee \bigvee_{i \in S} x_i\Big].$$

It follows from Proposition 2.2 that any n-ary w.l.p. function is entirely determined by 2^n parameters. That is, only 2^n parameters will be taken into account in the definition of any n-ary w.l.p. function, even if a larger number of parameters have been considered to construct it.

In the next section, we investigate the link between a given w.l.p. function and the parameters that define it.

Remark. Proposition 2.2 naturally includes the lattice polynomial functions. To see it, it suffices to consider nonconstant set functions $\alpha : 2^{[n]} \to \{0, 1\}$ and $\beta : 2^{[n]} \to \{0, 1\}$, with $\alpha(\varnothing) = 0$ and $\beta(\varnothing) = 1$.

3 Disjunctive and conjunctive forms

Let us denote by $p_\alpha^\vee$ (resp. $p_\beta^\wedge$) the w.l.p. function disjunctively (resp. conjunctively) generated by the set function $\alpha : 2^{[n]} \to L$ (resp. $\beta : 2^{[n]} \to L$), that is,

$$\begin{aligned} p_\alpha^\vee(x) &:= \bigvee_{S \subseteq [n]} \Big[\alpha(S) \wedge \bigwedge_{i \in S} x_i\Big], \\ p_\beta^\wedge(x) &:= \bigwedge_{S \subseteq [n]} \Big[\beta(S) \vee \bigvee_{i \in S} x_i\Big]. \end{aligned}$$

Of course, the set functions α and β are not uniquely determined. For instance, we have seen that $x_1 \vee (x_1 \wedge x_2)$ and x_1 represent the same function.

We now describe the class of all set functions that disjunctively (or conjunctively) generate a given w.l.p. function.

For any w.l.p. function $p : L^n \to L$, set $\alpha_p(S) := p(\mathbf{e}_S)$ and $\beta_p(S) := p(\mathbf{e}_{[n] \setminus S})$.

Proposition 3.1. *Let $p : L^n \to L$ be any w.l.p. function and consider two set functions $\alpha : 2^{[n]} \to L$ and $\beta : 2^{[n]} \to L$.*

1. *We have* $p^{\vee}_{\alpha} = p$ *if and only if* $\alpha^*_p \leqslant \alpha \leqslant \alpha_p$, *where the set function* $\alpha^*_p : 2^{[n]} \to L$ *is defined as*

$$\alpha^*_p(S) = \begin{cases} \alpha_p(S), & \text{if } \alpha_p(S) > \alpha_p(S \setminus \{i\}) \text{ for all } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

2. *We have* $p^{\wedge}_{\beta} = p$ *if and only if* $\beta_p \leqslant \beta \leqslant \beta^*_p$, *where the set function* $\beta^*_p : 2^{[n]} \to L$ *is defined as*

$$\beta^*_p(S) = \begin{cases} \beta_p(S), & \text{if } \beta_p(S) < \beta_p(S \setminus \{i\}) \text{ for all } i \in S, \\ 1, & \text{otherwise.} \end{cases}$$

Example 3.1. The possible disjunctive expressions of $x_1 \vee (x_1 \wedge x_2)$ as a 2-ary w.l.p. function are given by

$$x_1 \vee (c \wedge x_1 \wedge x_2) \qquad (c \in [0,1]).$$

For $c = 0$, we retrieve x_1 and, for $c = 1$, we retrieve $x_1 \vee (x_1 \wedge x_2)$.

We note that, from among all the set functions that disjunctively (or conjunctively) generate a given w.l.p. function p, only α_p (resp. β_p) is isotone (resp. antitone). Indeed, suppose for instance that α is isotone. Then, for all $S \subseteq [n]$, we have

$$\alpha(S) = \bigvee_{K \subseteq S} \alpha(K) = \alpha_p(S),$$

that is, $\alpha = \alpha_p$.

4 The discrete Sugeno integral

Certain w.l.p. functions have been considered in the area of nonlinear aggregation and integration. The best known instances are given by the discrete *Sugeno integral*, which is a particular discrete integration with respect to a *fuzzy measure* (see [7, 8]). In this section, we show the relationship between the discrete Sugeno integral and the w.l.p. functions. For a recent survey on the discrete Sugeno integral, see [2].

Definition 4.1. An L-valued *fuzzy measure* on $[n]$ is an isotone set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = 0$ and $\mu([n]) = 1$.

We now introduce the Sugeno integral in its disjunctive form, which is equivalent to the original definition (see [7]).

Definition 4.2. Let μ be an L-valued fuzzy measure on $[n]$. The *Sugeno integral* of a function $x : [n] \to L$ with respect to μ is defined by

$$\mathcal{S}_\mu(x) := \bigvee_{S \subseteq [n]} \Big[\mu(S) \wedge \bigwedge_{i \in S} x_i\Big].$$

Surprisingly, it appears immediately that any function $f : L^n \to L$ is an n-ary Sugeno integral if and only if it is a w.l.p. function fulfilling $f(\mathbf{e}_\varnothing) = 0$ and $f(\mathbf{e}_{[n]}) = 1$. Moreover, as the following proposition shows, any w.l.p. function can be easily expressed in terms of a Sugeno integral.

Proposition 4.1. *For any w.l.p. function $p : L^n \to L$, there exists a fuzzy measure $\mu : 2^{[n]} \to L$ such that*

$$p(x) = \text{median}\Big[p(\mathbf{e}_\varnothing), \mathcal{S}_\mu(x), p(\mathbf{e}_{[n]})\Big].$$

Corollary 4.1. *Consider a function $f : L^n \to L$. The following assertions are equivalent:*

1. *f is a Sugeno integral.*
2. *f is an idempotent w.l.p. function, i.e., such that $f(x, \ldots, x) = x$ for all $x \in L$.*
3. *f is a w.l.p. function fulfilling $f(\mathbf{e}_\varnothing) = 0$ and $f(\mathbf{e}_{[n]}) = 1$.*

5 The median based decomposition formula

Given a function $f : L^n \to L$ and an index $k \in [n]$, we define the functions $f_k^0 : L^n \to L$ and $f_k^1 : L^n \to L$ as

$$\begin{aligned} f_k^0(x) &= f(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \qquad (x \in L^n), \\ f_k^1(x) &= f(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) \qquad (x \in L^n). \end{aligned}$$

Clearly, if f is a w.l.p. function, so are f_k^0 and f_k^1.

Now consider the following system of n functional equations, which we will refer to as the *median based decomposition formula*:

$$f(x) = \text{median}\Big[f_k^0(x), x_k, f_k^1(x)\Big] \qquad (k \in [n]) \tag{1}$$

This remarkable functional system expresses that, for any index k, the variable x_k can be totally isolated in $f(x)$ by means of a median calculated over the variable x_k and the two functions f_k^0 and f_k^1, which are independent of x_k.

The following result shows that this system characterizes the n-ary w.l.p. functions.

Theorem 5.1. *The solutions of the median based decomposition formula (1) are exactly the n-ary w.l.p. functions.*

6 Conclusion

We have introduced the concept of weighted lattice polynomial functions, which generalize the lattice polynomial functions by allowing some variables to be regarded as parameters. We observed that these functions include the class of discrete Sugeno integrals, which have been extensively used not only in aggregation theory but also in fuzzy set theory. Finally, we have provided a median based system of functional equations that completely characterizes the weighted lattice polynomial functions.

Just as particular Sugeno integrals (such as the weighted minima, the weighted maxima, and their ordered versions) have already been investigated and axiomatized (see [2]), certain subclasses of weighted lattice polynomial functions deserve to be identified and investigated in detail. This is a topic for future research.

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